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DIVIDEND PAYMENTS IN THE DUAL MODEL WITH ERLANG(N) DISTRIBUTED OBSERVATION TIMES

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Abstract: In this paper, we consider the dividend payments in the dual model with Erlang(n) distributed observation times. We derive and solve the integral equations satisfied by the expected discounted dividends until ruin when the Laplace transform of a general gain distribution follows the rational case, which extends some corresponding results in [8].

Keywords: dual model; observation times; Laplace transform; discounted dividends

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1 Introduction

The dual ruin model is defined as

$$U(t) = u - ct + S(t) = u - ct + \sum_{i=1}^{N(t)} Y_i, \tag{1.1}$$

where $u \geq 0$ represents the initial surplus, c is expense rate and the the aggregate revenue $S(t)$ represents the compound Poisson process, given by the Poisson parameter λ. The gain amounts $\{Y_i, i \geq 1\}$ (independent of $\{N_t, t \geq 0\}$) is a sequence of independent and identically distributed (i.i.d) positive random variables with common density function $f_Y(y)$. The corresponding Laplace transform of common distribution Y is $\widetilde{f}_Y(s) = \displaystyle\int_0^\infty e^{-sy} f_Y(y)dy$.

A hot topic about risk model is the expected discounted dividends until ruin, which is studied thoroughly in many other papers. Avanzi et al. [1] studied the optimal dividends under the barrier strategy; Ng [2] considered discounted dividends in the dual model with a dividend threshold; Albrecher et al. [3] further discussed dividend payments with tax payments. These papers considered the model with exponential inter-event times while some other papers are based on Erlang(n) distributed inter-event times (see Albrecher et al. [4], Yang and Sendova [5] and Eugenio et al. [6]).

In practice, the company's board checks the surplus regularly and then decides whether to pay dividends to shareholders. Thus dividends may be paid to shareholders only in

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special times. As shown in Avanzi et al. [7], the dual model with Erlang(n) distributed observation times is provided. They assumed that the ruin happens as long as the surplus falls below the zero level. In fact, even if the surplus is negative, the management are no aware of bankruptcy and keep this business alive due to the continuity of business. Thus, only with negative assets in the special times can company go bankrupt (see Albrecher et al. [8]). Peng et al. [9] considered dividend payments in the dual model with exponentially distributed observation times, note that ruin and dividends can only be observed at these random observation times. In this paper, we consider the dual model based on the method of Albrecher et al. [8] who studied the classical risk model with random observation times.

We assume the dual model can only be observed at times $\{Z_k\}_{k=1}^{\infty}$, at which ruin and dividend occur. Constant dividend barrier strategy is implemented. If the surplus exceeds the barrier $b > 0$ at the times Z_k, the excess is paid out immediately as a dividend. Otherwise, there is no dividend payments.

Let $T_k = Z_k - Z_{k-1}$ (Z_0 is not assumed to be a dividend decision time), and assume that $\{T_k\}_{k=1}^{\infty}$ is an i.i.d. sequence distributed as a generic $r.v.$ T and independent of $\{N(t)\}_{t\geq 0}$ and $\{Y_i\}_{i=1}^{\infty}$. The common distribution T is Erlang(n) distributed with density

$$f_T(t) = \frac{\gamma^n t^{n-1} e^{-\gamma t}}{(n-1)!}, \ \ \gamma > 0, \ \ t > 0,$$

and corresponding Laplace transform has the form $\widetilde{f}_T(s) = \int_0^{\infty} e^{-st} f_T(t)dt = \big(\frac{\gamma}{\gamma + s}\big)^n$.

We denote the sequences of surplus levels at the time points $\{Z_k^-\}_{k=1}^{\infty}$ and $\{Z_k\}_{k=1}^{\infty}$ by $\{U_b(k)\}_{k=1}^{\infty}$ and $\{W_b(k)\}_{k=1}^{\infty}$, respectively, i.e., $\{U_b(k)\}$ and $\{W_b(k)\}$ are the surplus levels at the k-th observation before (after, respectively) potential dividends are paid.

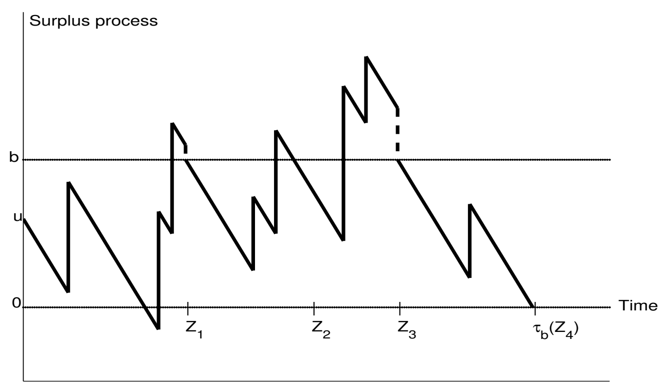

Figure 1: Typical sample path of the dual model under randomized observation

The time of ruin is defined by $\tau_b = Z_{k_b}$, where $k_b = \inf\{k \geq 1, W_b(k) \leq 0\}$ is the number of observation intervals before ruin. Then we have the recursive relation

$$U_b(k) = W_b(k-1) - cT_k + [S(Z_k) - S(Z_{k-1})],$$
$$W_b(k) = \min\{U_b(k), b\}, \quad k = 1, 2, 3, \cdots, \quad W_b(0) = u.$$

A sample path under the present model is depicted in Figure 1.

The total discounted dividend payments until ruin for a discount rate $\delta \geq 0$ are

$$\Delta_{M,\delta}(u,b) = \Big[\sum_{k=1}^{k_b} e^{-\delta Z_k}[U_b(k) - b]_+ | W_b(0) = u\Big], \quad u \in (-\infty, \infty).$$

With time 0 an observation time, the total discounted dividend payments until ruin are represented by

$$\Delta_\delta(u,b) = \begin{cases} 0, & u < 0, \\ \Delta_{M,\delta}(u,b), & 0 \leq u < b, \\ u - b + \Delta_{M,\delta}(b,b), & b \leq u. \end{cases}$$

In particular, the distribution of $\Delta_{M,\delta}(u,b)$ for $0 \leq u < b$ already determines $\Delta_\delta(u,b)$ for arbitrary u.

We assume that time 0 is not a dividend decision time. The total expected discounted dividends are

$$V(u,b) = E\Delta_{M,\delta}(u,b).$$

Depending on the value of the initial surplus, define

$$V(u,b) = \begin{cases} V_1(u,b), & u < 0, \\ V_2(u,b), & 0 \leq u < b, \\ V_3(u,b), & u \geq b. \end{cases} \tag{1.2}$$

The rest of this paper is organized as follows: in Section 2, we derive and solve the integral equations satisfied by the expected discounted dividends until ruin when the Laplace transform of a general gain distribution follows rational case. In Section 3, we obtain explicit form of the expected discounted dividends when jump sizes and inter-observation times follow an exponential distribution. In Section 4, we generalize the results to the case that the inter-observation times are Erlang(2) distributed. In addition, numerical illustrations for the effect of model parameters on the expected value of the discounted dividends are studied and image description is given.

2 Discounted Dividends $V(u,b)$

Due to the Markovian structure of $\{U_t\}_{t\geq 0}$, the sequence of pairs

$$\{(T_k, \sum_{i=1}^{N(T_k)} Y_i - cT_k), k \geq 1\}$$

is i.i.d with genetic distribution $(T, \sum_{i=1}^{N(T)} Y_i - cT)$ and joint Laplace transform

$$E\big[e^{-\delta T - s(\sum_{i=1}^{N(T)} Y_i - cT)}\big] = E\big[e^{-(\delta - cs)T} E\big[e^{-s\sum_{i=1}^{N(T)} Y_i} | T\big]\big] = E\big[e^{-[\lambda+\delta-cs-\lambda \widetilde{f}_Y(s)]T}\big]. \tag{2.1}$$

As in Albrecher et al. [8], we write

$$E\big[e^{-\delta T - s(\sum_{i=1}^{N(T)} Y_i - cT)}\big] = \int_{-\infty}^{\infty} e^{-sy} g_\delta(y) dy, \tag{2.2}$$

where $g_\delta(y)(-\infty < y < \infty)$ represents the discounted density of the increment $\sum_{i=1}^{N(T)} -cT$ between successive observation times, discounted at rate δ. According to the assumption that inter-observation T has an Erlang(n) distribution, eq.(2.1) is rewritten as

$$E\left[e^{-\delta T - s(\sum_{i=1}^{N(T)} Y_i - cT)}\right] = \left(\frac{\gamma}{\gamma + \lambda[1 - \widetilde{f}_Y(s)] + (\delta - cs)}\right)^n. \tag{2.3}$$

There are zeros in the denominator above, namely, the roots of the equation

$$\lambda \widetilde{f}_Y(s) - (\lambda + \gamma + \delta) + cs = 0 \tag{2.4}$$

in which there is a unique positive root $\rho_\gamma > 0$. To make calculation easier, we use the notation

$$g_\delta(y) = g_{\delta,-}(-y) I_{\{y<0\}} + g_{\delta,+}(y) I_{\{y \geq 0\}}, \quad -\infty < y < \infty. \tag{2.5}$$

By conditioning on the pair $(T_1, \sum_{i=1}^{N(T_1)} -cT_1)$ and using eq.(2.5), we get

$$V_1(u,b) = \int_{-u}^{-u+b} V_2(u+y) g_{\delta,+}(y) dy + \int_{-u+b}^{\infty} [u + y - b + V_2(b,b)] g_{\delta,+}(y) dy, \quad u < 0, \tag{2.6}$$

$$\begin{aligned} V_2(u,b) = & \int_0^u V_2(u-y) g_{\delta,-}(y) dy + \int_0^{b-u} V_2(u+y) g_{\delta,+}(y) dy \\ & + \int_{b-u}^{\infty} [u + y - b + V_2(b,b)] g_{\delta,+}(y) dy, \quad 0 \leq u < b, \end{aligned} \tag{2.7}$$

$$\begin{aligned} V_3(u,b) = & \int_{u-b}^{u} V_2(u-y) g_{\delta,-}(y) dy + \int_0^{u-b} [u - y - b + V_2(b,b)] g_{\delta,-}(y) dy \\ & + \int_0^{\infty} [u + y - b + V_2(b,b)] g_{\delta,+}(y) dy, \quad u \geq b \end{aligned} \tag{2.8}$$

with continuity condition $V_1(0,b) = V_2(0,b)$ and $V_2(b,b) = V_3(b,b)$.

The quantities $g_{\delta,-}(y)$ and $g_{\delta,+}(y)$ will not always have a tractable form, but if $f_Y(y)$ has a rational Laplace transform, i.e.,

$$f_Y(s) = \frac{Q_{2,m-1}(s)}{Q_{1,m}(s)},$$

where $Q_{1,m}(s)$ is a polynomial in s of degree exactly m with leading coefficient of 1 and $Q_{2,m-1}(s)$ is a polynomial in s of degree at most $m-1$ (and the two polynomials have distinct zeros). From Albrecher et al. [10], it follows that

$$g_{\delta,-}(y)=\sum_{j=1}^{n}B_j^*\frac{y^{j-1}e^{-\rho_\gamma y}}{(j-1)!}, \tag{2.9}$$

$$g_{\delta,+}(y)=\sum_{i=1}^{m}\sum_{j=1}^{n}B_{ij}\frac{y^{j-1}e^{-R_{\gamma,i}y}}{(j-1)!}, \tag{2.10}$$

where $-R_{\gamma,1}, -R_{\gamma,2}, \cdots, -R_{\gamma,m}$ are the m roots of eq.(2.4) with negative real parts and the constants B_j^* and B_{ij} are given by

$$B_j^*=(-1)^{n-j}\left(\frac{\gamma}{c}\right)^n\frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}\frac{[Q_{1,m}(s)]^n}{\prod\limits_{l=1}^{m}(s+R_{\gamma,l})^n}\Big|_{s=\rho_\gamma},\quad j=1,2,\cdots,n, \tag{2.11}$$

$$B_{ij}=\left(\frac{\gamma}{c}\right)^n\frac{1}{(n-j)!}\frac{d^{n-j}}{ds^{n-j}}\frac{[Q_{1,m}(s)]^n}{(\rho_\gamma-s)^n\prod\limits_{l=1,l\neq i}^{m}(s+R_{\gamma,l})^n}\Big|_{s=-R_{\gamma,i}},$$

$$i=1,2,\cdots,m,\quad j=1,2,\cdots,n. \tag{2.12}$$

In view of eqs.(2.6)–(2.8), the expression for $V_1(u,b)$ and $V_3(u,b)$ are closely associated with $V_2(u,b)$. So we derive $V_1(u,b)$ and $V_3(u,b)$ easily by substituting back the solution for $V_2(u,b)$ into eq.(2.6) and eq.(2.8).

Substitution of eq.(2.9) and eq.(2.10) into eq.(2.7) yields, after rearranging terms,

$$\begin{aligned}V_2(u,b)=&\sum_{j=1}^{n}B_j^*e^{-\rho_\gamma u}\int_0^u V_2(y,b)\frac{(u-y)^{j-1}e^{\rho_\gamma y}}{(j-1)!}dy\\&+\sum_{i=1}^{m}\sum_{j=1}^{n}B_{ij}e^{R_{\gamma,i}u}\int_u^b V_2(y,b)\frac{(y-u)^{j-1}}{(j-1)!}e^{-R_{\gamma,i}y}dy\\&+\sum_{i=1}^{m}\sum_{j=1}^{n}B_{ij}e^{R_{\gamma,i}u}\int_b^\infty [y-b+V_2(b,b)]\frac{(y-u)^{j-1}e^{-R_{\gamma,i}y}}{(j-1)!}dy.\end{aligned} \tag{2.13}$$

Applying the operator $(d/du+\rho_\gamma)^n\prod\limits_{i=1}^{m}(d/du-R_{\gamma,i})^n$ to both sides of the above, we obtain that $V_2(u,b)$ satisfies a homogeneous differential equation of order $n(m+1)$ in u with constant coefficients. A solution of eq.(2.13) is of the form

$$V_2(u,b)=\sum_{p=1}^{n(m+1)}A_pe^{\alpha_p u},\quad 0\le u<b, \tag{2.14}$$

where constants $\{A_p\}_{p=1}^{n(m+1)}$ and $\{\alpha_p\}_{p=1}^{n(m+1)}$ may be associated with b, but independent of u. When $\{\alpha_p\}_{p=1}^{n(m+1)}$ have multiple roots, the solution of eq.(2.13) is of the form

$$V_2(u,b)=\sum_{p=1}^{r}\sum_{j=1}^{k_p}A_{pj}u^{j-1}e^{\alpha_p u},$$

where k_p is the multiplicity of the root α_p and satisfies the equation $\sum\limits_{p=1}^{r} k_p = n(m+1)$. As for this case, the method is analogous as follows. Substituting eq.(2.14) into eq.(2.13), the first integral on the right-hand side of eq.(2.13) is evaluated as

$$\begin{aligned}
&\sum_{j=1}^{n} B_j^* e^{-\rho_\gamma u} \int_0^u V_2(y,b) \frac{(u-y)^{j-1} e^{\rho_\gamma y}}{(j-1)!} dy \\
=&\sum_{p=1}^{n(m+1)} A_p \sum_{j=1}^{n} B_j^* e^{\alpha_p u} \int_0^u \frac{y^{j-1} e^{-(\alpha_p+\rho_\gamma) y}}{(j-1)!} dy \\
=&\sum_{p=1}^{n(m+1)} A_p \left[\sum_{j=1}^{n} \frac{B_j^*}{(\rho_\gamma+\alpha_p)^j} \right] e^{\alpha_p u} - \sum_{j=1}^{n} \left[\sum_{p=1}^{n(m+1)} A_p \sum_{i=j}^{n} \frac{B_i^*}{(\rho_\gamma+\alpha_p)^i} \right] \frac{(\rho_\gamma+\alpha_p)^{j-1}}{(j-1)!} u^{j-1} e^{-\rho_\gamma u}.
\end{aligned} \tag{2.15}$$

Similarly, the second integral in eq.(2.13) is given by

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij} e^{R_{\gamma,i} u} \int_u^b V_2(y,b) \frac{(y-u)^{j-1} e^{-R_{\gamma,i} y}}{(j-1)!} dy \\
=&\sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij} \sum_{p=1}^{n(m+1)} A_p e^{\alpha_p u} \int_0^{b-u} \frac{y^{j-1} e^{-(R_{\gamma,i}-\alpha_p) y}}{(j-1)!} dy \\
=&\sum_{p=1}^{n(m+1)} A_p \left[\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{B_{ij}}{(R_{\gamma,i}-\alpha_p)^j} \right] e^{\alpha_p u} \\
&-\sum_{i=1}^{m}\sum_{j=1}^{n} \left[\sum_{p=1}^{n(m+1)} A_p \sum_{k=j}^{n} B_{ik} \sum_{l=j}^{k} \frac{b^{k-l} e^{\alpha_p b}}{(R_{\gamma,i}-\alpha_p)^l (k-l)!} \right] \frac{(\alpha_p - R_{\gamma,i})^{j-1} u^{j-1}}{(j-1)!} e^{R_{\gamma,i}(u-b)},
\end{aligned} \tag{2.16}$$

while the third integral in eq.(2.13) is written as

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij} e^{R_{\gamma,i} u} \int_b^\infty \left[y-b+V_2(b,b) \right] \frac{(y-u)^{j-1} e^{-R_{\gamma,i} y}}{(j-1)!} dy \\
=&\sum_{i=1}^{m}\sum_{j=1}^{n} \left[\sum_{k=j}^{n} B_{ik} \sum_{l=j}^{k} \frac{b^{k-l}(l+1-j)}{R_{\gamma,i}^{l+1}(k-l)!} \right] \frac{(-R_{\gamma,i} u)^{j-1}}{(j-1)!} e^{R_{\gamma,i}(u-b)} \\
&+\sum_{i=1}^{m}\sum_{j=1}^{n} \left[\sum_{p=1}^{n(m+1)} A_p \sum_{k=j}^{n} B_{ik} \sum_{l=j}^{k} \frac{b^{k-l} e^{\alpha_p b}}{R_{\gamma,i}^{l}(k-l)!} \right] \frac{(-R_{\gamma,i} u)^{j-1}}{(j-1)!} e^{R_{\gamma,i}(u-b)}.
\end{aligned} \tag{2.17}$$

Putting back eqs.(2.15)–(2.17) into eq.(2.13), equating coefficients of $e^{\alpha_p u}$ leads to

$$\sum_{j=1}^{n} \frac{B_j^*}{(\rho_\gamma+\alpha_p)^j} + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{B_{ij}}{(R_{\gamma,i}-\alpha_p)^j} = 1, \quad p=1,2,\cdots,n(m+1). \tag{2.18}$$

Substitution of eq.(2.9) and eq.(2.10) yields the requirement that

$$E\left[e^{-\delta T-s(\sum_{i=1}^{N(T)}Y_i-cT)}\right]=\int_{-\infty}^{\infty}e^{-sy}g_\delta(y)dy=\left(\frac{\gamma}{\gamma+\lambda[1-\tilde{f}_Y(s)]+(\delta-cs)}\right)^n$$
$$=\sum_{j=1}^{n}\frac{B_j^*}{(\rho_\gamma-s)^j}+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{B_{ij}}{(R_{\gamma,i}+s)^j}. \tag{2.19}$$

In comparison with eq.(2.18) above, we may conclude that $\{\alpha_p\}_{p=1}^{n(m+1)}$ are the opposite numbers to the roots of the equation

$$\sum_{j=1}^{n}\frac{B_j^*}{(\rho_\gamma-s)^j}+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{B_{ij}}{(R_{\gamma,i}+s)^j}=1.$$

In other words $\{\alpha_p\}_{p=1}^{n(m+1)}$ are also the opposite numbers to the roots of the equation

$$\left(\frac{\gamma}{\gamma+\lambda[1-\tilde{f}_Y(s)]+(\delta-cs)}\right)^n=1. \tag{2.20}$$

Equating coefficients of $u^{j-1}e^{R_{\gamma,i}(u-b)}$ leads to

$$\sum_{p=1}^{n(m+1)}A_p\sum_{k=j}^{n}B_{ik}\sum_{l=j}^{k}\frac{b^{k-l}e^{\alpha_p b}}{(R_{\gamma,i}-\alpha_p)^l(k-l)!}(\alpha_p-R_{\gamma,i})^{j-1}$$
$$=\sum_{p=1}^{n(m+1)}A_p\sum_{k=j}^{n}B_{ik}\sum_{l=j}^{k}\frac{b^{k-l}e^{\alpha_p b}}{R_{\gamma,i}^l(k-l)!}(-R_{\gamma,i})^{j-1}+\sum_{k=j}^{n}B_{ik}\sum_{l=j}^{k}\frac{b^{k-l}(l+1-j)}{R_{\gamma,i}^{l+1}(k-l)!}(-R_{\gamma,i})^{j-1},$$
$$i=1,2,\cdots,m,\ \ j=1,2,\cdots,n. \tag{2.21}$$

Finally, equating coefficients of $u^{j-1}e^{-\rho_\gamma u}$ yields

$$\sum_{p=1}^{n(m+1)}A_p\sum_{k=j}^{n}\frac{B_k^*}{(\rho_\gamma+\alpha_p)^{k+1-j}}=0,\quad j=1,2,\cdots,n. \tag{2.22}$$

Consequently, we have eq.(2.20) to solve for $\{\alpha_p\}_{p=1}^{n(m+1)}$. Moreover, notice that there is a system of $m\times n+n=n(m+1)$ equations for the constants $\{A_p\}_{p=1}^{n(m+1)}$ given by eq.(2.21) and eq.(2.22). Hence, the expression for $V_2(u,b)$ is obtained easily.

3 The Case That Jump Sizes and Inter-Observation Times are Exponential

In the case that $f_Y(y)=\beta e^{-\beta y}$, and $f_T(t)=\gamma e^{-\gamma t}$, eq.(2.4) reduces to

$$cs^2-(\lambda+\gamma+\delta-c\beta)s-(\gamma+\delta)\beta=0$$

in which there is a positive root ρ_γ and a negative root $-R_\gamma(R_\gamma > 0)$. Then we may simplify eq.(2.20) to

$$cs^2 - (\lambda + \delta - c\beta)s - \delta\beta = 0 \tag{3.1}$$

with a positive root ρ_0 and a negative root $-R_0$.

From eq.(2.9) and eq.(2.10), it follows that

$$g_{\delta,-}(y) = B^* e^{-\rho_\gamma y},\ \ B^* = \frac{\gamma(\beta + \rho_\gamma)}{c(\rho_\gamma + R_\gamma)}, \tag{3.2}$$

$$g_{\delta,+}(y) = Be^{-R_\gamma y},\ \ B = \frac{\gamma(\beta - R_\gamma)}{c(\rho_\gamma + R_\gamma)}. \tag{3.3}$$

Putting back $g_{\delta,-}(y)$ and $g_{\delta,+}(y)$ above into the original integral eqs. (2.6)–(2.8) yields

$$V_1(u,b) = e^{R_\gamma u}\int_0^b V_2(y,b)Be^{-R_\gamma y}dy + e^{R_\gamma u}\int_b^\infty [y - b + V_2(b,b)]\, Be^{-R_\gamma y}dy,\ \ u < 0, \tag{3.4}$$

$$\begin{aligned} V_2(u,b) =& e^{-\rho_\gamma u}\int_0^u V_2(y,b)B^* e^{\rho_\gamma y}dy + e^{R_\gamma u}\int_u^b V_2(y,b)Be^{-R_\gamma y}dy \\ &+ e^{R_\gamma u}\int_b^\infty [y - b + V_2(b,b)]\, Be^{-R_\gamma y}dy,\ \ 0 \le u < b, \end{aligned} \tag{3.5}$$

$$\begin{aligned} V_3(u,b) =& e^{-\rho_\gamma u}\int_0^b V_2(y,b)B^* e^{\rho_\gamma y}dy + e^{-\rho_\gamma u}\int_b^u [y - b + V_2(b,b)]\, B^* e^{\rho_\gamma y}dy \\ &+ e^{R_\gamma u}\int_u^\infty [y - b + V_2(b,b)]\, Be^{-R_\gamma y}dy,\ \ u \ge b. \end{aligned} \tag{3.6}$$

Furthermore, on combining with the conclusion mentioned at the end of Section 2 and the simplified eq.(3.1), the solution for $V_2(u,b)$ can be expressed as

$$V_2(u,b) = C_1 e^{-\rho_0 u} + C_2 e^{R_0 u}. \tag{3.7}$$

Substituting $V_2(u,b)$ above into eq.(3.5) and comparing the coefficients of $e^{-\rho_\gamma u}$ and $e^{R_\gamma(u-b)}$ leads to

$$\frac{C_1}{\rho_\gamma - \rho_0} + \frac{C_2}{\rho_\gamma + R_0} = 0 \tag{3.8}$$

and

$$\frac{C_1 \rho_0 e^{-\rho_0 b}}{\rho_0 + R_\gamma} + \frac{C_2 R_0 e^{R_0 b}}{R_0 - R_\gamma} = -\frac{1}{R_\gamma}. \tag{3.9}$$

Therefore, we have two linear equations satisfied by C_1 and C_2. After some calculations, we have

$$C_1 = \frac{(\rho_0 + R_\gamma)(\rho_\gamma - \rho_0)(R_0 - R_\gamma)}{R_0 R_\gamma(\rho_\gamma + R_0)(\rho_0 + R_\gamma)e^{R_0 b} - \rho_0 R_\gamma(\rho_\gamma - \rho_0)(R_0 - R_\gamma)e^{-\rho_0 b}},$$

$$C_2 = \frac{(\rho_0 + R_\gamma)(\rho_\gamma + R_0)(R_\gamma - R_0)}{R_0 R_\gamma(\rho_\gamma + R_0)(\rho_0 + R_\gamma)e^{R_0 b} - \rho_0 R_\gamma(\rho_\gamma - \rho_0)(R_0 - R_\gamma)e^{-\rho_0 b}}.$$

Hence we get

$$V_2(u,b)=\frac{(\rho_0+R_\gamma)(\rho_\gamma-\rho_0)(R_0-R_\gamma)e^{-\rho_0 u}+(\rho_0+R_\gamma)(\rho_\gamma+R_0)(R_\gamma-R_0)e^{R_0 u}}{R_0R_\gamma(\rho_\gamma+R_0)(\rho_0+R_\gamma)e^{R_0 b}-\rho_0R_\gamma(\rho_\gamma-\rho_0)(R_0-R_\gamma)e^{-\rho_0 b}}. \quad (3.10)$$

Putting $V_2(u,b)$ above into eqs.(3.4) and (3.6), we obtain

$$V_1(u,b)=\frac{\gamma(\rho_0+R_0)(\beta-R_\gamma)e^{R_\gamma u}}{cR_0R_\gamma(\rho_\gamma+R_0)(\rho_0+R_\gamma)e^{R_0 b}-c\rho_0R_\gamma(\rho_\gamma-\rho_0)(R_0-R_\gamma)e^{-\rho_0 b}}, \quad (3.11)$$

$$\begin{aligned}V_3(u,b)=&e^{\rho_\gamma(b-u)}\left[\frac{\delta}{\gamma+\delta}V_2(b,b)-\frac{\gamma(\lambda-c\beta)}{(\gamma+\delta)^2\beta}\right]+\frac{\gamma}{\gamma+\delta}u\\&+\left[V_2(b,b)-b\right]\frac{\gamma}{\gamma+\delta}+\frac{\gamma(\lambda-c\beta)}{(\lambda+\delta)^2\beta}.\end{aligned} \quad (3.12)$$

It should be mentioned that using different method we derive the same results as that given by Peng et al. [9].

4 The Case That Jump Sizes are Exponential and Inter-Observation Times are Erlang(2) Distributed

In the case that $f_Y(y)=\beta e^{-\beta y}$ and $f_T(t)=\gamma^2te^{-\gamma t}$, eq.(2.4) is equivalent to

$$cs^2-(\lambda+\gamma+\delta-c\beta)s-(\gamma+\delta)\beta=0$$

with two roots ρ_r and $-R_\gamma$ (the same as above). Then eq.(2.20) may be rewritten as

$$\left[cs^2-(\lambda+\delta-c\beta)s-\delta\beta\right]\left[cs^2-(\lambda+\delta+2\gamma-c\beta)s-(2\gamma+\delta)\beta\right]=0 \quad (4.1)$$

in which there are four roots ρ_0, $-R_0$, ρ_γ^* and $-R_\gamma^*$.

By eqs.(2.9) and (2.10), we immediately obtain

$$g_{\delta,-}(y)=B_1^*e^{-\rho_\gamma y}+B_2^*ye^{-\rho_\gamma y}, \quad (4.2)$$

$$g_{\delta,+}(y)=B_1e^{-R_\gamma y}+B_2ye^{-R_\gamma y}, \quad (4.3)$$

where the constants B_1^*, B_2^*, B_1, B_2 are given by

$$B_1^*=\left(\frac{\gamma}{c}\right)^2\frac{2(\beta+\rho_\gamma)(\beta-R_\gamma)}{(\rho_\gamma+R_\gamma)^3},\quad B_2^*=\left(\frac{\gamma}{c}\right)^2\frac{(\rho_\gamma+\beta)^2}{(\rho_\gamma+R_\gamma)^2},$$

$$B_1=\left(\frac{\gamma}{c}\right)^2\frac{2(\beta+\rho_\gamma)(\beta-R_\gamma)}{(\rho_\gamma+R_\gamma)^3},\quad B_2=\left(\frac{\gamma}{c}\right)^2\frac{(\beta-R_\gamma)^2}{(\rho_\gamma+R_\gamma)^2}.$$

Substituting back eqs.(4.2) and (4.3) into eq.(2.7) produces

$$\begin{aligned}V_2(u,b)=&e^{-\rho_\gamma u}\int_0^u V_2(y)\left[B_1^*e^{\rho_\gamma y}+B_2^*(u-y)e^{\rho_\gamma y}\right]dy\\&+e^{R_\gamma u}\int_u^b V_2(y)\left[B_1e^{-R_\gamma y}+B_2(y-u)e^{-R_\gamma y}\right]dy\\&+e^{R_\gamma u}\int_b^\infty \left[y-b+V_2(b,b)\right]\left[B_1e^{-R_\gamma y}+B_2(y-u)e^{-R_\gamma y}\right]dy.\end{aligned} \quad (4.4)$$

Applying the operator $(\frac{d}{du}+\rho_\gamma)^2(\frac{d}{du}-R_\gamma)^2$ on both sides, it implies that

$$V_2(u,b)=C_1e^{\alpha_1 u}+C_2e^{\alpha_2 u}+C_3e^{\alpha_3 u}+C_4e^{\alpha_4 u}.$$

Furthermore, the expression combined with the conclusion discussed before and eq.(4.1) indicates

$$\alpha_1=-\rho_0,\ \ \alpha_2=-(-R_0)=R_0,\ \ \alpha_3=-\rho_\gamma^*,\ \ \alpha_4=-(-R_\gamma^*)=R_\gamma^*.$$

Hence, the solution of eq.(4.4) has the explicit form

$$V_2(u,b)=C_1e^{-\rho_0 u}+C_2e^{R_0 u}+C_3e^{-\rho_\gamma^* u}+C_4e^{R_\gamma^* u}. \tag{4.5}$$

Putting $V_2(u,b)$ above into eq.(4.4) and equating coefficients of $ue^{-\rho_\gamma u}$ leads to

$$\frac{C_1}{\rho_\gamma-\rho_0}+\frac{C_2}{\rho_\gamma+R_0}+\frac{C_3}{\rho_\gamma-\rho_\gamma^*}+\frac{C_4}{\rho_\gamma+R_\gamma^*}=0. \tag{4.6}$$

Equating coefficients of $e^{-\rho_\gamma u}$ leads to

$$\frac{C_1}{(\rho_\gamma-\rho_0)^2}+\frac{C_2}{(\rho_\gamma+R_0)^2}+\frac{C_3}{(\rho_\gamma-\rho_\gamma^*)^2}+\frac{C_4}{(\rho_\gamma+R_\gamma^*)^2}=0. \tag{4.7}$$

Equating coefficients of $ue^{R_\gamma(u-b)}$ leads to

$$\frac{C_1\rho_0e^{-\rho_0 b}}{R_\gamma+\rho_0}+\frac{C_2(-R_0)e^{R_0 b}}{R_\gamma-R_0}+\frac{C_3\rho_\gamma^*e^{-\rho_\gamma^* b}}{R_\gamma+\rho_\gamma^*}+\frac{C_4(-R_\gamma^*)e^{R_\gamma^* b}}{R_\gamma-R_\gamma^*}=-\frac{1}{R_\gamma}. \tag{4.8}$$

Equating coefficients of $e^{R_\gamma(u-b)}$ leads to

$$\begin{aligned}&\frac{C_1(\rho_0^2+2\rho_0R_\gamma)e^{-\rho_0 b}}{(R_\gamma+\rho_0)^2}+\frac{C_2(R_0^2-2R_0R_\gamma)e^{R_0 b}}{(R_\gamma-R_0)^2}+\frac{C_3(\rho_\gamma^{*2}+2\rho_\gamma^*R_\gamma)e^{-\rho_\gamma^* b}}{(R_\gamma+\rho_\gamma^*)^2}\\&+\frac{C_4(R_\gamma^{*2}-2R_\gamma^*R_\gamma)e^{R_\gamma^* b}}{(R_\gamma-R_\gamma^*)^2}=-\frac{2}{R_\gamma}.\end{aligned} \tag{4.9}$$

Therefore, we have a system of linear eqs. (4.6)–(4.9) for the four remaining constants C_1, C_2, C_3, C_4 (only with given b).

Example 1 Let $T\sim$ Erlang(2, 2), $\lambda=1$, $c=0.8$, $\delta=0.05$ and $\beta=1$. The solution of eq.(4.1) are

$$\rho_0=0.4511,\ \ -R_0=-0.1386,\ \ \rho_\gamma^*=6.1374,\ \ -R_\gamma^*=-0.8249,$$

and we have

$$V_2(u,b)=c_1e^{-0.4511u}+c_2e^{0.1386u}+c_3e^{-6.1374u}+c_4e^{0.8249u}.$$

The coefficients $\{c_i\}_{i=1}^4$ can easily be determined by eqs.(4.6)–(4.9) and is a function of b.

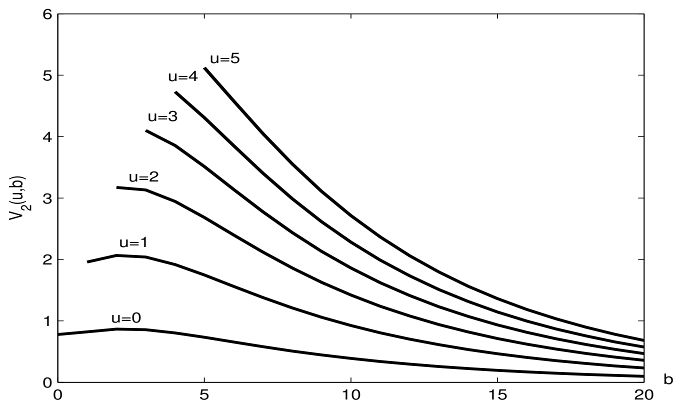

Figure 2: $V_2(u,b)$ as a function of b for $u = 0, 1, 2, 3, 4, 5$(from bottom to top)

At the end of this section, we use the following numerical examples to discuss the impact of the model parameters on the expected total dividend payments. Table 1 gives some numerical values of $V_2(u,b)$ and Figure 2 depicts the behavior of $V_2(u,b)$ as a function of b for some given values of initial capital u. The top curve corresponds to $u = 5$, and the next one corresponds to $u = 4$ and so on. Combining both together, we find that dividends increase as u increases for each fixed b. Observing carefully, $V_2(u,b)$ appears to increase with b initially and decrease afterwards if u is small. Further, with initial capital u bigger, $V_2(u,b)$ is a monotonically decreasing function of b.

Table 1: Exact values for the expectation $V_2(u,b)$ of the discounted dividend payment

$b \setminus u$	0	1	2	3	4	5	6	7	8	9
0	0.7774									
1	0.8209	1.9576								
2	0.8662	2.0654	3.1732							
3	0.8548	2.0384	3.1317	4.1000						
4	0.8038	1.9167	2.9450	3.8559	4.7292					
5	0.7323	1.7463	2.6832	3.5134	4.3098	5.1235				
6	0.6544	1.5606	2.3979	3.1400	3.8520	4.5801	5.3584			
7	0.5782	1.3789	2.1188	2.7745	3.4038	4.0476	4.7363	5.4941		
8	0.5076	1.2105	1.8600	2.4357	2.9882	3.5535	4.1586	4.8249	5.5711	
9	0.4440	1.0587	1.6268	2.1304	2.6137	3.1082	3.6376	4.2209	4.8747	5.6143

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观察时间服从Erlang(n) 分布的对偶模型红利支付

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摘要: 本文研究了观察时间服从Erlang(n) 分布的对偶模型红利支付问题. 在收益额的拉普拉斯变换是有理拉普拉斯变换的情况下, 获得了破产之前总贴现红利$V(u;b)$ 的求解方法. 该结果推广了文献[8] 的相应结论.

关键词: 对偶模型; 观察时间; 拉普拉斯变换; 贴现红利

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