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A NOTE ON HILBERT TRANSFORM OF A CHARACTERISTIC FUNCTION

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Abstract: Let E be a measurable subset in $\mathbb{R}$ and H be the Hilbert transform. I n this paper, we study some properties on the L^p integral and distribution function of $H(\chi_E)$. Based on elementary and accurate analysis, exact formulas are given for the above integral an distribution function. Our method adopted here gives a new proof of the results in [6].

Keywords: Hilbert transform; distribution function; L^p norm

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1 Introduction

The Hilbert transform is the operator H defined by

$$H(f)(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(x-y)}{y}dy$$

initially for $f \in \mathcal{S}(\mathbb{R})$. A very straight calculus via Fourier transform and Plancherel's equality show that H can be extended to an isomorphic on L^2; i.e.,

$$\|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \tag{1.1}$$

There were also several other ways to prove (1.1), see [2, 7] and references therein. H also satisfies so called Kolmogorov's inequality; i.e., for any $\lambda > 0$, there exists a positive constant C such that

$$|\{x \in \mathbb{R} : |H(f)(x)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R})}. \tag{1.2}$$

The best possible constant C in (1.2) was obtained by Davis in [4]. Moreover by interpolation technique and duality argument, H can be extended to a bounded operator on $L^p(\mathbb{R})$ for all $p > 1$. We can refer to the nice textbooks [3, 5] and [9] for more properties of Hilbert transform.

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Let E be a Lebesgue measurable set with $|E|<\infty$ and denote $H(\chi_E)$ be the Hilbert transform of the characteristic function of the set E. In 1959, Stein and Weiss [8] proved that the distribution function of $H(\chi_E)$ does not depend on the structure of the set E but only on its measure $|E|$. More precisely, for any $\lambda>0$,

$$|\{x\in\mathbb{R}:|H(\chi_E)(x)|>\lambda\}|=\frac{2|E|}{\sinh\pi\lambda}. \tag{1.3}$$

In [1], Colzani, Laeng and Monzón gave an exact formula for the L^p integral of $H(\chi_E)$. For $1<p<\infty$,

$$\int_{\mathbb{R}}|H(\chi_E)(x)|^p dx=\phi(p)|E|, \tag{1.4}$$

where $\phi(p)=\frac{1}{\pi^p}\int_{\mathbb{R}}\frac{|\log|x||^p}{(x-1)^2}dx=2p\int_0^\infty\frac{\lambda^{p-1}}{\sinh(\pi\lambda)}d\lambda$. With an ingenious calculus, ϕ can be represented by Gamma function and Riemann's Zeta function

$$\phi(p)=\frac{4(1-2^{-p})}{\pi^p}\zeta(p)\Gamma(p+1).$$

We recall that $\zeta(p)=\sum\limits_{k=0}^{+\infty}\frac{1}{(k+1)^p}$ and $\Gamma(p+1)=p\int_0^{+\infty}u^{p-1}e^{-u}du$. Recently, Laeng [6] gave a refinement of equations (1.3) and (1.4), respectively. Laeng's results reads the following two theorems.

Theorem 1.1 Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $|E|<\infty$ and let H be the Hilbert transform. For all $1<p<\infty$,

$$\int_E|H(\chi_E)(x)|^p dx = (2-\frac{1}{2^{p-2}})\frac{|E|}{\pi^p}\zeta(p)\Gamma(p+1), \tag{1.5}$$

$$\int_{\mathbb{R}\setminus E}|H(\chi_E)(x)|^p dx = 2\frac{|E|}{\pi^p}\zeta(p)\Gamma(p+1). \tag{1.6}$$

Theorem 1.2 Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $|E|<\infty$ and let H be the Hilbert transform. For any $\lambda>0$,

$$|\{x\in E:|H(\chi_E)(x)|>\lambda\}|=\frac{2|E|}{e^{\pi\lambda}+1}, \tag{1.7}$$

$$|\{x\in\mathbb{R}\setminus E:|H(\chi_E)(x)|>\lambda\}|=\frac{2|E|}{e^{\pi\lambda}-1}. \tag{1.8}$$

We note that in the proof of Theorem 1.2, Laeng used an argument taking Theorem 1.1 for granted. This argument (Lemma 1.4 in [1]) reads

> If $\|f\|_p=\|g\|_p$ for $p_1<p<p_2$ then the distrubtion functions of f and g equals; i.e., $|\{x\in E:|f(x)|>\lambda\}|=|\{x\in E:|f(x)|>\lambda\}|$ for all $\lambda>0$.

Also as pointed in [1], this argument is based on a Mellin transform. However as in the usual way, the $L^p(X)$ norm has layer cake representation

$$\|f\|^p_{L^p(X)}=\int_X|f(x)|^p dx=p\int_0^\infty\lambda^p|\{x\in X:|f(x)|>\lambda\}|d\lambda.$$

Once we proved the distribution function result (Theorem 1.2) in a direct way, Theorem 1.1 is proved with the help of “layer cake representation”.

This short note is just based on the upon argument. In Section 2, we prove Theorem 1.2 which relies on a refinement of the key lemma in [8] by Stein and Weiss. The proof of Theorem 1.2 also relies on a limiting argument. In Section 3, by using Theorem 1.2, we give the proof of Theorem 1.1 on the straight-forward way.

2 Proof of Theorem 1.2

We first recall the following result in [8].

Lemma 2.1 Let E be a compact set in $\mathbb{R}$ with $E = \cup_{i=1}^n [a_i, b_i]$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. Denote $f(x) = \prod\limits_{i=1}^n \frac{x-a_i}{x-b_i}$ be a rational function. Then for any $\xi > 1$,

$$|\{x \in \mathbb{R} : f(x) > \xi\}| = \frac{1}{\xi - 1} \sum_{i=1}^n (b_i - a_i), \tag{2.1}$$

$$|\{x \in \mathbb{R} : f(x) < -\xi\}| = \frac{1}{\xi + 1} \sum_{i=1}^n (b_i - a_i). \tag{2.2}$$

Remark 2.2 Let E as in Lemma 2.1, an observation is for any $\xi > 0$,

$$|\{x \in E : f(x) > \xi\}| = 0 \text{ for any } \xi > 0. \tag{2.3}$$

Indeed since $\{[a_i, b_i)\}_{i=1}^n$ intersect empty each other, for $x \in \cup_{i=1}^n [a_i, b_i)$ there exists only one i_0 such that $x \in [a_{i_0}, b_{i_0})$ with $\frac{x-a_{i_0}}{x-b_{i_0}} \leq 0$ but $\frac{x-a_i}{x-b_i} > 0 (i \neq i_0)$. So

$$f(x) = \frac{x - a_{i_0}}{x - b_{i_0}} \times \prod_{i=1, i \neq i_0}^n \frac{x - a_i}{x - b_i} \leq 0,\ x \in \cup_{i=1}^n [a_i, b_i),$$

which implies that the set $\{x \in E : f(x) > \xi\}$ is at most the collection of finite elements $b_1, \cdots, b_n$ and then is a set of measure zero.

With (2.1) and (2.3), we immediately have

$$|\{x \in \mathbb{R} \backslash E : f(x) > \xi\}| = \frac{1}{\xi - 1} \sum_{i=1}^n (b_i - a_i) \text{ for any } \xi > 1. \tag{2.4}$$

Similar way as discussed above, we also have

$$f(x) = \prod_{i=1}^n \frac{x - a_i}{x - b_i} \geq 0, x \in \mathbb{R} \backslash \cup_{i=1}^n (a_i, b_i]$$

and

$$|\{x \in \mathbb{R} \backslash E : f(x) < -\xi\}| = 0 \text{ for any } \xi > 0. \tag{2.5}$$

By (2.2) and (2.5), we have

$$|\{x \in E : f(x) < -\xi\}| = \frac{1}{\xi+1}\sum_{i=1}^{n}(b_i - a_i) \text{ for any } \xi > 1. \tag{2.6}$$

Similar to Lemma 2.1 and Remark 2.2, we immediate have

Lemma 2.3 Let E as in the Lemma 2.1. Denote $g(x) = f(x)^{-1} = \prod\limits_{k=1}^{n} \frac{x-b_k}{x-a_k}$, we have

$$|\{x \in E : g(x) > \xi\}| = |\{x \in \mathbb{R}\backslash E : g(x) < -\xi\}| = 0 \text{ for any } \xi > 0; \tag{2.7}$$

$$|\{x \in \mathbb{R}\backslash E : g(x) > \xi\}| = |\{x \in \mathbb{R} : g(x) > \xi\}| = \frac{1}{\xi-1}\sum_{i=1}^{n}(b_i - a_i) \text{ for any } \xi > 1; \tag{2.8}$$

$$|\{x \in E : g(x) < -\xi\}| = |\{x \in \mathbb{R} : g(x) < -\xi\}| = \frac{1}{\xi+1}\sum_{i=1}^{n}(b_i - a_i) \text{ for any } \xi > 1. \tag{2.9}$$

The following lemma asserts that Theorem 1.2 is right for a compact set $E \subset \mathbb{R}$.

Lemma 2.4 Let E be a compact set, equations (1.7) and (1.8) preserve.

Proof For any compact set E, we can write $E = \cup_{i=1}^{n}[a_i, b_i]$ with $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. Define $f(x) = \prod\limits_{i=1}^{n} \frac{x-a_i}{x-b_i}$ and $g(x) = f(x)^{-1} = \prod\limits_{i=1}^{n} \frac{x-b_i}{x-a_i}$ as introduced in Lemma 2.1. By the property of Hilbert transform $H\chi_{[a,b]}(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}$ (see Example 5.1.3 in [5]), we have

$$H(\chi_E)(x) = \frac{1}{\pi}\sum_{k=1}^{n}\log\frac{|x-a_k|}{|x-b_k|} = \frac{1}{\pi}\log|f(x)| = -\frac{1}{\pi}\log|g(x)|.$$

So for any $\lambda > 0$, the set $\{x \in \mathbb{R} : |H(\chi_E)(x)| > \lambda\}$ can be rewrite as

$$\begin{aligned}
& \{x \in \mathbb{R} : H(\chi_E)(x) > \lambda\} \cup \{x \in \mathbb{R} : H(\chi_E)(x) < -\lambda\} \\
= \ & \{x \in \mathbb{R} : |f(x)| > e^{\pi\lambda}\} \cup \{x \in \mathbb{R} : |g(x)| > e^{\pi\lambda}\} \\
= \ & \{x \in \mathbb{R} : f(x) > e^{\pi\lambda}\} \cup \{x \in \mathbb{R} : f(x) < -e^{\pi\lambda}\} \\
& \cup \{x \in \mathbb{R} : g(x) > e^{\pi\lambda}\} \cup \{x \in \mathbb{R} : g(x) < -e^{\pi\lambda}\} \\
=: \ & W_1 \cup W_2 \cup W_3 \cup W_4.
\end{aligned}$$

We note that W_i intersect empty each other. Therefore

$$E \cap \{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\} = (E \cap W_1) \cup (E \cap W_2) \cup (E \cap W_3) \cup (E \cap W_4).$$

By (2.3), (2.6), (2.7) and (2.9) with $\xi = e^{\pi\lambda}$,

$$\begin{aligned}
& |E \cap \{x \in \mathbb{R} : |H(\chi_E)(x)| > \lambda\}| \\
= \ & 0 + \frac{1}{e^{\pi\lambda}+1}\sum_{i=1}^{n}(b_i - a_i) + 0 + \frac{1}{e^{\pi\lambda}+1}\sum_{i=1}^{n}(b_i - a_i) \\
= \ & \frac{2}{e^{\pi\lambda}+1}\sum_{i=1}^{n}(b_i - a_i).
\end{aligned}$$

The same way, by (2.3), (2.5), (2.7) and (2.8) with $\xi = e^{\pi\lambda}$,

$$
\begin{aligned}
& |\mathbb{R}\backslash E \cap \{x \in \mathbb{R} : |H(\chi_E)(x)| > \lambda\}| \\
= \;& |(\mathbb{R}\backslash E) \cap W_1| + |(\mathbb{R}\backslash E) \cap W_2| + |(\mathbb{R}\backslash E) \cap W_3| + |(\mathbb{R}\backslash E) \cap W_4| \\
= \;& \frac{1}{e^{\pi\lambda}-1}\sum_{i=1}^{n}(b_i - a_i) + 0 + \frac{1}{e^{\pi\lambda}-1}\sum_{i=1}^{n}(b_i - a_i) + 0 \\
= \;& \frac{2}{e^{\pi\lambda}-1}\sum_{i=1}^{n}(b_i - a_i).
\end{aligned}
$$

The lemma is proved.

Now we turn to the proof of Theorem 1.2. Since E is finite measurable set, there exists a sequence of compact sets $\{F_n\}$, such that for any n, $F_n \subset E$ and $|E\backslash F_n| \le \frac{1}{n}$. With which we immediately get $\|\chi_E - \chi_{F_n}\|_2 \le \sqrt{\frac{1}{n}}$, and then $\|H(\chi_E) - H(\chi_{F_n})\|_2 \le \sqrt{\frac{1}{n}}$ by (1.1). Now for any fixed $\lambda > 0$, we write

$$\{x \in E : |H(\chi_E)(x)| > \lambda\} = \{x \in F_n : |H(\chi_E)(x)| > \lambda\} \cup \{x \in E\backslash F_n : |H(\chi_E)(x)| > \lambda\}.$$

Then for any $u \in (0,1)$, by Chebyshev's inequality and Lemma 2.4, we have

$$
\begin{aligned}
& |\{x \in E : |H(\chi_E)(x)| > \lambda\}| \\
= \;& |\{x \in F_n : |H(\chi_E)(x)| > \lambda\}| + |\{x \in E\backslash F_n : |H(\chi_E)(x)| > \lambda\}| \\
\le \;& |\{x \in F_n : |H(\chi_E)(x)| > \lambda\}| + |E\backslash F_n| \\
\le \;& |\{x \in F_n : |H(\chi_E)(x) - H(\chi_{F_n})(x)| > (1-u)\lambda\}| \\
& + |\{x \in F_n : |H(\chi_{F_n})(x)| > u\lambda\} + |E\backslash F_n| \\
\le \;& \frac{1}{n(1-u)^2\lambda^2} + \frac{2|F_n|}{e^{\pi u\lambda}+1} + |E\backslash F_n|.
\end{aligned}
$$

Let $n \to \infty$, and then let $u \to 1$, we have

$$|\{x \in E : |H(\chi_E)(x)| > \lambda\}| \le \frac{2|E|}{e^{\pi\lambda}+1}. \tag{2.10}$$

On the other hand, for any n and any $u \in (0,1)$, Chebyshev's inequality and Lemma 2.4 gives

$$
\begin{aligned}
& \mu\{x \in F_n : |H(\chi_{F_n})(x)| > \lambda\} \\
\le \;& |\{x \in E : |H(\chi_{F_n})(x)| > \lambda\}| \\
\le \;& |\{x \in E : |H(\chi_{F_n})(x) - H(\chi_E)(x)| > (1-u)\lambda\}| \\
& + |\{x \in E : |H(\chi_E)(x)| > u\lambda\}| \\
\le \;& \frac{1}{n(1-u)^2\lambda^2} + |\{x \in E : |H(\chi_E)(x)| > u\lambda\}|.
\end{aligned}
$$

Let $n \to \infty$ and then let $u \to 1$, we have

$$|\{x \in E : |H(\chi_E)(x)| > \lambda\}| \ge \frac{2|E|}{e^{\pi\lambda}+1}. \tag{2.11}$$

Both (2.10) and (2.11) give $|\{x \in E : |H(\chi_E)(x)| > \lambda\}| = \frac{2|E|}{e^{\pi\lambda}+1}$. This is just the equation (1.7). We end the proof of Theorem 1.2 since the proof of (1.8) is the similar one.

3 Proof of Theorem 1.1

Proof We only prove (1.5) since we can prove (1.6) in the similar way. For $p > 1$,

$$\int_E |H(\chi_E)(x)|^p dx = p\int_0^{+\infty} \lambda^{p-1}|\{x \in E : |H(\chi_E)(x)| > \lambda\}|d\lambda. \tag{3.1}$$

Then by Theorem 1.2, we have

$$\begin{aligned}
&p\int_0^{+\infty} \lambda^{p-1}|\{x \in E : |H(\chi_E)(x)| > \lambda\}|d\lambda \\
=&2p|E|\int_0^{+\infty} \frac{\lambda^{p-1}}{e^{\pi\lambda}+1}d\lambda = 2p\frac{|E|}{\pi^p}\int_0^{+\infty} \frac{\lambda^{p-1}}{e^{\lambda}+1}d\lambda \\
=&2p\frac{|E|}{\pi^p}\int_1^{+\infty} \frac{(\log t)^{p-1}}{t+1}\frac{1}{t}dt = 2p\frac{|E|}{\pi^p}\int_0^{1} \frac{(-\log t)^{p-1}}{t+1}dt
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\int_0^1 \frac{(-\log t)^{p-1}}{t+1}dt &= \int_0^1 (-\log t)^{p-1}(\sum_{k=0}^{+\infty}(-1)^k t^k)dt \\
&= \sum_{k=0}^{+\infty}(-1)^k \int_0^{+\infty} u^{p-1}e^{-u(k+1)}du \\
&= \sum_{k=0}^{+\infty}(-1)^k \frac{1}{(k+1)^p}\int_0^{+\infty} u^{p-1}e^{-u}du \\
&= (1-2^{1-p})\zeta(p)\Gamma(p).
\end{aligned} \tag{3.3}$$

In the last equality in (3.3), we use $\sum\limits_{k=0}^{+\infty}(-1)^k\frac{1}{(k+1)^p} = (1-2^{1-p})\zeta(p)$. Combining (3.1)–(3.3), (1.5) follows.

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关于特征函数的Hilbert变换的一个注记

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摘 要: 设E是$\mathbb{R}$中一可测子集, H为Hilbert变换. 本文研究了$H(\chi_E)$的L^p积分及其分布函数的相关性质. 利用初等但精细的分析, 给出了上述积分和分布函数的具体表达式. 本文所采用的方法给出了文献[6]中结果的一个新的证明.

关 键 词: Hilbert变换; 分布函数; L^p范数

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