Vol. 37 (2017)
No. 6

数 学 杂 志
J. of Math. (PRC)

BELL POLYNOMIALS AND ITS SOME IDENTITIES

GUO Jing[1], LI Xiao-xue[2]

(*1.School of Mathematics and Computer Science, Jiangxi Science & Technology Normal University, Nanchang 330038, China*)

(*2.School of Mathematics, Northwest University, Xi'an 710127, China*)

Abstract: In this paper, we introduce a new polynomial called Bell polynomials. By using the elementary and combinational methods, we prove some identities for this polynomials. As an application of these identities, we give an interesting congruence for Bell numbers.

Keywords: Bell numbers; Bell polynomials; identity; combinational method

2010 MR Subject Classification: 11B37; 11B83

Document code: A **Article ID:** 0255-7797(2017)06-1201-06

1 Introduction

For any integers $n \geq k \geq 0$, let $S(n,k)$ denote the number of partitions of a set with n elements into k nonempty blocks. It is clear that $S(n,k) > 0$ for all $1 \leq k \leq n$, and $S(n,k) = 0$ for $1 \leq n < k$. Put $S(0,0) = 1$ and $S(0,k) = 0$ for $k \geq 1$, $S(n,0) = 0$ for $n \geq 1$. These numbers were introduced by Stirling in his book "Methodus Differentialis" (see [3–5]). Now they are called as the Stirling numbers of the second kind. These numbers satisfy the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \ (n,k \geq 1).$$

The number of all partitions of a set with n elements is

$$B(n) = \sum_{k=1}^{n} S(n,k),$$

called also a Bell number (or exponential number), related contents can be found in many papers or books. For example, see [6–8].

These numbers satisfy the recurrence formula

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k), \tag{1.1}$$

* **Received date:** 2015-04-14 **Accepted date:** 2015-07-06

Foundation item: Supported by National Natural Science Foundation of China (11371291); Jiangxi Science and Technology Normal University (xjzd2015002).

Biography: Guo Jing (1973–), female, born at Jinxian, Jiangxi, associate professor, major in mathematics.

where $B(0) = 1$ by definition.

The generating function of $B(n)$ is given by

$$\sum_{n=0}^{\infty} \frac{B(n)}{n!} \cdot t^n = \exp\left(e^t - 1\right), \tag{1.2}$$

where $\exp(y) = e^y$.

The numbers $B(n)$ can be represented also as the sum of a convergent series (Dobinski's formula)

$$B(n) = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!} = \frac{1}{e}\left(\frac{1^n}{1!} + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots\right), \tag{1.3}$$

see Pólya and Szegö [9] for these basic properties.

In this paper, we introduce a new polynomials $B(x,n)$ (called Bell polynomials) as follows

$$\exp\left(x(e^t - 1)\right) = \sum_{n=0}^{\infty} \frac{B(n,x)}{n!} \cdot t^n. \tag{1.4}$$

It is clear that $B(0,x) = 1$, $B(1,x) = x$, $B(2,x) = x + x^2$, $B(3,x) = x + 3x^2 + x^3, \cdots$. If $x = 1$, then $B(n,1) = B(n)$, the well known Bell numbers. About the properties of $B(n,x)$, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because it can help us to further understand the properties of Bell numbers.

The main purpose of this paper is using the elementary and combinational methods to study the computational problem of the sums

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{B(a_1,x)}{a_1!} \cdot \frac{B(a_2,x)}{a_2!} \cdots \frac{B(a_k,x)}{a_k!}, \tag{1.5}$$

where $\sum\limits_{a_1+a_2+\cdots+a_k=n}$ denotes the summation over all k-tuples with non-negative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$, and give an exact computational formula for (1.5). That is, we shall prove the following.

Theorem 1 Let k be a positive integer with $k \geq 1$. Then for any positive integer $n \geq 1$, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{B(a_1,x)}{a_1!} \cdot \frac{B(a_2,x)}{a_2!} \cdots \frac{B(a_k,x)}{a_k!} = \frac{B(n,kx)}{n!},$$

where the polynomials $B(n,x)$ satisfy the recurrence formula $B(0,x) = 1$, $B(1,x) = x$, $B(2,x) = x + x^2$, $B(3,x) = x + 3x^2 + x^3$, and

$$B(n+1,x) = x \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot B(i,x) \text{ for all } n \geq 1.$$

For the polynomials $B(n,x)$, we also have a similar Dobinski's formula.

Theorem 2 For any positive integer $n \geq 1$, we have the identities

$$B(n,x) = \frac{1}{e^x}\sum_{m=0}^{\infty}\frac{x^m \cdot m^n}{m!} = \frac{1}{e^x}\left(\frac{x^1\cdot 1^n}{1!} + \frac{x^2\cdot 2^n}{2!} + \frac{x^3\cdot 3^n}{3!} + \cdots\right).$$

From Theorem 1 and the recurrence formula of $B(n,x)$, we may immediately deduce the following congruence.

Corollary 1 Let p be an odd prime. Then for any positive integer $k \geq 1$ with $(k,p) = 1$, we have the congruence

$$k\cdot B(p,x) \equiv B(p,kx) \bmod p \quad \text{and} \quad B'(p,x) \equiv 1 \bmod p.$$

Corollary 2 For any positive integer n, we have the identity

$$B(n+1,x) = x\cdot\left(B'(n,x) + B(n,x)\right) \quad \text{or} \quad B'(n,x) = \sum_{i=0}^{n-1}\binom{n}{i}\cdot B(i,x),$$

where $B'(n,x) = \frac{\partial B(n,x)}{\partial x}$.

2 Proof of the Theorems

In this section, we shall complete the proofs of our theorems. First we give a sample lemma, which are necessary in the proof of our theorems. Hereinafter, we shall use some elementary number theory contents and properties of power series, all of these can be found in references [1] and [2], so they will not be repeated here.

Lemma For any real number x, let function $f(t) = \exp\left(x(e^t-1)\right)$, then we have $f^{(n)}(0) = B(n,x)$ for all integers $n \geq 0$, where $f^{(n)}(t)$ denotes the n^{th} derivative of $f(t)$ for variable t.

Proof We prove this lemma by complete induction. It is clear that $f(0) = 1 = B(0,x)$, $f'(t) = xe^t\cdot\exp\left(x(e^t-1)\right) = xe^t\cdot f(t)$, and $f'(0) = x = B(1,x)$. So the lemma is true for $n = 0, 1$. Assume that $f^{(n)}(0) = B(n,x)$ for all $0 \leq n \leq r$. Then note that $f'(t) = xe^t\cdot f(t)$, so from the properties of derivative (Newton-Leibnitz formula), we have

$$\begin{aligned} f^{(r+1)}(t) &= \left(xe^t\cdot f(t)\right)^r = x\cdot\sum_{i=0}^{r}\binom{r}{i}\cdot\left(e^t\right)^{(r-i)}\cdot f^{(i)}(t) \\ &= x\cdot\sum_{i=0}^{r}\binom{r}{i}\cdot e^t\cdot f^{(i)}(t). \end{aligned} \tag{2.1}$$

Applying (2.1) and inductive hypothesis, we have

$$f^{(r+1)}(0) = x\cdot\sum_{i=0}^{r}\binom{r}{i}\cdot f^{(i)}(0) = x\cdot\sum_{i=0}^{r}\binom{r}{i}\cdot B(i,x) = B(r+1,x).$$

That is, $f^{(r+1)}(0) = B(r+1,x)$.

Now the lemma follows from the complete induction.

Proof of Theorem 1 For any positive integer $k \geq 2$, it is clear that $f^k(t) = \exp(kx(e^t-1))$, then from (1.4), we have

$$\begin{aligned} f^k(t) &= \left(\exp\left(x(e^t-1)\right)\right)^k = \left(\sum_{n=0}^{\infty}\frac{B(n,x)}{n!}\cdot t^n\right)^k \\ &= \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\frac{B(a_1,x)}{a_1!}\cdot\frac{B(a_2,x)}{a_2!}\cdots\frac{B(a_k,x)}{a_k!}\right)\cdot t^n. \end{aligned} \tag{2.2}$$

On the other hand, let $g(t) = f^k(t) = \exp(kx(e^t-1))$, then from the definition of the power series and lemma, we also have

$$f^k(t) = g(t) = \sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}\cdot t^n = \sum_{n=0}^{\infty}\frac{B(n,kx)}{n!}\cdot t^n. \tag{2.3}$$

Combining (2.2) and (2.3) we may immediately deduce the identity

$$\sum_{a_1+a_2+\cdots+a_k=n}\frac{B(a_1,x)}{a_1!}\cdot\frac{B(a_2,x)}{a_2!}\cdots\frac{B(a_k,x)}{a_k!} = \frac{B(n,kx)}{n!}.$$

This proves Theorem 1.

Proof of Theorem 2 Applying the power series $e^y = \sum\limits_{n=0}^{\infty}\frac{1}{n!}\cdot y^n$, we have

$$\begin{aligned} f(t) &= \frac{1}{e^x}\exp\left(xe^t\right) = \frac{1}{e^x}\cdot\sum_{m=0}^{\infty}\frac{x^m}{m!}\cdot e^{mt} = \frac{1}{e^x}\cdot\sum_{m=0}^{\infty}\frac{x^m}{m!}\cdot\left(\sum_{n=0}^{\infty}\frac{m^n}{n!}\cdot t^n\right) \\ &= \frac{1}{e^x}\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{\infty}\frac{x^m\cdot m^n}{m!}\right)\cdot t^n. \end{aligned} \tag{2.4}$$

Comparing the coefficients of t^n in (1.4) and (2.4), we may immediately deduce the identity

$$B(n,x) = \frac{1}{e^x}\sum_{m=0}^{\infty}\frac{x^m\cdot m^n}{m!} = \frac{1}{e^x}\left(\frac{x^1\cdot 1^n}{1!}+\frac{x^2\cdot 2^n}{2!}+\frac{x^3\cdot 3^n}{3!}+\cdots\right).$$

This proves Theorem 2.

Proof of Corollary 1 Let p be an odd prime, take $n = p+1$ in Theorem 1, then from the properties of $B(n,x)$ and Theorem 1, we have

$$\sum_{a_1+a_2+\cdots+a_k=p+1}\frac{B(a_1,x)}{a_1!}\cdot\frac{B(a_2,x)}{a_2!}\cdots\frac{B(a_k,x)}{a_k!} = \frac{B(p+1,kx)}{(p+1)!}$$

or

$$(p+1)!\sum_{a_1+a_2+\cdots+a_k=p+1}\frac{B(a_1,x)}{a_1!}\cdot\frac{B(a_2,x)}{a_2!}\cdots\frac{B(a_k,x)}{a_k!} = kx\cdot\sum_{i=0}^{p}\binom{p}{i}\cdot B(i,kx). \tag{2.5}$$

It is clear that $\binom{p}{i} \equiv 0 \bmod p$ for all $1 \le i \le p$. So we have

$$kx \cdot \sum_{i=0}^{p} \binom{p}{i} \cdot B(i, kx) \equiv kxB(0, kx) + kxB(p, kx) \equiv kx + kxB(p, kx) \bmod p. \tag{2.6}$$

Note that $k \ge 2$ and $a_1 + a_2 + \cdots + a_k = p + 1$, so if there are three of a_1, a_2, $\cdots$, a_k are positive integers, then

$$\frac{(p+1)!}{a_1! \cdot a_2! \cdots a_k!} \equiv 0 \bmod p. \tag{2.7}$$

If there are only two of a_1, a_2, $\cdots$, a_k are positive integers, and both of them are greater than one, then we also have

$$\frac{(p+1)!}{a_1! \cdot a_2! \cdots a_k!} \equiv 0 \bmod p. \tag{2.8}$$

If there are only two of a_1, a_2, $\cdots$, a_k are positive integers, and one is p, another is 1, then we also have

$$\frac{(p+1)!}{a_1! \cdot a_2! \cdots a_k!} \equiv 1 \bmod p. \tag{2.9}$$

If only one of a_1, a_2, $\cdots$, a_k are positive integers, then it must be $p+1$. This time we have

$$\frac{(p+1)!}{a_1! \cdot a_2! \cdots a_k!} \equiv 1 \bmod p. \tag{2.10}$$

Combining (2.5)–(2.10) and note that identity

$$B(n+1, x) = x \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot B(i, x) \text{ for all } n \ge 1,$$

we have

$$k \cdot B(p+1, x) + 2\binom{k}{2} \frac{(p+1)!}{p!} \cdot B(1, x) \cdot B(p, x) \equiv kx + kxB(p, kx) \bmod p$$

or

$$k \cdot B(p, x) \equiv B(p, kx) \bmod p.$$

This proves the first congruence of Corollary 1. The second congruence follows from the second identity of Corollary 2 with $n = p$.

Proof of Corollary 2 Let $f(t, x) = \exp\left(x(e^t - 1)\right)$, then from (1.4), we have

$$(e^t - 1) f(t, x) = \frac{\partial f(t, x)}{\partial x} = \sum_{n=0}^{\infty} \frac{B'(n, x)}{n!} \cdot t^n. \tag{2.11}$$

On the other hand, from the definition of $f(t, x)$, we also have

$$(e^t - 1) f(t, x) = e^t f(t, x) - f(t, x) = \frac{1}{x} \frac{\partial f(t, x)}{\partial t} - f(t, x)$$

$$= \frac{1}{x} \sum_{n=0}^{\infty} \frac{B(n+1, x)}{n!} \cdot t^n - \sum_{n=0}^{\infty} \frac{B(n, x)}{n!} \cdot t^n. \tag{2.12}$$

Comparing the coefficients of t^n in (2.11) and (2.12), we may immediately deduce the identity

$$B(n+1,x) = x \cdot (B'(n,x) + B(n,x)) \,. \tag{2.13}$$

Note that the recurrence formula $B(n+1,x) = x \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot B(i,x)$, from (2.13) we may immediately deduce the identity $B'(n,x) = \sum_{i=0}^{n-1} \binom{n}{i} \cdot B(i,x)$. This completes the proofs of our all results.

References

[1] Tom M Apostol. Introduction to analytic number theory[M]. New York: Springer-Verlag, 1976.

[2] Tom M Apostol. Mathematical analysis (2nd ed.)[M]. Boston: Addison-Wesley Publishing Co., 1974.

[3] Stirling J. Methodus differentialis[M]. Londini: Sive Tractatus de Summation et Interpolazione Serierum Infinitarum, 1730.

[4] Boole G. Calculus of finite differences[M]. London: Chelsea Publishing Company, 1860.

[5] Caralambides C A. On weighted Stirling and other related numbers and come combinatorial applications[J]. Fibonacci Quar., 1984, 22: 296–309.

[6] Conway H J, Guy R K. The book of numbers[M]. New York: Copernicus, 1996.

[7] Corcino C B. An asymptotic for the r-Bell numbers[J]. Matimyás Mat., 2001, 24: 9–18.

[8] Tan M H, Xiang Y H, Zha Z W. Some inifite summation identities of the second kind[J]. J. Math., 2013, 33(3): 388–392.

[9] Pólya G, Szegö G. Problems and theorems in analysis I[M]. New York: Springer-Verlag, 1972.

关于Bell多项式及其它的一些恒等式

过 静[1], 李小雪[2]

(1.江西科技师范大学数学与计算机科学学院, 江西 南昌 330038)

(2.西北大学数学学院, 陕西 西安 710127)

摘要: 本文引入了一个新的多项式, 即Bell多项式. 利用初等数论及组合方法, 证明了包含该多项式的一些恒等式. 作为这些恒等式的应用, 给出了关于Bell数的同余式.

关键词: Bell数; Bell多项式; 恒等式; 组合方法

MR(2010)**主题分类号**: 11B37; 11B83 **中图分类号**: O157.1