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A PROPER AFFINE SPHERE THEOREM RELATED TO HOMOGENEOUS FUNCTIONS

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Abstract: In this paper, we focus on the affine sphere theorem related to homogeneous function. Based on Hopf maximum principle, we obtain that the affine sphere theorem does hold for given elementary symmetric curvature problems under concavity conditions. In particular, it gives a new proof of Deicke's theorem on homogeneous functions.

Keywords: affine sphere theorem; homogeneous functions

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1 Main theorems

Let L be a positive function of class $C^4(\mathbb{R}^n/\{0\})$ with homogeneous of degree one. Introducing a matrix g of elements

$$g_{ij} = \frac{\partial^2\left(\frac{L^2}{2}\right)}{\partial x_i \partial x_j}, \tag{1.1}$$

Deicke [4] showed that the matrix g is positive and the following theorem, a short and elegant proof was presented in Brickell [1].

Theorem 1.1 Let $\det g$ be a constant on $\mathbb{R}^n/\{0\}$. Then g is a constant matrix on $\mathbb{R}^n/\{0\}$.

Theorem 1.1 is very important in affine geometry [10, 11, 13] and Finsler geometry [4]. There are lots of papers introducing the history and progress of these problems, for example [7]. A laplacian operator and Hopf maximum principle is the key point of Deicke [4] 's proof. However, our method depends on the concavity of the fully nonlinear operator, we give a new method to prove more generalized operator than Theorem 1.1, for considering operator $F(g)$, which including the operator of determinant.

Theorem 1.2 Let $F(g)$ be a constant on $\mathbb{R}^n/\{0\}$, $F(g)$ be concave with respect to matrix g, and the matrix $[F^{ij}]_{1\le i,j\le n} = [\frac{\partial F}{\partial g_{ij}}]_{1\le i,j\le n}$ be positive semi-definite. Then g is a constant matrix on $\mathbb{R}^n/\{0\}$.

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In fact

(1) If $F(g) = \log\det g$, Theorem 1.2 is just Theorem 1.1.

(2) An interesting example of Theorem 1.2 is $F(g) = (S_k(g))^{\frac{1}{k}}$, where $S_k(g)$ is the elementary symmetric polynomial of eigenvalues of g. The concavity of $F(g)$ was from Caffarelli-Nirenberg-Spruck [3]. A similar Liouville problem for the S_2 equation was obtained in [2].

It is easy to see that the method of Brickell [1] does not apply to our Theorem 1.2.

On the other hand, there are some remarkable results for homogeneous solution to partial differential equations. Han-Nadirashvili-Yuan [6] proved that any homogeneous order 1 solution to nondivergence linear elliptic equations in $\mathbb{R}^3$ must be linear, and Nadirashvili-Yuan [8] proved that any homogeneous degree other than 2 solution to fully nonlinear elliptic equations must be "harmonic". In fact, our methods can also be used to deal with the following hessian type equations

$$F(D^2u) = \text{constant}. \tag{1.2}$$

More recently, Nadirashvili-Vlăduţ [9] obtained the following theorem.

Theorem 1.3 Let u be a homogeneous order 2 real analytic function in $\mathbb{R}^4/\{0\}$. If u is a solution of the uniformly elliptic equation $F(D^2u) = 0$ in $\mathbb{R}^4/\{0\}$, then u is a quadratic polynomial.

However, our theorem say that above theorem holds provided F with some concavity/convexity property. Pingali [12] can show for 3-dimension, there is concave operator G form F without some concavity/convexity property, for example

$$F(D^2u) = \det D^2u + \Delta u$$

for $\lambda_1 \le \lambda_2 \le \lambda_3$ are eigenvalues of hessian matrix D^2u. Then

$$G(\lambda_1, \lambda_2, \lambda_3) = \int_{36}^{\lambda_1+\lambda_2+\lambda_3+\lambda_1\lambda_2\lambda_3} \exp(-\frac{t^2}{2})dt$$

has a uniformly positive gradient and is concave if $\lambda_1 > 3$. That is to say, using our methods, there is a simple proof of Theorem 1.3 if one can construct a concave operator with respect to F in Theorem 1.3.

2 Proof of Theorem 1.2

Here we firstly list the Hopf maximum principle to be used in our proof, see for example [5].

Lemma 2.1 Let u be a C^2 function which satisfies the differential inequality

$$Lu = a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b^i \frac{\partial u}{\partial x_i} \ge 0 \tag{2.1}$$

in an open domain Ω, where the symmetric matrix a^{ij} is locally uniformly positive definite in Ω and the coefficients a^{ij}, b^i are locally bounded. If u takes a maximum value M in Ω then $u \equiv M$.

Proof of Theorem 1.2 Differentiating this equation twice with respect to x

$$F(g) = \text{constant},$$

one has

$$F^{ij}g_{ijkl} + F^{ij,pq}g_{ijk}g_{pql} = 0,$$

where $F^{ij} = \frac{\partial F}{\partial g_{ij}}$, $F^{ij,pq} = \frac{\partial^2 F}{\partial g_{ij}\partial g_{pq}}$, $g_{ijk} = \frac{\partial g_{ij}}{\partial x_k}$, and $g_{ijkl} = \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}$.

The concavity of $F(g)$ with respect to g says that the matrix $J_{kl} = F^{ij}g_{ijkl}$ is positive semi-definite. In particular,

$$F^{ij}g_{ijkk} \geq 0. \tag{2.2}$$

We firstly consider (2.2) as an inequality in unit sphere S^{n-1},

$$F^{ij}g_{ijkk} \geq 0, \quad S^{n-1}, \tag{2.3}$$

that is to say using Hopf maximum principle of Lemma 2.1 and taking $\Omega = S^{n-1}$, it shows that g_{kk} is constant on S^{n-1}, and it is so on $\mathbb{R}^n/\{0\}$ because g_{kk} is positively homogeneous of degree zero. Then, owing to the matrix $F^{ij}g_{ijkl}$ be positive semi-definite

$$F^{ij}g_{ijkl} = 0.$$

Using Hopf maximum principle again and g_{kl} is positively homogeneous of degree zero, then the matrix g is constant matrix. We complete the proof of Theorem 1.2.

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齐次函数的一个仿射球定理

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摘要: 本文研究了相关齐次函数的仿射球定理. 利用Hopf极大值原理, 对任意给定的带凹性条件的初等对称曲率问题, 获得了此类仿射球定理. 特别地, 这也给出了Deicke 齐次函数定理的一个新证明.

关键词: 仿射球定理; 齐次函数

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