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A FUNDAMENTAL THEOREM ON F-SPACES AND ITS APPLICATION IN NUMERICAL ANALYSIS

DU Nai-lin, XIA Wei

(*School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)

Abstract: The main results of this paper are Theorems 1.1 and 1.2 in Section 1. Theorem 1.1 may be regarded as the basic tool in the theory of F-spaces, for it implies fundamental principles such as the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem. Theorem 1.2 can be viewed as an application of Theorem 1.1 in numerical analysis, which shows that an abstract approximation scheme (consisting of a sequence of closed operators with closed ranges in F-space setting) is convergent if and only if it is stable.

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1 Introduction and Main Results

The uniform boundedness theorem, the closed graph theorem and the open mapping theorem are usually referred to as fundamental principles in functional analysis (see e.g. [5]). In this paper, based upon the Baire-Hausdorff theorem, we prove a theorem which contains the above principal theorems as its simple corollaries (see Theorem 1.1). As a more profound application of the theorem, a useful result in numerical analysis is established, which may be viewed as an abstract generalization of the well-known Lax-Richtmyer equivalene theorem [3] (see Theorem 1.2).

We need to recall the definition of F-spaces and to illustrate the notation used in the paper. A linear space X is called a quasi-normed linear space, if for every $x \in X$ there is associated a real number $\|x\|$, the quasi-norm of the vector x, which satisfies

$$\begin{aligned}
&\|x\| \geq 0 \text{ and } \|x\| = 0 \text{ if and only if } x = 0,\\
&\|x+y\| \leq \|x\| + \|y\| \quad \text{(triangle inequality)},\\
&\|-x\| = \|x\|, \quad \lim_{\alpha_n \to 0} \|\alpha_n x\| = 0 \quad \text{and} \quad \lim_{\|x_n\| \to 0} \|\alpha x_n\| = 0.
\end{aligned}$$

A quasi-normed linear space X is called an F-space if it is complete. Next, let X be an F-space. For a point $x \in X$ and a real number $r > 0$, by $B_X(x, r)$ we denote the open ball in X with the center at x and the radius r, namely,

$$B_X(x, r) := \{y \in X \mid \|y - x\| < r\}.$$

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Biography: Du Nailin (1962–), male, born at Handan, Hebei, professor, major in ill-posed problems and generalized inverses.

For a subset A of X, let the symbols A° and $\overline{A}$ denote the interior and the closure of A respectively. The set A is said to be bounded if it is absorbed by any open ball $B_X(0,\varepsilon)$ with center at 0, i.e., if there exists a positive constant α such that $\alpha^{-1}A \subseteq B_X(0,\varepsilon)$, where

$$\alpha^{-1}A := \left\{x \in X \mid x = \alpha^{-1}a, a \in A\right\}.$$

For a sequence of nonempty subsets $\{A_n\}$ of X, we set

$$s\text{-}\lim_{n\to\infty} A_n := \left\{x \in X \mid \lim_{n\to\infty} \operatorname{dist}(x, A_n) = 0\right\},$$

where

$$\operatorname{dist}(x, A_n) := \inf\{\|y - x\| \mid y \in A_n\}.$$

Now, let Y be an F-space on the same scalar field as the F-space X, $T : D \subset X \to Y$ a linear mapping from the subspace D of X into Y. By $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{N}(T)$ and $\mathcal{G}(T)$, we denote the domain, the range, the null space, and the graph of T, respectively, i.e.,

$$\begin{aligned}&\mathcal{D}(T) := D, \quad \mathcal{R}(T) := \{Tx \mid x \in D\},\\ &\mathcal{N}(T) := \{x \in D \mid Tx = 0\}, \quad \mathcal{G}(T) := \{(x, Tx) \mid x \in D\}.\end{aligned}$$

For an $y \in Y$, the preimage of the point y is denoted by $T^{-1}(y)$, namely,

$$T^{-1}(y) := \{x \in \mathcal{D}(T) \mid Tx = y\}.$$

In addition, $X \times Y$ is also an F-space by the algebraic operations

$$(x_1, y_1) + (x_2, y_2) := (x_1, y_1) + (x_2, y_2), \quad \alpha(x, y) := (\alpha x, \alpha y),$$

and the quasi-norm

$$\|(x, y)\| := \left(\|x\|^2 + \|y\|^2\right)^{1/2}.$$

The main results in this paper are the two following theorems.

Theorem 1.1 Let X be an F-space. Let $p : X \to \mathbb{R}$ be a real-valued function on X with the following properties:

a) $p(x) \geq 0$ for all $x \in X$ (nonnegativity);

b) $p(-x) = p(x)$ for every $x \in X$ (symmetry);

c) $\lim\limits_{n\in\mathbb{N}\to\infty} p(n^{-1}x) = 0$ for each $x \in X$ (absorbability);

d) $p\left(\sum\limits_1^\infty x_n\right) \leq \sum\limits_1^\infty p(x_n)$ if $\sum\limits_1^\infty \|x_n\| < \infty$ (countable subadditivity).

Then p is continuous on X.

Theorem 1.2 Let X and Y be F-spaces. Let $\{T_n : \mathcal{D}(T_n) \subset X \to Y\}$ be a sequence of closed operators with closed ranges. Then the following three properties of $\{T_n\}$ are equivalent:

A) if $x_n \in \mathcal{D}(T_n)$ $(n \in \mathbb{N})$ with $\lim_{n\to\infty} \|T_n x_n\| = 0$,

$$\varliminf_{k\in\mathbb{N}\to\infty} \sup_{n\in\mathbb{N}} \operatorname{dist}\left(k^{-1}x_n, \mathcal{N}(T_n)\right) = 0;$$

B) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$B_Y(0,\delta) \cap \mathcal{R}(T_n) \subseteq T_n\left(B_X(0,\varepsilon) \cap \mathcal{D}(T_n)\right) \quad \text{for all } n \in \mathbb{N};$$

C) if $y_n \in \mathcal{R}(T_n)$ $(n \in \mathbb{N})$ with $\lim_{n\to\infty} y_n = y$,

$$\underset{n\to\infty}{s\text{-}\lim} T_n^{-1}(y_n) = \left\{x \in X \mid (x,y) \in \underset{n\to\infty}{s\text{-}\lim} \mathcal{G}(T_n)\right\}.$$

Theorem 1.1 may be regarded as the basic tool in the theory of F-spaces, for it implies some fundamental principles as simple corollaries, such as the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem. It must be noted that Theorem 1.1 contains Theorem 1.2 as a more profound application.

Theorem 1.2 can be interpreted as a generalization of the well-known Lax-Richtmyer equivalence theorem [3] (which states that, for linear well-posed initial value problems, a consistent difference scheme is convergent if and only if it is stable): for an oprater equation of the first kind $Tx = y$, where $T : \mathcal{D}(T) \subset X \to Y$ is a closed linear operator, we assume that the equation is with a consistent approximation scheme $\{T_n\}$ as given in Theorem 1.2, here “consistent” means

$$\mathcal{G}(T) = \underset{n\to\infty}{s\text{-}\lim} \mathcal{G}(T_n).$$

Then condition A or B stands for stability of the scheme $\{T_n\}$, condition C stands for convergence of $\{T_n\}$, and the conclusion is

$$\text{convergence C } \left(\underset{n\to\infty}{s\text{-}\lim} T_n^{-1}(y_n) = T^{-1}(y)\right) \Longleftrightarrow \text{stability A} \Longleftrightarrow \text{stability B}.$$

This is also a generalization of [1, Theorem 2.1] and [2, Theorem 1].

The paper is organized as follows: in Section 2, we present the proof of Theorem 1.1, and in Section 3, we show that some fundamental principles as simple corollaries of the theorem. In Section 4, we present the proof of Theorem 1.2 with remarks and examples of application.

2 The Proof of Theorem 1.1

We recall the Baire-Hausdorff theorem before proving Theorem 1.1: A non-void complete metric space is of the second category (see, e.g., [5]). As is well known, the completeness of an F-space enables us to apply the Baire-Hausdorff theorem and to obtain such fundamental principles in functional analysis as the uniform boundedness theorem, the closed graph theorem and open mapping theorem. Here, we are to apply the Baire-Hausdorff theorem to establish a more general principle, Theorem 1.1.

Proof of Theorem 1.1 The proof is to be carried out in three steps.

Step 1 Prove that for any $\varepsilon > 0$ there exists a sequence $\{\delta_n\}$ of positive real numbers such that

$$B_X(0, \delta_n) \subset \overline{p^{-1}([0, \varepsilon/2^n])} \quad \text{for each } n \in \mathbb{N} \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \delta_n < +\infty. \tag{2.2}$$

It is obvious by properties a) and c) of p that

$$X = \bigcup_{n=1}^{\infty} np^{-1}([0, \varepsilon/2]) \quad \text{for every } \varepsilon > 0.$$

Since X is a non-void complete metric space, X is of the second category by the Baire-Hausdorff theorem, so that there must be a natural number k such that $\overline{kp^{-1}([0, \varepsilon/2])}^{\circ} \neq \varnothing$. Hence, there exist $x_0 \in X$ and $r > 0$ such that$B_X(x_0, r) \subset \overline{kp^{-1}([0, \varepsilon/2])}$, that is,

$$k^{-1} B_X(x_0, r) \subset \overline{p^{-1}([0, \varepsilon/2])}. \tag{2.3}$$

Note that, by properties b) and d) of p, the set $\overline{p^{-1}([0, \varepsilon/2])}$ is with the following two properties

$$\left.\begin{array}{l} x \in \overline{p^{-1}([0, \varepsilon/2])} \Longrightarrow \quad -x \in \overline{p^{-1}([0, \varepsilon/2])}; \\ \{x, y\} \subset \overline{p^{-1}([0, \varepsilon/2])} \quad \Longrightarrow \quad x + y \in \overline{p^{-1}([0, \varepsilon])}. \end{array}\right\} \tag{2.4}$$

Therefore from (2.3) and (2.4), we obtain $B_X(0, r/k) \subset \overline{p^{-1}([0, \varepsilon])}$. Thus for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$B_X(0, \delta) \subset \overline{p^{-1}([0, \varepsilon])}. \tag{2.5}$$

From (2.5), we obtain that for any $\varepsilon > 0$ there exists a sequence $\{\delta_n\}$ of positive real numbers such that (2.1) and (2.2) hold.

Step 2 Prove that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B_X(0, \delta) \subset p^{-1}([0, \varepsilon]). \tag{2.6}$$

By the conclusion of Step 1, for any $\varepsilon > 0$ there exists a sequence $\{\delta_n\}$ of positive real numbers such that (2.1) and (2.2) hold. Let $\delta = \delta_1$ and $x \in B_X(0, \delta)$. Then we have that $x \in \overline{p^{-1}([0, \varepsilon/2])}$ from (2.1), and hence there exists $x_1 \in p^{-1}([0, \varepsilon/2])$ such that$\|x - x_1\| < \delta_2$, i.e., $x - x_1 \in B(0, \delta_2)$. Assume $n \geq 1$ and $x_k \in p^{-1}\left([0, \varepsilon/2^k]\right)$ $(k = 1, \cdots, n)$ are chosen to satisfy

$$\left\| x - \sum_{j=1}^{k} x_j \right\| < \delta_{k+1}, \quad \text{i.e.,} \quad x - \sum_{j=1}^{k} x_j \in B_X(0, \delta_{k+1}).$$

Then by (2.1), there exists $x_{n+1} \in p^{-1}([0, \varepsilon/2^{n+1}])$ such that

$$\left\| x - \sum_{j=1}^{n+1} x_j \right\| < \delta_{n+2}, \quad \text{i.e.,} \quad x - \sum_{j=1}^{n+1} x_j \in B(0, \delta_{n+2}).$$

Thus we obtain a sequence $\{x_n\}$ of X such that

$$x_n \in p^{-1}\left([0, \varepsilon/2^n]\right) \tag{2.7}$$

and

$$\left\| x - \sum_{k=1}^{n} x_k \right\| < \delta_{n+1} \quad \text{and} \quad \|x\| < \delta_1. \tag{2.8}$$

Note that (2.8) with (2.2) implies that

$$\sum_{n=1}^{\infty} \|x_n\| \leq 2 \sum_{n=1}^{\infty} \delta_n < +\infty \tag{2.9}$$

and

$$x = \sum_{n=1}^{\infty} x_n. \tag{2.10}$$

Therefore, from (2.7), (2.9), (2.10), and the property d) of p, we conclude that

$$p\left(x\right) \leq \sum_{n=1}^{\infty} p\left(x_n\right) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus (2.6) follows.

Step 3 Prove that p is continuous on X, we need only to show that p is continuous at 0. This is obviously true by the conclusion of Step 2.

3 Simple Corollaries of Theorem 1.1

In this section, we will see that the uniform boundedness theorem, the closed graph theorem, and the open mapping theorem can be proved as simple corollaries of Theorem 1.1. In order to emphasize the role of Theorem 1.1, the above three theorems are stated in a little more generality than is usually needed. By doing that, we get more interesting versions.

Corollary 3.1 (The Uniform Boundedness Theorem) Let X be an F-space, Y a quasi-normed linear space and $\{T_\lambda\}_{\lambda\in\Lambda}$ a family of continuous mappings defined on X into Y. Assume that

1) $\|T_\lambda\left(x+y\right)\| \leq \|T_\lambda\left(x\right)\| + \|T_\lambda\left(y\right)\|$ for any $\lambda \in \Lambda$ and $x, y \in X$, and

2) $\lim\limits_{n\in\mathbb{N}\to\infty} \sup\limits_{\lambda\in\Lambda} \|T_\lambda\left(n^{-1}x\right)\| = 0$ for any $x \in X$.

Then $\lim\limits_{x\to 0} \sup\limits_{\lambda\in\Lambda} \|T_\lambda\left(x\right)\| = 0$.

Proof By Assumption 1), we have

$$\frac{1}{n} \|T_\lambda\left(x\right)\| \leq \left\| T_\lambda\left(\frac{1}{n}x\right) \right\| \quad \text{for each } n \in \mathbb{N}.$$

Therefore, by Assumption 2), we obtain

$$\lim_{n\to\infty} \left\{ \frac{1}{n} \sup_{\lambda\in\Lambda} \|T_\lambda\left(x\right)\| \right\} = 0 \quad \text{for any } x \in X. \tag{3.1}$$

It follows from (3.1) that

$$\sup_{\lambda\in\Lambda}\|T_\lambda(x)\|<\infty \quad \text{for any } x\in X. \tag{3.2}$$

Now, define $p: X\to\mathbb{R}$ as follows

$$p(x) := \max\left\{\sup_{\lambda\in\Lambda}\|T_\lambda(x)\|,\ \sup_{\lambda\in\Lambda}\|T_\lambda(-x)\|\right\}.$$

It is clear that p is well-defined (by (3.2)) and is with nonnegativity, symmetry, absorbability and countable subadditivity. Hence, by Theorem 1.1, p is continuous on X, which implies

$$\lim_{x\to0}\sup_{\lambda\in\Lambda}\|T_\lambda(x)\|=0.$$

Corollary 3.2 (The Generalized Closed Graph Theorem) Let X and Y be F-spaces. Let $T:\mathcal{D}(T)\subset X\to Y$ be a mapping which satisfies the following conditions

1) $\mathcal{D}(T)$ is a closed subspace of X;
2) $\|T(-x)\|=\|T(x)\|$ for every $x\in\mathcal{D}(T)$;
3) $\underline{\lim}_{n\to\infty}\|T(n^{-1}x)\|=0$ for every $x\in\mathcal{D}(T)$; and
4) $\left\|T\left(\sum\limits_1^\infty x_n\right)\right\|\le\sum\limits_1^\infty\|T(x_n)\|$ for any sequence $\{x_n\}$ of $\mathcal{D}(T)$ with

$$\sum_1^\infty\|x_n\|<\infty \quad\text{and}\quad \sum_1^\infty\|T(x_n)\|<\infty.$$

Then

(a) $\lim\limits_{x\to0}T(x)=0$;
(b) $\|T(x)\|$ is continuous on $\mathcal{D}(T)$.

Proof Define $p:\mathcal{D}(T)\to\mathbb{R}$ as follows $p(x):=\|T(x)\|$. It is not difficult to see that p satisfies all the conditions in Theorem 1.1. Hence, by Theorem 1.1, p is continuous on $\mathcal{D}(T)$, which implies (a) and (b).

Corollary 3.3 (The Open Mapping Theorem) Let X and Y be F-spaces. Let $T:\mathcal{D}(T)\subset X\to Y$ be a closed linear operator with $\mathcal{R}(T)=Y$. Then T is an open mapping, i.e., $T(U)$ is open in Y whenever U is open in $\mathcal{D}(T)$.

Proof Define $p: Y\to\mathbb{R}$ as follows

$$p(y):=\inf_{x\in T^{-1}(y)}\|x\|. \tag{3.3}$$

Then p is obviously with nonnegativity, symmetry, and absorbability on Y. We now verify that p is with countable subadditivity. Let a sequence $\{y_n\}$ of Y satisfy $\sum\limits_1^\infty\|y_n\|<+\infty$. Then, by the completeness of Y, there exists a vector $y_\infty\in Y$ such that $y_\infty=\sum\limits_1^\infty y_n$ in Y. We need to show that

$$p(y_\infty)\le\sum_1^\infty p(y_n). \tag{3.4}$$

If $\sum\limits_1^\infty p(y_n) = +\infty$, (3.4) is automatically satisfied. Assume $\sum\limits_1^\infty p(y_n) < +\infty$. For every $\varepsilon > 0$ and every $n \in \mathbb{N}$, by (3.3), there exists a vector $x_n \in T^{-1}(y_n)$ such that $\|x_n\| < p(y_n) + \frac{\varepsilon}{2^n}$, and therefore

$$\sum_1^\infty \|x_n\| < \sum_1^\infty p(y_n) + \varepsilon. \tag{3.5}$$

Since X is an F-space, $\sum\limits_1^\infty x_n$ is convergent to a vector x_∞ of X. By the closedness of T, we have that $x_\infty \in \mathcal{D}(T)$ and $Tx_\infty = y_\infty$. Hence, by (3.5),

$$p(y_\infty) \le \|x_\infty\| \le \sum_1^\infty \|x_n\| < \sum_1^\infty p(y_n) + \varepsilon. \tag{3.6}$$

Note that the ε is arbitrary, so (3.6) implies (3.4). Now, by Theorem 1.1, p is continuous on Y.

To prove that T is an open mapping, we have only to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B_Y(0,\delta) \subset T(\mathcal{D}(T) \cap B_X(0,\varepsilon)). \tag{3.7}$$

By the continuity of p, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$p(y) < \varepsilon \quad \text{for all } y \in B_Y(0,\delta).$$

Hence, for every $y \in B_Y(0,\delta)$, there exists a vector $x \in T^{-1}(y) \cap B_X(0,\varepsilon)$ by the definition of p, and therefore (3.7) is true.

4 The Proof of Theorem 1.2

To prove Theorem 1.2, we prepare the following lemma.

Lemma 4.1 Let Y be an F-space, $\{Y_n\}$ a sequence of closed subspaces of Y. Let $\mathfrak{Y}$ be the set of all vectors $\xi = \{y_n\}$ with $y_n \in Y_n$ (for every $n \in \mathbb{N}$) and $\lim\limits_{n\to\infty} y_n = 0$, i.e.,

$$\mathfrak{Y} = \left\{\xi \mid \xi = \{y_n\},\ y_n \in Y_n \ \ (\forall n \in \mathbb{N}),\ \lim_{n\to\infty} y_n = 0\right\}.$$

Then $\mathfrak{Y}$ is an F-space by the algebrac operations $\{y'_n\} + \{y''_n\} := \{y'_n + y''_n\}$, $\alpha\{y_n\} := \{\alpha y_n\}$ and the quasi-norm $\|\{y_n\}\| := \sup\limits_{n\in\mathbb{N}} \|y_n\|$.

Proof It is easy to show that $\mathfrak{Y}$ is a linear space and

$$\|\xi\| \ge 0 \quad \text{and} \quad \|\xi\| = 0 \iff \xi = \mathbf{0}, \quad \|\xi + \eta\| \le \|\xi\| + \|\eta\|, \quad \|-\xi\| = \|\xi\|.$$

To prove that $\|\cdot\|$ is a quasi-norm on $\mathfrak{Y}$, we need only to show that

$$\lim_{\alpha_k \to 0} \|\alpha_k \xi\| = 0 \tag{4.1}$$

and

$$\lim_{\left\|\xi^{(k)}\right\|\to 0}\left\|\alpha\xi^{(k)}\right\|=0. \tag{4.2}$$

Noting that $\xi=\{y_n\}$, as a subset of Y, is bounded and that $\alpha_k\to 0$, we obtain (4.1). Put $\xi^{(k)}:=\left\{y_n^{(k)}\right\}_{n=1}^{\infty}$ $(k\in\mathbb{N})$. Then $\left\|\xi^{(k)}\right\|\to 0$ implies $\lim\limits_{k\to\infty}\sup\limits_{n\in\mathbb{N}}\left\|y_n^{(k)}\right\|=0$ and therefore

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\left\|\alpha y_n^{(k)}\right\|=0,$$

since $\lim\limits_{\|y\|\to 0}\|\alpha y\|=0$. Thus, (4.2) holds.

Next, we show that $\mathfrak{Y}$ is complete. Let $\lim\limits_{k,l\to\infty}\left\|\xi^{(k)}-\xi^{(l)}\right\|=0$ in $\mathfrak{Y}$. Put $\xi^{(k)}:=\left\{y_n^{(k)}\right\}$ $(k\in\mathbb{N})$. Then $\lim\limits_{k,l\to\infty}\sup\limits_{n\in\mathbb{N}}\left\|y_n^{(k)}-y_n^{(l)}\right\|=0$. Hence there exists a sequence $\left\{y_n^{(\infty)}\right\}$ of Y such that

$$\lim_{k\to\infty}\sup_{n\in\mathbb{N}}\left\|y_n^{(k)}-y_n^{(\infty)}\right\|=0 \tag{4.3}$$

by the completeness of Y. Note that (4.3) implies that $\lim\limits_{n\to\infty}\left\|y_n^{(\infty)}\right\|=0$. Put $\xi^{(\infty)}:=\left\{y_n^{(\infty)}\right\}$, then $\xi^{(\infty)}\in\mathfrak{Y}$ and $\xi^{(k)}\to\xi^{(\infty)}$ as $k\to\infty$ in $\mathfrak{Y}$.

Proof of Theorem 1.2 The proof will be carried out in three steps.

Step 1 A) $\Rightarrow$ B): Assume A) and put

$$\mathfrak{Y}=\left\{\xi\mid\xi=\{y_n\},\ y_n\in\mathcal{R}(T_n)\ (\forall n\in\mathbb{N}),\ \lim_{n\to\infty}y_n=0\right\}.$$

Then

$$\varliminf_{k\to\infty}\sup_{n\in\mathbb{N}}\operatorname{dist}\left(k^{-1}x_n,\mathcal{N}(T_n)\right)=0$$

for any $\xi=\{y_n\}\in\mathfrak{Y}$ and $\{x_n\}$ with $x_n\in T_n^{-1}(y_n)$ $(n\in\mathbb{N})$. Since

$$\operatorname{dist}\left(k^{-1}x_n,\mathcal{N}(T_n)\right)=\inf_{x\in T_n^{-1}(y_n)}\left\|k^{-1}x\right\|,\quad\text{where } y_n=T_n(x_n),$$

and since

$$\inf_{x\in T_n^{-1}(y_n)}\|x\|\le k\inf_{x\in T_n^{-1}(y_n)}\left\|k^{-1}x\right\|\quad(n,k\in\mathbb{N}),$$

we have that

$$\sup_{n\in\mathbb{N}}\inf_{x\in T_n^{-1}(y_n)}\|x\|<+\infty\quad\text{for any }\xi=\{y_n\}\in\mathfrak{Y} \tag{4.4}$$

and that

$$\varliminf_{k\to\infty}\sup_{n\in\mathbb{N}}\inf_{x\in T_n^{-1}(y_n)}\left\|k^{-1}x\right\|=0\quad\text{for any }\xi=\{y_n\}\in\mathfrak{Y}. \tag{4.5}$$

Noting that $\mathcal{R}(T_n)\,(n\in\mathbb{N})$ are all closed, by Lemma 4.1, $\mathfrak{Y}$ is an F-space by the algebrac operations $\{y_n'\}+\{y_n''\}:=\{y_n'+y_n''\}$, $\alpha\{y_n\}:=\{\alpha y_n\}$ and the quasi-norm$\|\{y_n\}\|:=\sup\limits_{n\in\mathbb{N}}\|y_n\|$. Define $p:\mathfrak{Y}\to\mathbb{R}$ as follows

$$p(\xi):=\sup_{n\in\mathbb{N}}\inf_{x\in T_n^{-1}(y_n)}\|x\|\quad\text{for any }\xi=\{y_n\}\in\mathfrak{Y}. \tag{4.6}$$

By (4.4), (4.6) and (4.5), p is well-defined on $\mathfrak{Y}$, and is with nonnegativity, symmetry and absorbability. Next we verify that p is with countable subadditivity.

Let $\left\{\xi^{(k)}\right\}_{k=1}^{\infty}$ be a sequence of $\mathfrak{Y}$ with $\sum\limits_{k=1}^{\infty}\left\|\xi^{(k)}\right\| < +\infty$. Then, by the completeness of $\mathfrak{Y}$, there exists a vector $\xi^{(\infty)} \in \mathfrak{Y}$ such that

$$\xi^{(\infty)} = \sum_{k=1}^{\infty}\xi^{(k)} \quad \text{in } \mathfrak{Y}. \tag{4.7}$$

We have to show that

$$p\left(\xi^{(\infty)}\right) \le \sum_{k=1}^{\infty} p\left(\xi^{(k)}\right). \tag{4.8}$$

If $\sum\limits_{k=1}^{\infty} p\left(\xi^{(k)}\right) = +\infty$, (4.8) holds automatically. Now, assume $\sum\limits_{k=1}^{\infty} p\left(\xi^{(k)}\right) < +\infty$ and put

$$\xi^{(k)} = \left\{y_n^{(k)}\right\}, \quad \xi^{(\infty)} = \left\{y_n^{(\infty)}\right\}. \tag{4.9}$$

By the definition of p (see (4.6)),

$$\inf_{x\in T_n^{-1}\left(y_n^{(k)}\right)} \|x\| \le p\left(\xi^{(k)}\right) \qquad \text{for any } k \in \mathbb{N} \text{ and } n \in \mathbb{N}.$$

Therefore, for every $\varepsilon > 0$, every $k \in \mathbb{N}$ and every $n \in \mathbb{N}$, there exists a point $x_n^{(k)}(\varepsilon) \in T_n^{-1}\left(y_n^{(k)}\right)$ such that

$$\left\|x_n^{(k)}(\varepsilon)\right\| < p\left(\xi^{(k)}\right) + \frac{\varepsilon}{2^k}.$$

Hence, for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there is a sequence $\left\{x_n^{(k)}(\varepsilon)\right\}_{k=1}^{\infty}$ of $\mathcal{D}(T_n)$ such that

$$T_n\left(x_n^{(k)}(\varepsilon)\right) = y_n^{(k)} \quad \text{for every } k \in \mathbb{N} \tag{4.10}$$

and

$$\sum_{k=1}^{\infty}\left\|x_n^{(k)}(\varepsilon)\right\| < \sum_{k=1}^{\infty} p\left(\xi^{(k)}\right) + \varepsilon. \tag{4.11}$$

By (4.11) and the completeness of X, there exists a point $x_n^{(\infty)}(\varepsilon) \in X$ such that

$$x_n^{(\infty)}(\varepsilon) = \sum_{k=1}^{\infty} x_n^{(k)}(\varepsilon) \quad \text{in } X. \tag{4.12}$$

On the other hand, (4.10) with (4.7) and (4.9) implies that for every $n \in \mathbb{N}$, there holds

$$y_n^{(\infty)} = \sum_{k=1}^{\infty} y_n^{(k)} = \sum_{k=1}^{\infty} T_n\left(x_n^{(k)}(\varepsilon)\right) \quad \text{in } Y. \tag{4.13}$$

Since, for every $n \in \mathbb{N}$, T_n is closed, it follows from (4.12) and (4.13) that $x_n^{(\infty)}(\varepsilon) \in \mathcal{D}(T_n)$ and $x_n^{(\infty)}(\varepsilon) \in T_n^{-1}\left(y_n^{(\infty)}\right)$. Hence, by (4.12) and (4.11), for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, we have that

$$\inf_{x \in T_n^{-1}\left(y_n^{(\infty)}\right)} \|x\| \le \left\| x_n^{(\infty)}(\varepsilon) \right\| \le \sum_{k=1}^{\infty} \left\| x_n^{(k)}(\varepsilon) \right\| < \sum_{k=1}^{\infty} p\left(\xi^{(k)}\right) + \varepsilon.$$

So for every $\varepsilon > 0$, we have

$$p\left(\xi^{(\infty)}\right) \le \sum_{k=1}^{\infty} p\left(\xi^{(k)}\right) + \varepsilon.$$

Thus (4.8) is true.

We have proved that p satisfies all the conditions of Theorem 1.1. Hence, p is continuous on $\mathfrak{Y}$, and is continuous at 0. This implies that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\inf_{x \in T_n^{-1}(y)} \|x\| < \varepsilon \quad \text{for every } n \in \mathbb{N} \text{ and every } y \in \mathcal{R}(T_n) \text{ with } \|y\| < \delta.$$

Therefore we obtain that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$B_Y(0,\delta) \cap \mathcal{R}(T_n) \subseteq T_n\left(B_X(0,\varepsilon) \cap \mathcal{D}(T_n)\right) \quad \text{for all } n \in \mathbb{N}.$$

That shows that A) implies B).

Step 2 B) $\Rightarrow$ C) Assume that $\{T_n\}$ satisfies B) and that $y_n \in \mathcal{R}(T_n)$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} y_n = y$. We have to show that

$$s\text{-}\lim_{n\to\infty} T_n^{-1}(y_n) = \left\{ x \in X \mid (x,y) \in s\text{-}\lim_{n\to\infty} \mathcal{G}(T_n) \right\}. \tag{4.14}$$

If $x \in s\text{-}\lim_{n\to\infty} T_n^{-1}(y_n)$, there exists a sequence $\{x_n\}$ with $x_n \in T_n^{-1}(y_n)$ $(n \in \mathbb{N})$ such that $x_n \to x,\ T_n x_n \to y,$ as $n \to \infty$. This implies that $(x,y) \in s\text{-}\lim_{n\to\infty} \mathcal{G}(T_n)$. Hence we obtain

$$s\text{-}\lim_{n\to\infty} T_n^{-1}(y_n) \subseteq \left\{ x \in X \mid (x,y) \in s\text{-}\lim_{n\to\infty} \mathcal{G}(T_n) \right\}. \tag{4.15}$$

On the other hand, if $x \in X$ such that $(x,y) \in s\text{-}\lim_{n\to\infty} \mathcal{G}(T_n)$, then there exists a sequence $\{x_n\}$ of X with $x_n \in \mathcal{D}(T_n)$ $(n \in \mathbb{N})$ such that $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} T_n x_n = y$, and therefore

$$\lim_{n\to\infty} (y_n - T_n x_n) = 0. \tag{4.16}$$

Since $\{T_n\}$ satisfies B), i.e., for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$B_Y(0,\delta) \cap \mathcal{R}(T_n) \subseteq T_n\left(B_X(0,\varepsilon) \cap \mathcal{D}(T_n)\right) \quad \text{for all } n \in \mathbb{N},$$

it follows from (4.16) that for every $\varepsilon > 0$, there exists a natural number N such that

$$y_n - T_n x_n \in T_n\left(B_X(0,\varepsilon) \cap \mathcal{D}(T_n)\right) \quad \text{for all } n > N.$$

This implies that $\lim\limits_{n\to\infty} \inf\limits_{v\in T_n^{-1}(y_n-T_nx_n)} \|v\| = 0$. So we obtain that

$$\begin{aligned} \operatorname{dist}(x, T_n^{-1}(y_n)) &= \inf_{v\in T_n^{-1}(y_n-T_nx_n)} \|v + x_n - x\| \\ &\leq \|x_n - x\| + \inf_{v\in T_n^{-1}(y_n-T_nx_n)} \|v\| \to 0, \quad \text{as } n\to\infty. \end{aligned}$$

Thus $x \in s\text{-}\lim\limits_{n\to\infty} T_n^{-1}(y_n)$. Hence we have that

$$\left\{x \in X \mid (x,y) \in s\text{-}\lim_{n\to\infty}\mathcal{G}(T_n)\right\} \subseteq s\text{-}\lim_{n\to\infty}T_n^{-1}(y_n). \tag{4.17}$$

(4.14) follows from (4.15) and (4.17).

Step 3 C) $\Rightarrow$ A) Assume C) and let $\{x_n\}$ be a sequence of X with $x_n \in \mathcal{D}(T_n)$ $(n \in \mathbb{N})$ and $\lim\limits_{n\to\infty} T_nx_n = 0$. Then

$$s\text{-}\lim_{n\to\infty}T_n^{-1}(T_nx_n) = \left\{x \in X \mid (x, 0_Y) \in s\text{-}\lim_{n\to\infty}\mathcal{G}(T_n)\right\} \ni 0_X .$$

That implies that $\lim\limits_{n\to\infty} \inf\limits_{x\in T_n^{-1}(T_nx_n)} \|x\| = 0$. Hence, there exists a sequence $\{x_n'\}$ of X with $x_n' \in T_n^{-1}(T_nx_n)$ $(n \in \mathbb{N})$ such that $\lim\limits_{n\to\infty} x_n' = 0$; therefore, $\{x_n'\}$ is a bounded subset of X and this impies that $\varliminf\limits_{k\to\infty} \sup\limits_{n\in\mathbb{N}} \|k^{-1}x_n'\| = 0$. Noting that

$$\operatorname{dist}(k^{-1}x_n, \mathcal{N}(T_n)) = \operatorname{dist}(k^{-1}x_n', \mathcal{N}(T_n)) \leq \|k^{-1}x_n'\| \quad \text{for all } n \in \mathbb{N},$$

we obtain that $\varliminf\limits_{k\to\infty} \sup\limits_{n\in\mathbb{N}} \operatorname{dist}(k^{-1}x_n, \mathcal{N}(T_n)) = 0$, that is, A) holds.

Remark 4.1 In Theorem 1.2, if X and Y are Banach spaces, then the following three properties of $\{T_n\}$ are equivalent:

A) For any $x_n \in \mathcal{D}(T_n)$ $(n \in \mathbb{N})$ with $\lim\limits_{n\to\infty} \|T_nx_n\| = 0$, there holds

$$\sup_{n\in\mathbb{N}} \operatorname{dist}\ (x_n, \mathcal{N}(T_n)) < +\infty.$$

B) There exists a $\delta > 0$ such that

$$B_Y(0,\delta) \cap \mathcal{R}(T_n) \subseteq T_n(B_X(0,1) \cap \mathcal{D}(T_n)) \quad \text{for all } n \in \mathbb{N}.$$

C) If $y_n \in \mathcal{R}(T_n)$ $(n \in \mathbb{N})$ with $\lim\limits_{n\to\infty} y_n = y$,

$$s\text{-}\lim_{n\to\infty}T_n^{-1}(y_n) = \left\{x \in X \mid (x,y) \in s\text{-}\lim_{n\to\infty}\mathcal{G}(T_n)\right\}.$$

Remark 4.2 The following case is of practical interest for the applications of Theorem 1.2: There exists a closed linear operator $T : \mathcal{D}(T) \subset X \to Y$ such that $\{T_n\}$ is its consistent approximation scheme, that is, $\mathcal{G}(T) = s\text{-}\lim\limits_{n\to\infty}\mathcal{G}(T_n)$. Here we provide two examples with the above case.

Example 4.1 Let T, T_n be all self-adjoint operators in a Hilbert space H, and let

$$s\text{-}\lim_{n\to\infty} R_\lambda(T_n) = R_\lambda(T) \quad (\forall\lambda\in\mathbb{C}\backslash\mathbb{R}),$$

where $R_\lambda(T)$ and $R_\lambda(T_n)$ denote the resolvent operator of T and T_n, respectively. Then $\{T_n\}$ is a consistent approximation scheme of T. See [6, pp.152–153] or [4, pp. 148–149] for the proof of the statement. Now, by Theorem 1.2 we conclude that: if $\mathcal{R}(T_n)$ $(n\in\mathbb{N})$ are closed, and if

$$\sup_{n\in\mathbb{N}} \left\|T_n^\dagger\right\| < \infty \quad (\text{where } T_n^\dagger \text{ is the Moore-Penrose inverse of } T_n),$$

then for any $y_n\in\mathcal{R}(T_n)$ $(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty} y_n = y$, there holds $s\text{-}\lim\limits_{n\to\infty} T_n^{-1}(y_n) = T^{-1}(y)$; especially, there holds $s\text{-}\lim\limits_{n\to\infty}\mathcal{N}(T_n) = \mathcal{N}(T)$.

Example 4.2 Let $A, A_n \in \mathcal{B}(X,Y)$, where X and Y be Banach spaces, and let $s\text{-}\lim\limits_{n\to\infty} A_n = A$. Then $\{A_n\}$ is a consistent approximation scheme of A. By Theorem 1.2, we conclude that: If $\mathcal{R}(A_n)$ $(n\in\mathbb{N})$ are closed, and if for any $x_n\in\mathcal{D}(A_n)$ $(n\in\mathbb{N})$ with $\lim\limits_{n\to\infty}\|A_n x_n\| = 0$, there holds $\sup\limits_{n\in\mathbb{N}} \operatorname{dist}(x_n, \mathcal{N}(A_n)) < +\infty$, then the conclusion of Example 4.1 holds.

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F-空间中的一个基本定理及其在数值分析中的应用

杜乃林, 夏 炜

(武汉大学数学与统计学院, 湖北 武汉 430072)

摘要: 本文针对F-空间中闭算子方程的一般逼近格式, 研究其相容性、收敛性和稳定性之间的关系. 所得的主要结果是: 这种一般逼近格式在相容性条件下, 其收敛性与稳定性是等价的. 此定理可以看作是对Lax等价原理的推广, 是求解第一类闭算子方程的一般逼近格式的基本定理. 为得到这一主要结果, 本文还给出了F-空间中的一条基本定理, 众所周知的一致有界原理, 闭图像定理和开映像定理是其简单推论.

关键词: F-空间; 闭算子; 收敛性; 稳定性

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