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THE NEHARI MANIFOLD FOR A QUSILINEAR SUB-ELLIPTIC EQUATION WITH A SIGN-CHANGING WEIGHT FUNCTION ON THE HEISENBERG GROUP

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Abstract: In this paper, we investigate the Dirichlet problem for the following quasilinear sub-elliptic equation on the Heisenberg group $-\Delta_{\mathbb{H},p}u = \lambda f(\xi)|u|^{p-2}u + g(\xi)|u|^{r-2}u$. Using the Nehari manifold and fibrering maps, we obtain the existence and multiplicity results of positive weak solution of the equation and show how existence results for positive solutions of the equation are linked to properties of the Nehari manifold, which generalize the corresponding results in Euclidean space.

Keywords: Heisenberg group; Nehari manifold; fibrering maps; sub-p-Laplacian; indefinite weight functions

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1 Introduction

In this paper, we shall discuss the existence and multiplicity of non-negative solutions for the following nonlinear boundary value problem

$$-\Delta_{\mathbb{H},p}u = \lambda f(\xi)|u|^{p-2}u + g(\xi)|u|^{r-2}u \quad \text{in } \Omega\,; \tag{1.1}$$

$$u(\xi) = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is a bounded region with smooth boundary in $\mathbb{H}^N$, $1 < p < r < \frac{pQ}{Q-p}$, $\lambda > 0$ is a real parameter and $f, g : \Omega \to \mathbb{R}$ are given functions which change sign on Ω, i.e. f, g are indefinite weight functions. We assume that $f(\xi),\ g(\xi) \in L^\infty(\Omega)$, $\{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega f(\xi)|u|^p d\xi > 0\} \neq \emptyset$ and $\{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g(\xi)|u|^r d\xi > 0\} \neq \emptyset$.

Problems (1.1)–(1.2) are studied in connection with the corresponding eigenvalue problem for the p-sub-Laplacian

$$\begin{cases} -\Delta_{\mathbb{H},p}u = \lambda f(\xi)|u|^{p-2}u & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.3}$$

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Over the recent past decade, several authors used the Nehari manifold and fibering maps (i.e., maps of the form $t \to J_\lambda(tu)$, where J_λ is the Euler function associated to the equation) to solve semilinear and quasilinear problems (see [3–9, 12]). By the fibering method, Drabek and Pohozaev [9], Bozhkov and Mitidieri [12] studied respectively the existence of multiple solutions to the following p-Laplacian equation

$$\begin{cases} -\Delta_p u = \lambda f(x)|u|^{p-2}u + g(x)|u|^{r-2}u & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

In [6], from the viewpoint of the Nehari manifold, the authors studied the following subcritical semilinear elliptic equation with a sign-changing weight function

$$\begin{cases} -\Delta u = \lambda f(x)u + g(x)|u|^{r-2}u & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $2 < r < \frac{2N}{N-2}$, λ is constant, and $f(x), g(x)$ are smooth functions which may change sign in Ω. Exploiting the relationship between the Nehari manifold and fibering maps, they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. In [8], the author dealt with the similar problem for the case $1 < r < 2$ and discussed the existence and multiplicity of non-negative solutions of (1.5) from a variational viewpoint making use of the Nehari manifold.

The Dirichlet problems (1.1)–(1.2) on the Heisenberg group is a natural generalization of the classical problem on $\mathbb{R}^N$, see [6–11] and their references. It is well known that (1.4) and (1.5) are counterparts of (1.1)–(1.2) in $\mathbb{R}^N$. In this work, we use a variational method which is similar to the fibering method (see [9]) to prove the existence and multiplicity of positive weak solution for problems (1.1)–(1.2), particularly, by using the method of [6].

This paper, except for the introduction, is divided into four sections. In Section 2, we firstly recall some basic facts and necessary known results on the Heisenberg group, and then we consider the eigenvalue problem (1.3). In Section 3, we focus on the Nehari manifold and the connection between the Nehari manifold and the fibrering maps. In Section 4, we discuss the Nehari manifold when $\lambda < \lambda_1(f)$ and show how the behaviour of the manifold as $\lambda \to \lambda_1^-(f)$ depends on the sign of $\displaystyle\int_\Omega g(\xi)\phi_1^r\, d\xi$. In Section 5, using the properties of Nehari manifold we give simple proofs of the existence of two positive solutions.

2 Notations and Preliminaries

Let $\xi = (x_1, \cdots, x_N, y_1, \cdots, y_N, t) = (x, y, t) = (z, t) \in \mathbb{R}^{2N+1}$ with $N \geq 1$. The Heisenberg group $\mathbb{H}^N$ is the set $\mathbb{R}^{2N+1}$ equipped with the group law

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle x',\ y\rangle - \langle x,\ y'\rangle)),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. This group multiplication endows $\mathbb{H}^N$ with a structure of a Lie group. A family of dilations on $\mathbb{H}^N$ is defined as $\delta_\tau(x, y, t) = (\tau x, \tau y, \tau^2 t)$,

$\tau > 0$. The homogeneous dimension with respect to dilations is $Q = 2N + 2$. The sub-Laplacian $\Delta_{\mathbb{H}}$ is obtained from the vector fields $X_i = \partial_{x_i} + 2y_i\partial_t$, $Y_i = \partial_{y_i} - 2x_i\partial_t$, $i = 1, \cdots, N$, as

$$\Delta_{\mathbb{H}} := \nabla_{\mathbb{H}} \cdot \nabla_{\mathbb{H}} = \sum_{i=1}^{N} X_i \circ X_i + Y_i \circ Y_i, \tag{2.1}$$

i.e.,

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \partial_{x_i}^2 + \partial_{y_i}^2 + 4y_i\partial_{x_i}\partial_t - 4x_i\partial_{y_i}\partial_t + 4(x_i^2 + y_i^2)\partial_t^2, \tag{2.2}$$

where $\nabla_{\mathbb{H}}$ is the $2n$-vector $(X_1, \cdots, X_N, Y_1, \cdots, Y_N)$.

For $p > 1$, the sub-p-Laplacian $\Delta_{\mathbb{H},p}$ is defined as

$$\Delta_{\mathbb{H},p} := \nabla_{\mathbb{H}}(|\nabla_{\mathbb{H}}|^{p-2}\nabla_{\mathbb{H}}u)\,. \tag{2.3}$$

For more details concerning the Heisenberg group, see [1, 2].

The space $\mathcal{D}_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \left(\int_\Omega |\nabla_{\mathbb{H}}u|^p d\xi\right)^{\frac{1}{p}}.$$

For notational convenience, we denote $X := D_0^{1,p}(\Omega)$ and define the norm in $L^p(\Omega)$ by $\|u\|_p$. The following lemma will be referred to as the Folland-Stein embedding theorem.

Lemma 2.1 (see [19]) Let $\Omega \subset \mathbb{H}^N$ be a bounded domain. Then the following inclusion is compact

$$\mathcal{D}_0^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \text{for} \quad 1 < q < \frac{pQ}{Q-p}.$$

According to the continuity of the Nemytskii operator (see [20, 22]) and Lemma 2.1, $f \in L^\infty(\Omega)$ implies that

(f) the functional

$$u \mapsto \int_\Omega f(\xi)|u|^p d\xi$$

is weakly continuous on X.

Analogously, it follows from $g \in L^\infty(\Omega)$ and $1 < r < \frac{pQ}{Q-p}$ that

(g) the functional

$$u \mapsto \int_\Omega g(\xi)|u|^r d\xi$$

is weakly continuous on X.

Now, we consider the nonlinear eigenvalue problem (1.3). This eigenvalue problem is also of independent interest (see [13–15]). Set

$$I(u) = \int_\Omega |\nabla_{\mathbb{H}}u|^p d\xi, \quad \mathcal{M}(f) = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega f(\xi)|u|^p d\xi = 1\}.$$

From the definition of the space $\mathcal{D}_0^{1,p}(\Omega)$, it is obvious that I is coercive and weakly lower semi-continuous. We have the following theorem.

Theorem 2.1 If $1 < p < Q$ and $f(\xi)$ satisfies the conditions above, then

(i) there exists the first positive eigenvalue $\lambda_1(f)$ of (1.3) which is variationally expressed as

$$\lambda_1(f) = \inf_{u\in\mathcal{M}(f)} I(u)\,; \tag{2.4}$$

(ii) $\lambda_1(f)$ is simple, i.e., the eigenfunctions associated to $\lambda_1(f)$ are merely a constant multiple of each other;

(iii) $\lambda_1(f)$ is unique, i.e., if $v \geq 0$ is an eigenfunction associated with an eigenvalue λ with $\int_\Omega f(\xi)|v|^p d\xi = 1$, then $\lambda = \lambda_1(f)$.

A key point of the proof of Theorem 2.1 lies on the following lemma.

Lemma 2.2 (see [16]) Let $u \geq 0$ and $v > 0$ be differentiable functions on $\Omega \subset \mathbb{H}^N$, where Ω is a bounded or unbounded domain in $\mathbb{H}^N$. Then we have

$$L(u,v) = R(u,v) \geq 0\,, \tag{2.5}$$

where

$$\begin{aligned} L(u,v) &= |\nabla_{\mathbb{H}} u|^p + (p-1)\frac{u^p}{v^p}|\nabla_{\mathbb{H}} v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla_{\mathbb{H}} v|^{p-2}\nabla_{\mathbb{H}} u\cdot\nabla_{\mathbb{H}} v,\\ R(u,v) &= |\nabla_{\mathbb{H}} u|^p - |\nabla_{\mathbb{H}} v|^{p-2}\nabla_{\mathbb{H}}\Big(\frac{u^p}{v^{p-1}}\Big)\cdot\nabla_{\mathbb{H}} v \end{aligned}$$

for $p > 1$. Moreover, $L(u,v) = 0$ a.e. on Ω if and only if $\nabla_{\mathbb{H}}(\frac{u}{v}) = 0$ a.e. on Ω.

A direct consequence of Theorem 2.1 is

Corollary 2.1 If $1 < p < Q$, $0 < \lambda < \lambda_1(f)$ and $f(\xi)$ satisfies the conditions above, then the eigenvalue problem

$$\begin{cases} -\Delta_{\mathbb{H},p} u - \lambda f(\xi)|u|^{p-2}u = \mu|u|^{p-2}u & \text{in } \Omega\,;\\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

has the first positive eigenvalue $\mu_1(\lambda)$ which is variationally expressed as

$$\mu_1(\lambda) = \inf_{u\in\mathcal{M}(1)} \int_\Omega (|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p)d\xi\,. \tag{2.7}$$

Moreover, $\mu_1(\lambda)$ is simple and unique.

3 The Nehari Manifold

The Euler-Lagrange functional associated with (1.1)–(1.2) is

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla_{\mathbb{H}} u|^p\,d\xi - \frac{\lambda}{p}\int_\Omega f(\xi)|u|^p\,d\xi - \frac{1}{r}\int_\Omega g(\xi)|u|^r\,d\xi \tag{3.1}$$

for all $u \in X$. J_λ may be not bounded from below on whole X, since $p < r$. In order to obtain existence results in this case, we introduce the Nehari manifold

$$\mathcal{N}(\lambda) = \{u \in X : \langle J'_\lambda(u), u\rangle = 0\},$$

where$\langle\, ,\, \rangle$ denotes the usual duality between X and X^*. Thus $u \in \mathcal{N}(\lambda)$ if and only if

$$\int_\Omega |\nabla_{\mathbb{H}} u|^p \, d\xi - \lambda \int_\Omega f(\xi)|u|^p \, d\xi - \int_\Omega g(\xi)|u|^r \, d\xi = 0\,.$$

Clearly $\mathcal{N}(\lambda)$ is a much smaller set than X and, as we will see below, J_λ is much better behaved on $\mathcal{N}(\lambda)$. In particular, on $\mathcal{N}(\lambda)$ we have that

$$J_\lambda(u) = (\frac{1}{p} - \frac{1}{r}) \int_\Omega (|\nabla_{\mathbb{H}} u|^p - \lambda f|u|^p) d\xi = (\frac{1}{p} - \frac{1}{r}) \int_\Omega g|u|^r d\xi\,. \tag{3.2}$$

The Nehari manifold is closely linked to the behaviour of the functions of the form $\phi_u : t \to J_\lambda(tu)$ $(t > 0)$. Such maps are known as fibrering maps and were introduced by Drabek and Pohozaev in [9]. They were also discussed in [6] and [8]. If $u \in X$, we have

$$\phi_u(t) = \frac{t^p}{p} \int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda\, f(\xi)|u|^p] \, d\xi - \frac{t^r}{r} \int_\Omega g(\xi)|u|^r d\xi\,, \tag{3.3}$$

$$\phi'_u(t) = t^{p-1} \int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda\, f(\xi)|u|^p] \, d\xi - t^{r-1} \int_\Omega g(\xi)|u|^r d\xi\,, \tag{3.4}$$

$$\phi''_u(t) = (p-1)t^{p-2} \int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p] \, d\xi - (r-1)t^{r-2} \int_\Omega g(\xi)|u|^r d\xi\,. \tag{3.5}$$

It is easy to see that $u \in \mathcal{N}(\lambda)$ if and only if $\phi'_u(1) = 0$. More generally, $\phi'_u(t) = 0$ if and only if $tu \in \mathcal{N}(\lambda)$, i.e., elements in $\mathcal{N}(\lambda)$ correspond to stationary points of fibering maps. Thus it is natural to subdivide $\mathcal{N}(\lambda)$ into sets corresponding to local minima, local maxima and points of inflection, respectively. It follows from (3.4) and (3.5) that, $\phi'_u(t) = 0$ implies

$$\phi''_u(t) = (p-r)t^{p-2} \int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p] \, d\xi = (p-r)t^{r-2} \int_\Omega g(\xi)|u|^r d\xi. \tag{3.6}$$

Thus we define

$$\mathcal{N}^+(\lambda) = \{u \in \mathcal{N}(\lambda) : \int_\Omega g(\xi)|u|^r d\xi < 0\},$$
$$\mathcal{N}^-(\lambda) = \{u \in \mathcal{N}(\lambda) : \int_\Omega g(\xi)|u|^r d\xi > 0\},$$
$$\mathcal{N}^0(\lambda) = \{u \in \mathcal{N}(\lambda) : \int_\Omega g(\xi)|u|^r d\xi = 0\},$$

so that $\mathcal{N}^+(\lambda)$, $\mathcal{N}^-(\lambda)$, $\mathcal{N}^0(\lambda)$ correspond to minima, maxima and points of inflection, respectively.

Let $u \in X$. Then

(i) if $\int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p]\, d\xi$ and $\int_\Omega g(\xi)|u|^r d\xi$ have the same sign, ϕ_u has exactly one turning point at

$$t(u) = \left[\frac{\displaystyle\int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p]\, d\xi}{\displaystyle\int_\Omega g(\xi)|u|^r d\xi} \right]^{\frac{1}{r-p}},$$

this turning point is a minimum (maximum) so that $t(u)u \in \mathcal{N}^+(\lambda)$ $(\mathcal{N}^-(\lambda))$ if and only if $\int_\Omega g(\xi)|u|^r d\xi < 0\, (> 0)$;

(ii) if $\int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p]\, d\xi$ and $\int_\Omega g(\xi)|u|^r d\xi$ have opposite sign, ϕ_u has no turning points and so no multiples of u lie in $\mathcal{N}(\lambda)$.

Hence we define

$$\begin{aligned} \mathcal{L}^+ &= \{u \in X : \|u\| = 1, \int_\Omega [|\nabla_{\mathbb{H}} u|^p - \lambda f(\xi)|u|^p]\, d\xi > 0\}, \\ \mathcal{B}^+ &= \{u \in X : \|u\| = 1, \int_\Omega g(\xi)|u|^r d\xi > 0\}. \end{aligned} \tag{3.7}$$

Analogously we can define $\mathcal{L}^-$, $\mathcal{L}^0$, $\mathcal{B}^-$, $\mathcal{B}^0$ by replacing '> 0' in (3.7) by '< 0' or '$= 0$', respectively. Then we have

(i) if $u \in \mathcal{L}^+ \cap \mathcal{B}^+$, then the fibering map ϕ_u has a unique critical point which is a local maximum. Moreover, $t(u)u \in \mathcal{N}^-(\lambda)$;

(ii) if $u \in \mathcal{L}^- \cap \mathcal{B}^-$, then the fibering map ϕ_u has a unique critical point which is a local minimum. Moreover, $t(u)u \in \mathcal{N}^+(\lambda)$;

(iii) if $u \in \mathcal{L}^+ \cap \mathcal{B}^-$, then the fibering map ϕ_u is strictly increasing and no multiple of u lies in $\mathcal{N}(\lambda)$;

(iv) if $u \in \mathcal{L}^- \cap \mathcal{B}^+$, then the fibering map ϕ_u is strictly decreasing and no multiple of u lies in $\mathcal{N}(\lambda)$.

Thus the following theorem holds.

Theorem 3.1 If $u \in X\backslash\{0\}$, then

(a) a multiple of u lies in $\mathcal{N}^-(\lambda)$ if and only if $\frac{u}{\|u\|}$ lies in $\mathcal{L}^+ \cap \mathcal{B}^+$;

(b) a multiple of u lies in $\mathcal{N}^+(\lambda)$ if and only if $\frac{u}{\|u\|}$ lies in $\mathcal{L}^- \cap \mathcal{B}^-$;

(c) no multiple of u lies in $\mathcal{N}(\lambda)$ if $u \in \mathcal{L}^+ \cap \mathcal{B}^-$ or $u \in \mathcal{L}^- \cap \mathcal{B}^+$.

The following lemma was stated in [6] (see also [17]) which showed that minimizers on $\mathcal{N}(\lambda)$ are also critical points for J_λ on X.

Lemma 3.1 Suppose that u_0 is a local maximum or minimum for J_λ on $\mathcal{N}(\lambda)$. Then if $u_0 \notin \mathcal{N}^0(\lambda)$, u_0 is a critical point of J_λ.

4 The Case When $\lambda < \lambda_1(f)$

In this section, we discuss the Nehari manifold when $\lambda < \lambda_1(f)$ and show how the behaviour of the manifold as $\lambda \to \lambda_1^-(f)$ depends on the sign of $\int_\Omega g(\xi)\phi_1^r d\xi$. As a consequence, we obtain that equations (1.1)–(1.2) have at least one positive solution in this case.

Suppose $0<\lambda<\lambda_1(f)$. It follows from (2.7) that there exists $\delta(\lambda)>0$ such that

$$\int_\Omega (|\nabla_{\mathbb{H}}u|^p-\lambda f(\xi)|u|^p)d\xi \geq \delta(\lambda)\|u\|^p, \quad \forall u\in X. \tag{4.1}$$

Thus by (4.1), we have

Lemma 4.1 If $1<\lambda<\lambda_1(f)$, then $\mathcal{L}^-$, $\mathcal{L}^0$ and $\mathcal{N}^+(\lambda)$ are empty and $\mathcal{N}^0(\lambda)=\{0\}$. Moreover, $\mathcal{N}^-(\lambda)=\{t(u)u: u\in\mathcal{B}^+\}$ and $\mathcal{N}(\lambda)=\mathcal{N}^-(\lambda)\cap\{0\}$.

We now investigate the behavior of J_λ on $\mathcal{N}^-(\lambda)$. In view of the preceding lemma, we have

Theorem 4.1 If $0<\lambda<\lambda_1(f)$, then $\inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u)>0$.

Proof By (3.2) and the structure of $\mathcal{N}^-(\lambda)$, we easily obtain $J_\lambda(u)>0$ whenever $u\in\mathcal{N}^-(\lambda)$ and so $J_\lambda(u)$ is bounded from below by 0 on $\mathcal{N}^-(\lambda)$. Let $u\in\mathcal{N}^-(\lambda)$, then $v=\frac{u}{\|u\|}\in\mathcal{L}^+\cap\mathcal{B}^+$ and $u=t(v)v$ where $t(v)=\left[\dfrac{\int_\Omega[|\nabla_{\mathbb{H}}v|^p-\lambda f(\xi)|v|^p]\,d\xi}{\int_\Omega g(\xi)|v|^r d\xi}\right]^{\frac{1}{r-p}}$. Denote $b^*=\sup\limits_{\xi\in\Omega} g(\xi)$ and $K^{1/r}$ is a Folland-Stein embedding constant. Then $b^*>0$ and

$$\int_\Omega g(\xi)|v|^r d\xi \leq b^*\int_\Omega |v|^r d\xi\leq b^*K\|v\|^r=b^*K\,. \tag{4.2}$$

Combining (4.1) and (4.2), it yields that

$$\begin{aligned} J_\lambda(u)&=J_\lambda(t(v)v)=(\frac{1}{p}-\frac{1}{r})t^p(v)\int_\Omega(|\nabla_{\mathbb{H}}v|^p-\lambda f|v|^p)d\xi\\ &=(\frac{1}{p}-\frac{1}{r})\frac{\left(\int_\Omega(|\nabla_{\mathbb{H}}v|^p-\lambda f|v|^p)\,d\xi\right)^{\frac{r}{r-p}}}{\left(\int_\Omega g|v|^p d\xi\right)^{\frac{p}{r-p}}}\geq(\frac{1}{p}-\frac{1}{r})\frac{\delta(\lambda)^{\frac{r}{r-p}}}{(b^*K)^{\frac{p}{r-p}}}, \end{aligned}$$

and $\inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u)>0$.

Next, we will show that there exists a minimizer on $\mathcal{N}^-(\lambda)$ which is a critical point of J_λ and also a nontrivial solution of (1.1)–(1.2).

Theorem 4.2 If $0<\lambda<\lambda_1(f)$, then there exists a minimizer of J_λ on $\mathcal{N}^-(\lambda)$ which is a critical point of J_λ.

Proof Let $\{u_m\}\subset\mathcal{N}^-(\lambda)$ be a minimizing sequence, i.e.,

$$\lim_{m\to\infty}J_\lambda(u_m)=\inf_{u\in\mathcal{N}^-(\lambda)}J_\lambda(u).$$

Using (3.2) and (4.1), we obtain

$$J_\lambda(u_m)=(\frac{1}{p}-\frac{1}{r})\int_\Omega(|\nabla_{\mathbb{H}}u_m|^p-\lambda f|u_m|^p)d\xi\geq(\frac{1}{p}-\frac{1}{r})\delta(\lambda)\|u_m\|^p\,, \tag{4.3}$$

thus the sequence $\{u_m\}$ is bounded in X, and so we may assume passing to a subsequence that $u_m \rightharpoonup u_0$ in X. Since

$$J_\lambda(u_m) = (\frac{1}{p} - \frac{1}{r}) \int_\Omega g|u_m|^r d\xi \le b^* K(\frac{1}{p} - \frac{1}{r}) \|u_m\|^r$$

together with (4.3) implies that $\|u_m\| \ge \varepsilon$ for some $\varepsilon > 0$. Using (4.3) again, we deduce that

$$(\frac{1}{p} - \frac{1}{r})\delta(\lambda)\varepsilon^p \le \lim_{m\to\infty} J_\lambda(u_m) = \lim_{m\to\infty} (\frac{1}{p} - \frac{1}{r}) \int_\Omega g|u_m|^r d\xi = (\frac{1}{p} - \frac{1}{r}) \int_\Omega g|u_0|^r d\xi \,,$$

which implies that $u_0 \ne 0$. By (4.1), we get

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f|u_0|^p) d\xi \ge \delta(\lambda)\|u_0\|^p > 0 \,. \tag{4.4}$$

Hence $\frac{u_0}{\|u_0\|} \in \mathcal{L}^+ \cap \mathcal{B}^+$.

We claim that $u_m \to u_0$ in X. Suppose the contradiction, then we have $\|u_0\| < \liminf\limits_{m\to\infty} \|u_m\|$. Together with (f) and (g) in Section 2, it yields

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f|u_0|^p - g|u_0|^r) d\xi < \liminf_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f|u_m|^p - g|u_m|^r) d\xi = 0 \,,$$

which means $\phi'_{u_0}(1) = \displaystyle\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda\, f|u_0|^p - g|u_0|^r)\, d\xi < 0$. Since

$$\phi'_{u_0}(t) = t^{p-1} \left(\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda\, f|u_0|^p)\, d\xi - t^{r-p} \int_\Omega g|u_0|^r d\xi \right) \,,$$

it follows from (4.4) that $\phi'_{u_0}(t) > 0$ for t sufficiently small. Then there exists $0 < \alpha < 1$ such that $\phi'_{u_0}(\alpha) = 0$, i.e., $\alpha u_0 \in \mathcal{N}^-(\lambda)$. In virtue of $u_m \in \mathcal{N}^-(\lambda)$, we conclude that $\phi_{u_m}(t)$ attains its maximum at $t = 1$. Hence

$$J_\lambda(t u_m) = \phi_{u_m}(t) \le \phi_{u_m}(1) = J_\lambda(u_m), \quad \forall t > 0,$$

and $J_\lambda(\alpha u_m) \le J_\lambda(u_m)$. Note that $\alpha u_m \rightharpoonup \alpha u_0$ and $\|u_0\| < \liminf\limits_{m\to\infty} \|u_m\|$, we obtain

$$J_\lambda(\alpha u_0) < \liminf_{m\to\infty} J_\lambda(\alpha u_m) \le \lim_{m\to\infty} J_\lambda(u_m) = \inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) \,,$$

i.e., it is a contradiction. Therefore $u_m \to u_0$ in X. This implies that

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f|u_0|^p - g|u_0|^r) d\xi = 0 \tag{4.5}$$

and

$$J_\lambda(u_0) = \lim_{m\to\infty} J_\lambda(u_m) = \inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) \,. \tag{4.6}$$

From (4.5), (4.4) and (4.6), we conclude that u_0 is a minimizer for J_λ on $\mathcal{N}^-(\lambda)$. Since $\displaystyle\int_\Omega g(\xi)|u_0|^r d\xi > 0$, $u_0 \notin \mathcal{N}^0(\lambda)$. By Lemma 3.1 u_0 is a critical point of J_λ. Since $J_\lambda(|u|) = J_\lambda(u)$, by applying Harnack inequality [18], we may assume that u_0 is positive.

As a direct consequence of Theorem 4.2, we obtain the following existence theorem.

Theorem 4.3 Equations (1.1)–(1.2) have at least one positive solution whenever $0 < \lambda < \lambda_1(f)$.

We conclude this section by proving some properties of the branch of solutions bifurcated from $\lambda_1(f)$ whenever the condition $\int_\Omega g(\xi)\phi_1^r\,d\xi > 0$ is satisfied. The case where $\int_\Omega g(\xi)\phi_1^r\,d\xi < 0$ which gives rise to multiple solutions when $\lambda > \lambda_1(f)$ will be discussed in the next section.

Theorem 4.4 Suppose $\int_\Omega g(\xi)\phi_1^r\,d\xi > 0$. Then

(i) $\lim\limits_{\lambda\to\lambda_1^-(f)}\inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) = 0$;

(ii) if $\lambda_m \to \lambda_1^-(f)$ and u_m is a minimizer of J_{λ_m} on $\mathcal{N}^-(\lambda)$ (we may assume that $u_m > 0$), then $\lim\limits_{m\to\infty} u_m = 0$. Moreover, $\lim\limits_{m\to\infty}\frac{u_m}{\|u_m\|} = \phi_1$.

Proof (i) We may assume, without loss of generality, that $\|\phi_1\| = 1$. Since $\int_\Omega g(\xi)\phi_1^r\,d\xi > 0$ and $\lambda < \lambda_1(f)$, we have $\phi_1 \in \mathcal{L}^+\cap\mathcal{B}^+$. Hence $t(\phi_1)\phi_1 \in \mathcal{N}^-(\lambda)$, where

$$t(\phi_1) = \left[\frac{\int_\Omega(|\nabla_{\mathbb{H}}\phi_1|^p - \lambda f(\xi)\phi_1^p)d\xi}{\int_\Omega g(\xi)\phi_1^r d\xi}\right]^{\frac{1}{r-p}} = \left[(\lambda_1(f)-\lambda)\frac{\int_\Omega f(\xi)\phi_1^p d\xi}{\int_\Omega g(\xi)\phi_1^r d\xi}\right]^{\frac{1}{r-p}}.$$

Therefore we have

$$\begin{aligned} J_\lambda(t(\phi_1)\phi_1) &= (\frac{1}{p}-\frac{1}{r})\int_\Omega g(\xi)|t(\phi_1)\phi_1|^r d\xi \\ &= (\frac{1}{p}-\frac{1}{r})(\lambda_1(f)-\lambda)^{\frac{r}{r-p}}\frac{\left(\int_\Omega f(\xi)\phi_1^p d\xi\right)^{\frac{r}{r-p}}}{\left(\int_\Omega g(\xi)\phi_1^r d\xi\right)^{\frac{p}{r-p}}} \\ &\to 0, \quad \text{as}\,\lambda\to\lambda_1^-(f). \end{aligned}$$

Since $0 < \inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) \le J_\lambda(t(\phi_1)\phi_1)$, it follows from the above relation that

$$\lim_{\lambda\to\lambda_1^-(f)}\inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) = 0\,.$$

(ii) First, we show that every minimizing sequence $\{u_m\}$ on $\mathcal{N}^-(\lambda)$ is bounded. Suppose otherwise, then we may assume without loss of generality that $\|u_m\|\to\infty$. Let $v_m = \frac{u_m}{\|u_m\|}$, we may assume that $v_m \rightharpoonup v_0$ in X. By (f) and (g), we have

$$\lim_{m\to\infty}\int_\Omega f(\xi)|v_m|^p d\xi = \int_\Omega f(\xi)|v_0|^p d\xi, \quad \lim_{m\to\infty}\int_\Omega g(\xi)|v_m|^r d\xi = \int_\Omega g(\xi)|v_0|^r d\xi\,.$$

Using (i), we obtain

$$(\frac{1}{p}-\frac{1}{r})\int_\Omega(|\nabla_{\mathbb{H}} u_m|^p - \lambda_m f(\xi)|u_m|^p)d\xi = (\frac{1}{p}-\frac{1}{r})\int_\Omega g(\xi)|u_m|^r d\xi = J_{\lambda_m}(u_m)\to 0$$

as $m \to \infty$. Divided by $\|u_m\|^p$, we have

$$\lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda_m f(\xi)|v_m|^p) d\xi = 0, \quad \lim_{m\to\infty} \|u_m\|^{r-p} \int_\Omega g(\xi)|v_m|^r d\xi = 0\,.$$

Hence $\int_\Omega g(\xi)|v_0|^r d\xi = \lim_{m\to\infty} \int_\Omega g(\xi)|v_m|^r d\xi = 0$. Now we show that $v_m \to v_0$ in X. Suppose not, and then

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda_1 f(\xi)|v_0|^p) d\xi < \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda_m f(\xi)|v_m|^p) d\xi = 0\,, \tag{4.7}$$

which is a contradiction for (2.4). Hence $v_m \to v_0$ in X. Thus we have

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda_1 f(\xi)|v_0|^p) d\xi = \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda_m f(\xi)|v_m|^p) d\xi = 0\,.$$

Since $\|v_m\| = 1$, we have $\|v_0\| = 1$. It then follows from Theorem 2.1 that $v_0 = \phi_1$ and

$$\int_\Omega g(\xi)\phi_1^r d\xi = \int_\Omega g(\xi)|v_0|^r d\xi = 0\,,$$

a contradiction. Therefore $\{u_n\}$ is bounded.

Thus we may assume, without loss of generality, that $u_m \rightharpoonup u_0$. Then by analogous argument above on $\{u_m\}$, it follows that $u_m \to u_0$ and $u_0 = 0$. Moreover, $\frac{u_m}{\|u_m\|} \to \phi_1$ and so the proof is complete.

5 The Case When $\lambda > \lambda_1(f)$

In this section, with the properties of Nehari manifold, we shall give simple proofs of the existence of two positive solutions, one in $\mathcal{N}^-(\lambda)$ and the other in $\mathcal{N}^+(\lambda)$.

If $\lambda > \lambda_1(f)$, then

$$\int_\Omega (|\nabla_{\mathbb{H}} \phi_1|^p - \lambda f(\xi)\phi_1^p) d\xi = (\lambda_1(f) - \lambda) \int_\Omega f(\xi)\phi_1^p d\xi < 0\,.$$

This yields $\phi_1 \in \mathcal{L}^-$. Hence, $\phi_1 \in \mathcal{L}^- \cap \mathcal{B}^-$ and $\mathcal{N}^+(\lambda) \neq \emptyset$ if $\int_\Omega g(\xi)\phi_1^r d\xi < 0$. As we shall see, $\mathcal{N}(\lambda)$ may consist of two distinct components. Problems (1.1)–(1.2) have at least two positive solutions, if we show that J_λ has an appropriate minimizer on each component.

The following lemma provides a useful property of the positive solutions to our problem.

Lemma 5.1 Suppose $\int_\Omega g(\xi)\phi_1^r d\xi < 0$. Then there exists $\delta > 0$ such that $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$ whenever $\lambda_1(f) \leq \lambda < \lambda_1(f) + \delta$.

Proof Suppose that the result is false. Then there exist sequences $\{\lambda_m\}$ and $\{u_m\}$ such that $\|u_m\| = 1$, $\lambda_m \to \lambda_1^+(f)$ and

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda_m f(\xi)|u_m|^p) d\xi \leq 0\,, \quad \int_\Omega g(\xi)|u_m|^r d\xi \geq 0. \tag{5.1}$$

Since $\{u_m\}$ is bounded, we assume without loss of generality that $u_m \rightharpoonup u_0$ in X.

We now show that $u_m \to u_0$ in X. Suppose otherwise, then $\|u_0\| < \liminf\limits_{m\to\infty} \|u_m\|$ and (f) implies that

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1(f) f|u_0|^p)d\xi < \liminf_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda_m f|u_m|^p)d\xi \le 0\,,$$

which is a contradiction to (2.4). It follows from (5.1), (f) and (g) that

(i) $\displaystyle\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1 f(\xi)|u_0|^p)d\xi \le 0\,,$

(ii) $\displaystyle\int_\Omega g(\xi)|u_0|^r d\xi \ge 0.$

Using (i) and Theorem 2.1, we obtain $u_0 = k\phi_1$ for some constant k. Hence, from (ii) we deduce that $k = 0$ which is impossible as $\|u_0\| = 1$.

We next show that, if $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$, it is possible to obtain more information about the nature of the Nehari manifold.

Theorem 5.1 Suppose $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$, then

(i) $\mathcal{N}^0(\lambda) = \{0\}$;

(ii) $0 \notin \overline{\mathcal{N}^-(\lambda)}$ and $\mathcal{N}^-(\lambda)$ is closed;

(iii) $\mathcal{N}^-(\lambda)$ and $\mathcal{N}^+(\lambda)$ are separated, i.e., $\mathcal{N}^-(\lambda) \cap \overline{\mathcal{N}^+(\lambda)} = \emptyset$;

(iv) $\mathcal{N}^+(\lambda)$ is bounded.

Proof (i) Suppose $u_0 \in \mathcal{N}^0(\lambda)\backslash\{0\}$, then $\frac{u_0}{\|u_0\|} \in \mathcal{L}^0 \cap \mathcal{B}^0 \subset \mathcal{L}^0 \cap \overline{\mathcal{B}^+} = \emptyset$. Hence $\mathcal{N}^0(\lambda) = \{0\}$.

(ii) Suppose $0 \in \overline{\mathcal{N}^-(\lambda)}$, then there exists $\{u_m\} \subseteq \mathcal{N}^-(\lambda)$ such that $u_m \to 0$ in X. Hence

$$0 < \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p)d\xi = \int_\Omega g(\xi)|u_m|^r d\xi \to 0\,.$$

Let $v_m = \frac{u_m}{\|u_m\|}$, then we may assume that $v_m \rightharpoonup v_0$ in X. Clearly

$$0 < \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f(\xi)|v_m|^p)d\xi = \|u_m\|^{r-p} \int_\Omega g(\xi)|v_m|^r d\xi \to 0\,.$$

Thus by (f), we have

$$0 = \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f|v_m|^p)d\xi = 1 - \lim_{m\to\infty} \lambda \int_\Omega f|v_m|^p d\xi = 1 - \lambda \int_\Omega f|v_0|^p d\xi\,,$$

and then $v_0 \ne 0$. Moreover, by the weak lower semicontinuity of the norm and (f), we get

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda f|v_0|^p)d\xi \le \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f|v_m|^p)d\xi = 0\,,$$

and then $\frac{v_0}{\|v_0\|} \in \mathcal{L}^0 \cup \mathcal{L}^-$. Since $\displaystyle\int_\Omega g(\xi)|v_m|^r d\xi > 0$, it follows that $\displaystyle\int_\Omega g(\xi)|v_0|^r d\xi \ge 0$ and $\frac{v_0}{\|v_0\|} \in \overline{\mathcal{B}^+}$. Hence $\frac{v_0}{\|v_0\|} \in \overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+}$, and this is a contradiction. Thus $0 \notin \overline{\mathcal{N}^-(\lambda)}$.

By (i), $\overline{\mathcal{N}^-(\lambda)} \subseteq \mathcal{N}^-(\lambda) \cup \mathcal{N}^0(\lambda) = \mathcal{N}^-(\lambda) \cup \{0\}$. Since $0 \notin \overline{\mathcal{N}^-(\lambda)}$, it follows that $\overline{\mathcal{N}^-(\lambda)} = \mathcal{N}^-(\lambda)$, i.e., $\mathcal{N}^-(\lambda)$ is closed.

(iii) By (i) and (ii),

$$\overline{\mathcal{N}^-(\lambda)} \cap \overline{\mathcal{N}^+(\lambda)} \subseteq \mathcal{N}^-(\lambda) \cap (\mathcal{N}^+(\lambda) \cup \mathcal{N}^0(\lambda)) = (\mathcal{N}^-(\lambda) \cap \mathcal{N}^+(\lambda)) \cup (\mathcal{N}^-(\lambda) \cap \{0\}) = \emptyset,$$

and then $\mathcal{N}^-(\lambda)$ and $\mathcal{N}^+(\lambda)$ are separated.

(iv) Suppose that $\mathcal{N}^+(\lambda)$ is unbounded, then there exists $\{u_m\} \subseteq \mathcal{N}^+(\lambda)$ such that $\|u_m\| \to \infty$ as $m \to \infty$. By definition,

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi = \int_\Omega g(\xi)|u_m|^r d\xi < 0\,.$$

Let $v_m = \frac{u_m}{\|u_m\|}$. According to the above formula, we get

$$\int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f(\xi)|v_m|^p) d\xi = \|u_m\|^{r-p} \int_\Omega g(\xi)|v_m|^r d\xi\,. \tag{5.2}$$

We can assume that $v_m \rightharpoonup v_0$ in X. Since the left-hand side (l.h.s) of (5.2) is bounded but $\|u_m\| \to \infty$, it follows that $\lim\limits_{m\to\infty} \displaystyle\int_\Omega g(\xi)|v_m|^r d\xi = 0$. Then (g) implies $\displaystyle\int_\Omega g(\xi)|v_0|^r d\xi = 0$.

Now we prove that $v_m \to v_0$ in X. Suppose otherwise, then $\|v_0\| < \liminf\limits_{m\to\infty} \|v_m\|$ and

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda f|v_0|^p) d\xi < \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f|v_m|^p) d\xi \le 0\,.$$

Thus $\frac{v_0}{\|v_0\|} \in \mathcal{L}^- \cap \overline{\mathcal{B}^+}$ which is impossible. Hence $v_m \to v_0$ in X.

Since $v_m \to v_0$, we have $\|v_0\| = 1$. Hence $v_0 \in \mathcal{B}^0$ and moreover $v_0 \in \overline{\mathcal{B}^+}$. By (f), we have

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda f|v_0|^p) d\xi = \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f|v_m|^p) d\xi \le 0\,,$$

and then $v_0 \in \overline{\mathcal{L}^-}$. Thus $v_0 \in \overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+}$ which is again impossible. Hence $\mathcal{N}^+(\lambda)$ is bounded.

When $\mathcal{N}^-(\lambda)$ and $\mathcal{N}^+(\lambda)$ are separated and $\mathcal{N}^0(\lambda) = \{0\}$, any non-zero minimizer for J_λ on $\mathcal{N}^-(\lambda)$ (or on $\mathcal{N}^+(\lambda)$) is also a local minimizer on $\mathcal{N}(\lambda)$ which is a critical point for J_λ on $\mathcal{N}(\lambda)$ and a solution of (1.1)–(1.2).

Theorem 5.2 Suppose $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$. Then

(i) every minimizing sequence for J_λ on $\mathcal{N}^-(\lambda)$ is bounded;

(ii) $\inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) > 0$;

(iii) there exists a minimizer of J_λ on $\mathcal{N}^-(\lambda)$.

Proof (i) Suppose that $\{u_m\} \in \mathcal{N}^-(\lambda)$ is a minimizing sequence of J_λ . Then

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi = \int_\Omega g(\xi)|u_m|^r d\xi \to c\,, \tag{5.3}$$

where $c \ge 0$.

Assume that $\{u_m\}$ is unbounded, i.e., $\|u_m\| \to \infty$ as $m \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|}$. Divided (5.3) by $\|u_m\|^p$ gives

$$\int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f(\xi)|v_m|^p) d\xi = \|u_m\|^{r-p} \int_\Omega g(\xi)|v_m|^r d\xi\,. \tag{5.4}$$

Since $\|v_m\| = 1$, we may assume that $v_m \rightharpoonup v_0$ in X. Since the l.h.s of (5.4) is bounded, it follows that $\lim\limits_{m\to\infty}\int_\Omega g(\xi)|v_m|^r d\xi = 0$ and therefore $\int_\Omega g(\xi)|v_0|^r d\xi = 0$.

We now show that $v_m \to v_0$ in X. Suppose otherwise, then

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda f|v_0|^p)d\xi < \lim_{m\to\infty}\int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda f|v_m|^p)d\xi = 0\,.$$

Thus $v_0 \neq 0$ and $\frac{v_0}{\|v_0\|} \in \mathcal{L}^- \cap \mathcal{B}^0$ which is impossible. Hence $v_m \to v_0$ in X. It follows that $\|v_0\| = 1$. Moreover, by (f)

$$\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda f|v_0|^p)d\xi = \int_\Omega g|v_0|^r d\xi = 0\,.$$

This implies that $v_0 \in \mathcal{L}^0 \cap \mathcal{B}^0$ which contradicts to the assumption $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$. Hence u_m is bounded.

(ii) Since $J_\lambda(u) > 0$ on $\mathcal{N}^-(\lambda)$, we have $\inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) \geq 0$. Suppose $\inf\limits_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) = 0$. Let $\{u_m\} \in \mathcal{N}^-(\lambda)$ is a minimizing sequence, then

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f|u_m|^p)d\xi = \int_\Omega g|u_m|^r d\xi = (\frac{1}{p} - \frac{1}{r})^{-1} J_\lambda(u_m) \to 0\,.$$

By (i) we know that $\{u_m\}$ is bounded and we may suppose $u_m \rightharpoonup u_0$ in X. By using exactly the same argument on $\{v_m\}$ in (i), it may be shown that $u_m \to u_0$ in X. By Theorem 5.1 we know that $0 \notin \overline{\mathcal{N}^-(\lambda)}$ and so $u_0 \neq 0$. It then follows exactly as in the proof in (i) that $\frac{u_0}{\|u_0\|} \in \mathcal{L}^0 \cap \mathcal{B}^0$ and this contradicts the assumption $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$.

(iii) Let $\{u_m\} \in \mathcal{N}^-(\lambda)$ is a minimizing sequence of J_λ , then

$$\begin{aligned} J_\lambda(u_m) &= (\frac{1}{p} - \frac{1}{r})\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p)d\xi \\ &= (\frac{1}{p} - \frac{1}{r})\int_\Omega g(\xi)|u_m|^r d\xi \to \inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) > 0, \end{aligned}$$

and then

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f|u_m|^p)d\xi = \int_\Omega g|u_m|^r d\xi \to (\frac{1}{p} - \frac{1}{r})^{-1}\inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u) > 0\,.$$

By (i) we know that $\{u_m\}$ is bounded. We may assume that $u_m \rightharpoonup u_0$ in X. Then by (g), we have $\int_\Omega g(\xi)|u_0|^r d\xi = \lim\limits_{m\to\infty}\int_\Omega g(\xi)|u_m|^r d\xi > 0$ and so $\frac{u_0}{\|u_0\|} \in \mathcal{B}^+$. Since $(\mathcal{L}^0 \cup \mathcal{L}^-) \cap \mathcal{B}^+ = \emptyset$, it follows that $\frac{u_0}{\|u_0\|} \in \mathcal{B}^+ \subseteq \mathcal{L}^+$. Hence $\frac{u_0}{\|u_0\|} \in \mathcal{B}^+ \cap \mathcal{L}^+$ and $\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f|u_0|^p)d\xi > 0$.

Furthermore, we have $t(u_0)u_0 \in \mathcal{N}^-(\lambda)$ where $t(u_0) = \left[\dfrac{\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p)d\xi}{\int_\Omega g(\xi)|u_0|^r d\xi}\right]^{\frac{1}{r-p}}$.

We will show that $u_m \to u_0$ in X. Suppose not, then

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1 f(\xi)|u_0|^p) d\xi < \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi$$
$$= \lim_{m\to\infty} \int_\Omega g(\xi)|u_m|^r d\xi = \int_\Omega g(\xi)|u_0|^r d\xi\,,$$

so $t(u_0) < 1$. Since $t(u_0)u_m \rightharpoonup t(u_0)u_0$ and the map $t \mapsto J_\lambda(tu_m)$ attains its maximum value at $t = 1$, we obtain

$$J_\lambda(t(u_0)u_0) < \liminf_{m\to\infty} J_\lambda(t(u_0)u_m) \le \lim_{m\to\infty} J_\lambda(u_m) = \inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u)\,,$$

a contradiction. Hence $u_m \to u_0$.

We can easily deduce that

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p) d\xi = \int_\Omega g(\xi)|u_0|^r d\xi\,,$$

and therefore $u_0 \in \mathcal{N}(\lambda)$. Since $\int_\Omega g(\xi)|u_0|^r d\xi > 0$, $u_0 \in \mathcal{N}^-(\lambda)$. Also

$$J_\lambda(u_0) = \lim_{m\to\infty} J_\lambda(u_m) = \inf_{u\in\mathcal{N}^-(\lambda)} J_\lambda(u),$$

which implies u_0 is a minimizer for $J_\lambda(u)$ on $\mathcal{N}^-(\lambda)$.

We now turn our attention to $\mathcal{N}^+(\lambda)$.

Theorem 5.3 Suppose $\mathcal{L}^-$ is non-empty and $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$, then there exists a minimizer of $J_\lambda(u)$ on $\mathcal{N}^+(\lambda)$.

Proof Since $\overline{\mathcal{L}^-} \cap \overline{\mathcal{B}^+} = \emptyset$, then $\mathcal{L}^- \cap \mathcal{B}^- = \mathcal{L}^- \neq \emptyset$ and so $\mathcal{N}^+(\lambda) \neq \emptyset$. Denote $b_0 = \inf_{\xi\in\Omega} g(\xi)$. Then $\mathcal{N}^+(\lambda) \neq \emptyset$ implies that $b_0 < 0$. As $\mathcal{N}^+(\lambda)$ is bounded, there exists $M > 0$ such that $\|u\| \le M$ for all $u \in \mathcal{N}^+(\lambda)$. Hence by Lemma 1.2, for $u \in \mathcal{N}^+(\lambda)$, we obtain

$$J_\lambda(u) = (\frac{1}{p} - \frac{1}{r}) \int_\Omega g|u|^r d\xi \ge (\frac{1}{p} - \frac{1}{r}) b_0 \int_\Omega |u|^r d\xi$$
$$\ge (\frac{1}{p} - \frac{1}{r}) b_0 K \|u\|^r \ge (\frac{1}{p} - \frac{1}{r}) b_0 K M^r.$$

It follows that $J_\lambda(u)$ is bounded from below on $\mathcal{N}^+(\lambda)$ and $\inf_{u\in\mathcal{N}^+(\lambda)} J_\lambda(u)$ exists. Clearly $\inf_{u\in\mathcal{N}^+(\lambda)} J_\lambda(u) < 0$. Suppose that $\{u_m\} \subseteq \mathcal{N}^+(\lambda)$ is a minimizing sequence of J_λ , then

$$J_\lambda(u_m) = (\frac{1}{p} - \frac{1}{r}) \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi$$
$$= (\frac{1}{p} - \frac{1}{r}) \int_\Omega g(\xi)|u_m|^r d\xi \to \inf_{u\in\mathcal{N}^+(\lambda)} J_\lambda(u) < 0$$

as $m \to \infty$. Since $\mathcal{N}^+(\lambda)$ is bounded, we may assume $u_m \rightharpoonup u_0$ in X. Then by (g) and (f), we have

$$\int_\Omega g(\xi)|u_0|^r d\xi = \lim_{m\to\infty} \int_\Omega g(\xi)|u_m|^r d\xi < 0$$

and

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p) d\xi \le \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi < 0.$$

Hence $\frac{u_0}{\|u_0\|} \in \mathcal{L}^- \cap \mathcal{B}^-$ and $t(u_0)u_0 \in \mathcal{N}^+(\lambda)$.

Suppose $u_m \not\to u_0$ in X. Then we get

$$\begin{aligned}\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p) d\xi &< \lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda f(\xi)|u_m|^p) d\xi \\ &= \lim_{m\to\infty} \int_\Omega g(\xi)|u_m|^r d\xi = \int_\Omega g(\xi)|u_0|^r d\xi\end{aligned}$$

and $t(u_0) = \left[\dfrac{\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p) d\xi}{\int_\Omega g(\xi)|u_0|^r d\xi}\right]^{\frac{1}{r-p}} > 1$. It follows that $J_\lambda(t(u_0)u_0) \le J_\lambda(u_0) < \lim\limits_{m\to\infty} J_\lambda(u_m) = \inf\limits_{u\in\mathcal{N}^+(\lambda)} J_\lambda(u)$ and this is impossible. Hence $u_m \to u_0$ in X. We thus deduce that

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda f(\xi)|u_0|^p) d\xi = \int_\Omega g(\xi)|u_0|^r d\xi < 0\,,$$

which shows $u_0 \in \mathcal{N}^+(\lambda)$ and then $J_\lambda(u_0) = \lim\limits_{m\to\infty} J_\lambda(u_m) = \inf\limits_{u\in\mathcal{N}^+(\lambda)} J_\lambda(u)$. Hence u_0 is a minimizer for $J_\lambda(u)$ on $\mathcal{N}^+(\lambda)$

Corollary 5.1 Suppose $\displaystyle\int_\Omega g(\xi)\phi_1^r d\xi < 0$ and δ is as in Lemma 5.1. Then equations (1.1)–(1.2) have at least two positive solutions whenever $\lambda_1(f) < \lambda < \lambda_1(f) + \delta$.

Proof Since $\lambda > \lambda_1(f)$, we have that $\phi_1 \in \mathcal{L}^-$. By Theorems 5.2 and 5.3, there exist minimizers u_λ^+ and u_λ^- of $J_\lambda(u)$ on $\mathcal{N}^+(\lambda)$ and $\mathcal{N}^-(\lambda)$, respectively. According to Theorem 5.1, $\mathcal{N}^-(\lambda)$ and $\mathcal{N}^+(\lambda)$ are separated and $\mathcal{N}^0(\lambda) = \{0\}$. Hence there exist at least two minimizers which are local minimizers for J_λ on $\mathcal{N}(\lambda)$. These minimizers does not belong to $\mathcal{N}^0(\lambda)$. Moreover $J_\lambda(u_\lambda^\pm) = J_\lambda(|u_\lambda^\pm|)$ and $|u_\lambda^\pm| \in \mathcal{N}^\pm(\lambda)$, so we may assume $u_\lambda^\pm \ge 0$. By Lemma 3.1, $u_\lambda^\pm$ are critical points of J_λ on X and hence are weak solutions of (1.1)–(1.2). Finally, by the Harnack inequality [18], we obtain that $u_\lambda^\pm$ are positive solutions of (1.1)–(1.2).

Finally in this section, we investigate the nature of $\mathcal{N}^+(\lambda)$ as $\lambda \to \lambda_1^+(f)$.

Theorem 5.4 Suppose $\displaystyle\int_\Omega g(\xi)\phi_1^r d\xi < 0$, $\lambda_m \to \lambda_1^+(f)$ and $u_m \in \mathcal{N}^+(\lambda)$ is a critical point of $J_\lambda(u)$ corresponding to $\lambda = \lambda_m$ (we may assume that $u_m > 0$). Then as $m \to \infty$,

(i) $u_m \to 0\,$;

(ii) $\frac{u_m}{\|u_m\|} \to \phi_1$ in X.

Proof (i) Since $u_m \in \mathcal{N}^+(\lambda)$ is a critical point of $J_{\lambda_m}(u)$, we have

$$\int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda_m f(\xi)|u_m|^p)d\xi = \int_\Omega g(\xi)|u_m|^r d\xi < 0\,.$$

By Lemma 5.1 and Theorem 5.2, we get $\mathcal{N}^+(\lambda)$ is bounded, and so is $\{u_m\}$. We may suppose that $u_m \rightharpoonup u_0$ in X. Suppose $u_m \not\to u_0$, then

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1 f|u_0|^p)d\xi < \liminf_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} u_m|^p - \lambda_m f|u_m|^p)d\xi \le 0,$$

which is impossible because of (2.4). Hence $u_m \to u_0$ as $m \to \infty$. This, together with (f) and (g), implies that

$$\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1 f|u_0|^p)d\xi = \int_\Omega g|u_0|^r d\xi \le 0\,.$$

Combining with (2.4), it yields $\int_\Omega (|\nabla_{\mathbb{H}} u_0|^p - \lambda_1 f|u_0|^p)d\xi = 0$. Thus by Theorem 2.1, we have $u_0 = k\phi_1$ for some k. But, as $\int_\Omega g(\xi)|\phi_1|^r d\xi < 0$, it follows that $k = 0$. Hence $u_m \to 0$ in X.

(ii) Let $v_m = \frac{u_m}{\|u_m\|}$. We may assume that $v_m \rightharpoonup v_0$ in X. Clearly

$$\int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda_m f(\xi)|v_m|^p)d\xi = \|u_m\|^{r-p} \int_\Omega g(\xi)|v_m|^r d\xi$$

and so, since $\|u_m\| \to 0$, $\lim_{m\to\infty} \int_\Omega (|\nabla_{\mathbb{H}} v_m|^p - \lambda_m f(\xi)|v_m|^p)d\xi = 0$. Suppose $v_m \not\to v_0$, then $\|v_0\| < \lim_{m\to\infty} \|v_m\|$ and therefore $\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda_1 f|v_0|^p)d\xi < 0$ which gives us a contradiction. Hence $v_m \to v_0$, so $\|v_0\| = 1$ and $\int_\Omega (|\nabla_{\mathbb{H}} v_0|^p - \lambda_1 f|v_0|^p)d\xi = 0$. It then follows from Theorem 2.1 that $v_0 = \phi_1$ and the proof is completed.

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References

[1] Stein E M. Harmonic analysis[M]. Princeton: Princeton Univ. Press, 1993.

[2] Bonfiglioli A, Lanconelli E, Uguzzoni F. Stratified Lie groups and potential theory for their sub-Laplacians[M]. New York: Springer, 2007.

[3] Alves C O, Hamidi A El. The Nehari manifold and existence of positive solutions to a class of quassilinear problems[J]. Nonl. Anal., 2005, 60(4): 611–624.

[4] Fan Haining, Liu Xiaochun. Multiple positive solutions to a class of quasi-linear elliptic equations involving critical Sobolev exponent[J]. Monatshefte Für Mathematik, 2014, 174(3): 427–447.

[5] Fan Haining, Liu Xiaochun. Multiple positive solutions for a class of quasi-linear elliptic equations involving critical sobolev exponent[J]. Acta Math. Scientia: English, 2014, 34(4): 1111–1126.
[6] Brown K J, Zhang Yangping. The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function[J]. J. Diff. Equ., 2003, 193(2): 481–499.
[7] Brown K J, Wu Tsung-fang. A fibrering map approach to a semilinear elliptic boundary value problem[J]. Electron. J. Diff. Equ., 2007, 69: 1–9.
[8] Brown K J. The Nehari manifold for a semilinear elliptic equation involving a sublinear term[J]. Calc.Var., 2005, 22(4): 483–494.
[9] Drabek P, Pohozaev S I. Positive solutions for the p-Laplacian: Application of the fibering method[J]. Proc. Roy. Soc. Edinburgh Sect., 1997, A127: 703–727.
[10] Zhang Jinguo, Liu Xiaochun. The existence of solutions for a class of elliptic hemivariational inequality[J]. J. Math., 2012, 32(4): 571–581.
[11] Luo Lin, Peng Shuangjie. The concentration behavior of the ground state solutions for Henon equation[J]. J. Math., 2003, 23(4): 417–422.
[12] Bozhkov Y D, Mitodieri E. Existence of multiple solutions for quasilinear systems via Fibering Method[J]. J. Diff. Equ., 2003, 190(1): 239–267.
[13] Dou Jingbo, Niu Pengcheng, Yuan Zixia. A Hardy inequality with remainder terms in the Heisenberg group and the weighted eigenvalue problem[J]. J. Ineq. Appl., 2007, 2(2): 295–314.
[14] Huang Guangyue, Chen Wenyi. Inequalities of eigenvalues for bi-kohn laplacian on heisenberg group[J]. Acta Math. Sci., 2010, 30(1): 125–131.
[15] Dou Jingbo, Han Yazhou. Eigenvalue problems of the p-sub-Laplacian with indefinite weight on the Heisenberg group[J]. Adv. Math., 2011, 40(6): 687–696.
[16] Niu Pengcheng, Zhang Huiqing, Wang Yong. Hardy type and Rellich type inequalities on the heisenberg group[J]. Proc. Amer. Math. Soc., 2001, 129(12): 3623–3630.
[17] Binding P A, Drabek P, Huang Yinxing. On Neumann boundary value problems for some quasilinear elliptic equations[J]. Electron. J. Diff. Equ., 1997, 5: 1–11.
[18] Capogna L, Danielli D, Garofalo N. An embedding theorem and the Harnack inequality for nonlinear subelliptic equations[J]. Comm. P.D.E., 1993, 18: 1765–1794.
[19] Folland G B, Stein E M. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group[J]. Comm. Pure Appl. Math., 1974, 27: 429–522.
[20] Fučik S, Kufner A. Nonlinear differential equations[M]. Amsterdam: Elsevier, 1980.
[21] Struwe M. Variational methods[M]. New York: Springerr-Verlag, 1984.
[22] Willem M. Minimax theorem[M]. Berlin: Birkhauser, 1996.

Heisenberg群上一类具变号权函数的拟线性次椭圆型方程的Nehari流形方法

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摘要: 本文研究了 Heisenberg 群上带有Dirichlet边界条件的拟线性次椭圆方程$-\Delta_{\mathbb{H},p}u = \lambda f(\xi)|u|^{p-2}u + g(\xi)|u|^{r-2}u$. 利用Nehari 流形和纤维映射方法, 获得了方程解的存在性以及多解性结果, 同时说明了上述方程解的存在性是如何随着Nehari 流形的性质而相应地改变, 推广了欧氏空间中相应的结果.

关键词: Heisenberg 群; Nehari 流形; 纤维映射; 次 p-Laplacian; 不定加权函数

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