Vol. 37 (2017)
No. 6

数 学 杂 志
J. of Math. (PRC)

T-STRUCTURES INDUCED BY HALF RECOLLEMENTS

YIN You-qi

(*Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China*)
(*Department of Mathematics, Shaoxing College of Arts and Sciences, Shaoxing 312000, China*)

Abstract: Let $\mathcal{C}', \mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. In this paper, we consider how to induce t-structures on $\mathcal{C}'$ and $\mathcal{C}''$ from a t-structure on $\mathcal{C}$ given an upper (resp. lower) recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$. By the concept of left(right) t-exact, we give a sufficient condition such that a t-structure on $\mathcal{C}$ may induce t-structures on $\mathcal{C}'$ and $\mathcal{C}''$, which generalizes some results concerning recollements to upper (resp. lower) recollements.

Keywords: triangulated category; upper(lower) recollement; stable t-structure
2010 MR Subject Classification: 18A40; 18E35; 18E30
Document code: A **Article ID:** 0255-7797(2017)06-1215-05

1 Introduction

Recollements of triangulated categories play an important role in algebraic geometry (see [1]), representation theory (see [2–5]), etc. A recollement $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ of triangulated categories provides a platform for various questions concerning the three terms in a recollement. For examples, given a recollement of a triangulated category $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, t-structures $(\mathcal{C}'^{\leq 0}, \mathcal{C}'^{\geq 0})$ and $(\mathcal{C}''^{\leq 0}, \mathcal{C}''^{\geq 0})$ of $\mathcal{C}'$ and $\mathcal{C}''$, respectively, Beilinson, Bernstein and Deligne [1] proved that $\mathcal{C}$ also has a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, where

$$\mathcal{C}^{\leq 0} := \{A \in \mathcal{C} \mid j^*A \in \mathcal{C}''^{\leq 0}, \quad i^*A \in \mathcal{C}'^{\leq 0}\},$$
$$\mathcal{C}^{\geq 0} := \{B \in \mathcal{C} \mid j^*B \in \mathcal{C}''^{\geq 0}, \quad i^!B \in \mathcal{C}'^{\geq 0}\}.$$

On the other hand, Lin [6] proved that certain t-structure on $\mathcal{C}$ may induce t-structures on $\mathcal{C}'$ and $\mathcal{C}''$. Chen [7] studied the relationship of cotorsion pairs among three triangulated categories in a recollement. She proved the following results: cotorsion pairs on $\mathcal{C}$ may be obtained from cotorsion pairs on $\mathcal{C}'$ and $\mathcal{C}''$ and certain cotorsion pairs on $\mathcal{C}$ may induce cotorsion pairs on $\mathcal{C}'$ and $\mathcal{C}''$. More relevant results can be seen in [8–11], etc.

In a viewpoint of Beilinson, Ginsburg and Schechtman (see [12]), upper and lower recollements are more fundamental than a recollement (upper and lower recollements are

* **Received date:** 2015-11-11 **Accepted date:** 2016-02-18

Foundation item: Supported by National Natural Science Foundation of China (11271251; 11431010; 11571239); Zhejiang Provincial Natural Science Foundation (LY14A010006).

Biography: Yin Youqi (1979–), female, born at Shengzhou, Zhejiang, lecturer, major in represent theory of algebras.

called steps in [8]). For a given upper (lower) recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, a sufficient condition that t-structures on $\mathcal{C}'$ and $\mathcal{C}''$ may be induced by a t-structure on $\mathcal{C}$ is given in this paper.

2 Preliminaries

Recall the following definitions.

Definition 2.1 Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories.

(1) [1] A recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a diagram of triangle functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.1}$$

such that

(R1) $(i^*, i_*), (i_*, i^!), (j_!, j^*)$ and (j^*, j_*) are adjoint pairs;

(R2) $i_*, j_!$ and j_* are fully faithful;

(R3) $j^* i_* = 0$;

(R4) for each $X \in \mathcal{C}$, there are distinguished triangles

$$j_!j^*X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_*i^*X \longrightarrow (j_!j^*X)[1],$$
$$i_*i^!X \xrightarrow{\omega_X} X \xrightarrow{\zeta_X} j_*j^*X \longrightarrow (i_*i^!X)[1],$$

where ϵ_X is the counit of $(j_!, j^*)$, η_X is the unit of (i^*, i_*), ω_X is the counit of $(i_*, i^!)$, and ζ_X is the unit of (j^*, j_*).

(2) [5, 12, 13] Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. An upper recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a diagram of triangle functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"'] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[r, "j^*"'] & \mathcal{C}'' \arrow[l, bend right, "j_!"'] \end{tikzcd} \tag{2.2}$$

such that the conditions involved $i^*, i_*, j_!, j^*$ in (1) are satisfied.

(3) [5, 12, 13] Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. An lower recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is a diagram of triangle functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.3}$$

such that the conditions involved $i_*, i^!, j^*, j_*$ in (1) are satisfied.

For short, we denote respectively the recollement (2.1), upper recollement (2.2) and lower recollement (2.3) by $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, i^!, j_!, j^*, j_*)$, $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i^*, i_*, j_!, j^*)$ and $(\mathcal{C}', \mathcal{C}, \mathcal{C}'', i_*, i^!, j^*, j_*)$, or uniformly by $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$.

We need the following fact.

Lemma 2.2 (see [14]) Let $(\mathcal{C}',\mathcal{C},\mathcal{C}'')$ be an upper recollement. Then there exists a triangle-equivalence $\widetilde{j^*}:\mathcal{C}/i_*\mathcal{C}'\cong\mathcal{C}''$ such that $\widetilde{j^*}V=j^*$, where $V:\mathcal{C}\to\mathcal{C}/i_*\mathcal{C}'$ is the Verdier functor.

The subcategories in this section are full subcategories closed under isomorphisms.

Definition 2.3 [1] Let $\mathcal{C}$ be a triangulated category with the shift functor [1]. A t-structure on $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$ with the following properties:

If we put $\mathcal{D}^{\leq n}:=\mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n}:=\mathcal{D}^{\geq 0}[-n]$, $\forall\ n\in\mathbb{Z}$, we have

(t1) $\mathrm{Hom}_{\mathcal{D}}(X,Y)=0,\ \forall\ X\in\mathcal{D}^{\leq 0},\ Y\in\mathcal{D}^{\geq 1}$;

(t2) $\mathcal{D}^{\leq 0}\subseteq\mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1}\subseteq\mathcal{D}^{\geq 0}$;

(t3) For each $X\in\mathcal{D}$, there is a distinguished triangle

$$A\longrightarrow X\longrightarrow B\longrightarrow A[1],$$

where $A\in\mathcal{D}^{\leq 0},\ \ B\in\mathcal{D}^{\geq 1}$.

Let $(\mathcal{U},\mathcal{V})$ be a t-structure on $\mathcal{C}$. We call $(\mathcal{U},\mathcal{V})$ a stable t-structure, if $\mathcal{U}$ and $\mathcal{V}$ are triangulated subcategories of $\mathcal{C}$ (see [15, Definition 0.2]).

Here are basic properties of stable t-structures.

Lemma 2.4 (see [15]) Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a thick subcategory of $\mathcal{D}$, and $Q:\mathcal{D}\to\mathcal{D}/\mathcal{C}$ the canonical quotient. For a stable t-structure $(\mathcal{U},\mathcal{V})$ on $\mathcal{D}$, the following are equivalent.

(i) $(Q(\mathcal{U}),Q(\mathcal{V}))$ is a stable t-structure on $\mathcal{D}/\mathcal{C}$, where $Q(\mathcal{U})$ (resp. $Q(\mathcal{V})$) is the full subcategory of $\mathcal{D}/\mathcal{C}$ consisting of objects $Q(\mathcal{X})$ for $X\in\mathcal{U}$ (resp. $Q(\mathcal{Y})$ for $Y\in\mathcal{V}$);

(ii) $(\mathcal{U}\cap\mathcal{C},\mathcal{V}\cap\mathcal{C})$ is a stable t-structure on $\mathcal{C}$.

Definition 2.5 [1] Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories with t-structures $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ and $(\mathcal{D}^{\leq 0},\mathcal{D}^{\geq 0})$. An triangle functor $F:\mathcal{C}\longrightarrow\mathcal{D}$ is

(i) left t-exact if $F(\mathcal{C}^{\geq 0})\subset\mathcal{D}^{\geq 0}$;

(ii) right t-exact if $F(\mathcal{C}^{\leq 0})\subset\mathcal{D}^{\leq 0}$.

3 t-Structure Induced by Upper Recollement

This section aims to prove the main result of this paper. Let $\mathcal{C}',\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. Given a upper recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, a t-structure on $\mathcal{C}$ induces t-structures on $\mathcal{C}'$ and $\mathcal{C}''$ under some conditions.

Proposition 3.6 Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories, let diagram (2.2) be an upper recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, and let $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ be a t-structure on $\mathcal{C}$. If i_*i^* is left t-exact and $j_!j^*$ is right t-exact, then

(i) $(i^*(\mathcal{C}^{\leq 0}),i^*(\mathcal{C}^{\geq 0}))$ is a t-structure on $\mathcal{C}'$;

(ii) $(j^*(\mathcal{C}^{\leq 0}),j^*(\mathcal{C}^{\geq 0}))$ is a t-structure on $\mathcal{C}''$;

(iii) If $(\mathcal{C}^{\leq 0},\mathcal{C}^{\geq 0})$ and $(i^*(\mathcal{C}^{\leq 0}),i^*(\mathcal{C}^{\geq 0}))$ are stable t-structures on $\mathcal{C}$ and $\mathcal{C}'$, respectively, then $(j^*(\mathcal{C}^{\leq 0}),j^*(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $\mathcal{C}''$.

Proof (i) For $X \in \mathcal{C}^{\leq 0}$, $Y \in \mathcal{C}^{\geq 1}$, since (i^*, i_*) is an adjoint pair and i_*i^* is left t-exact, we have $\mathrm{Hom}_{\mathcal{C}'}(i^*X, i^*Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, i_*i^*Y) = 0$. Thus (t1) hold.

Condition (t2) follows from the closure of $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ under the shifts [1] and [-1], respectively.

Let $X^{'} \in \mathcal{C}'$. There is a distinguished triangle $A \to i_*X' \to B \to A[1]$ in $\mathcal{C}$, where $A \in \mathcal{C}^{\leq 0}$, $B \in \mathcal{C}^{\geq 1}$. Applying i^* to this triangle, we have $i^*A \to i^*i_*X' \to i^*B \to i^*A[1]$, where $i^*A \in i^*(\mathcal{C}^{\leq 0})$, $i^*B \in i^*(\mathcal{C}^{\geq 1})$. Since i_* is fully faithful and (i^*, i_*) is an adjoint pair, we have $i^*i_*X' \cong X'$. Therefore, the distinguished triangle $i^*A \to X' \to i^*B \to i^*A[1]$ is the t-decomposition of X'. We have condition (t3).

(ii) Similarly, we obtain argument (ii).

(iii) We prove the last statement by three steps.

Step 1 $j_!j^*$ is right t-exact $\Rightarrow i_*i^*$ is right t-exact.

Let $X \in \mathcal{C}^{\leq 0}$, for $Y \in \mathcal{C}^{\geq 1}$. Applying cohomological functor $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ to the distinguished triangle

$$j_!j^*X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_*i^*X \longrightarrow (j_!j^*X)[1],$$

we get an exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(X[1], Y) \to \mathrm{Hom}_{\mathcal{C}}(j_!j^*X[1], Y) \to \mathrm{Hom}_{\mathcal{C}}(i_*i^*X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \cdots.$$

Since $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X[1], Y) = 0$, we get $\mathrm{Hom}_{\mathcal{C}}(i_*i^*X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(j_!j^*X[1], Y) = 0$.

Step 2 We claim $i_*i^*(\mathcal{C}^{\leq 0}) = i_*\mathcal{C}' \cap \mathcal{C}^{\leq 0}$ and $i_*i^*(\mathcal{C}^{\geq 0}) = i_*\mathcal{C}' \cap \mathcal{C}^{\geq 0}$.

By Step 1 we have i_*i^* is right t-exact, i.e. $i_*i^*(\mathcal{C}^{\leq 0}) \subseteq \mathcal{C}^{\leq 0}$. Therefore, $i_*i^*(\mathcal{C}^{\leq 0}) \subseteq i_*\mathcal{C}' \cap \mathcal{C}^{\leq 0}$. Conversely, for $X \in i_*\mathcal{C}' \cap \mathcal{C}^{\leq 0}$, there exists a distinguished triangle $j_!j^*X \to X \to i_*i^*X \to (j_!j^*X)[1]$. Since $X \in i_*\mathcal{C}'$, it follows $j_!j^*X = 0$. Since X is in $\mathcal{C}^{\leq 0}$, we have $X \cong i_*i^*X \subseteq i_*i^*(\mathcal{C}^{\leq 0})$.

Similarly we have $i_*i^*(\mathcal{C}^{\geq 0}) = i_*\mathcal{C}' \cap \mathcal{C}^{\geq 0}$.

Therefore, $(i_*\mathcal{C}' \cap \mathcal{C}^{\leq 0}, i_*\mathcal{C}' \cap \mathcal{C}^{\geq 0}) = (i_*i^*(\mathcal{C}^{\leq 0}), i_*i^*(\mathcal{C}^{\geq 0}))$.

Step 3 Assume that $(i^*(\mathcal{C}^{\leq 0}), i^*(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $\mathcal{C}'$. Since i_* is fully faithful, $(i_*i^*(\mathcal{C}^{\leq 0}), i_*i^*(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $i_*\mathcal{C}'$. By Step 2, $(i_*\mathcal{C}' \cap \mathcal{C}^{\leq 0}, i_*\mathcal{C}' \cap \mathcal{C}^{\geq 0})$ is a stable t-structure on $i_*\mathcal{C}'$. Hence $(Q(\mathcal{C}^{\leq 0}), Q(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $\mathcal{C}/i_*\mathcal{C}'$ by Lemma 2.4. There exists a triangle-equivalence $\widetilde{j^*} : \mathcal{C}/i_*\mathcal{C}' \cong \mathcal{C}''$ such that $j^* = \widetilde{j^*}Q$, so $(j^*(\mathcal{C}^{\leq 0}), j^*(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $\mathcal{C}''$. The proof is completed.

By the similar argument we have statements for lower recollements.

Corollary 3.7 Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories, let diagram (2.3) be a lower recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, and let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ a t-structure on $\mathcal{C}$. If $i_*i^!$ is right t-exact and j_*j^* is left t-exact, then

(i) $(i^!(\mathcal{C}^{\leq 0}), i^!(\mathcal{C}^{\geq 0}))$ is a t-structure on $\mathcal{C}'$;

(ii) $(j^*(\mathcal{C}^{\leq 0}), j^*(\mathcal{C}^{\geq 0}))$ is a t-structure on $\mathcal{C}''$;

(iii) If $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ and $(i^!(\mathcal{C}^{\leq 0}), i^!(\mathcal{C}^{\geq 0}))$ are stable t-structures on $\mathcal{C}$ and $\mathcal{C}'$, respectively, then $(j^*(\mathcal{C}^{\leq 0}), j^*(\mathcal{C}^{\geq 0}))$ is a stable t-structure on $\mathcal{C}''$.

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半粘合诱导的t-结构

尹幼奇

(上海交通大学数学系, 上海 200240)

(绍兴文理学院数学系, 浙江 绍兴 312000)

摘要: 本文研究了对于给定的一个三角范畴的上 (下) 粘合 $(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$, 如何由 $\mathcal{C}$ 的一个 t-结构诱导 $\mathcal{C}'$ 和 $\mathcal{C}''$ 的 t-结构的问题. 利用左 (右) t-正合函子的概念, 给出了由 $\mathcal{C}$ 的一个t-结构可诱导出 $\mathcal{C}'$ 和 $\mathcal{C}''$ 的 t-结构的充分条件. 将粘合的一些相关结果推广到了上 (下) 粘合的情形.

关键词: 三角范畴; 上(下)粘合; 稳定 t-结构

MR(2010)**主题分类号**: 18A40; 18E35; 18E30 **中图分类号**: O153.3