STABILITY OF GORENSTEIN MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract: This paper mainly study the stability of Gorenstein modules with respect to a semidualizing module C (i.e., Gorenstein C-projective, Gorenstein C-injective and Gorenstein C-flat modules). With the method of homology, we gain the result that Gorenstein C-projective (resp., C-injective, C-flat) modules have a nice stability, which generalizes the result that Gorenstein projective (resp., injective, flat) modules have a nice stability.

Keywords: Gorenstein C-projective (injective, flat) modules; stability

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1 Introduction

Gorenstein homological theory originated from the works of Auslander and Bridger [1], where they introduced modules of G-dimension 0. Enochs et al. [2, 3] extended these ideas and introduced Gorenstein projective, Gorenstein injective and Gorenstein flat modules and corresponding dimensions over arbitrary rings. From then, Gorenstein homological theory was extensively studied and developed, see for example, [4, 5] and references therein.

Recall from White [6] that an R-module C is said to be semidualizing if C admits a degreewise finite projective resolution, the natural homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}_R^{\geq 1}(C,C) = 0$. Examples include the rank one free modules and a dualizing (Canonical) modules when one exists. With this notion, Gorenstein homological theory with respect to a semidualizing module was extensively studied, see for example, [7] and [8].

Let R be a ring. White [6] initiated the study of G_C-projective R-modules as the modules of the form $\operatorname{Im}(P_0 \to C \otimes_R P^0)$ for some exact sequence of R-modules

$$\mathbf{P} : \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots ,$$

where each P_i and P^i is projective, and such that the complex $\operatorname{Hom}_R(\mathbf{P}, C \otimes_R Q)$ is exact for each projective R-module Q. The complex $\mathbf{P}$ is called a complete PC-resolution. When

$C=R$, the definition corresponds to the definition of Gorenstein projective modules. In [6], Proposition 2.6 showed that if P is R-projective, then P and $C \otimes_R P$ are G_C-projective, and Proposition 2.9 showed that every cokernel in a complete PC-resolution is G_C-projective, so every image and kernel in it is also G_C-projective.

For G_C-projective R-modules, we give another generalization in this paper, which is different from the definition above and that in [9]. An R-module M is called Gorenstein C-projective if there exists an exact sequence of R-modules

$$\mathbf{X} : \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots ,$$

where each P_i and P^i is projective with $M \cong \mathrm{Im}(C \otimes_R P_0 \to C \otimes_R P^0)$, such that the complex $\mathrm{Hom}_R(\mathbf{X}, C \otimes_R Q)$ is exact for each projective R-module Q. It is easy to see that, for every projective R-module P, $C \otimes_R P$ is Gorenstein C-projective from definition, while the projective module P is not Gorenstein C-projective, so Gorenstein C-projective modules are a generalization of P_C-projective modules not projective modules, which distinguishes G_C-projective modules. When $C=R$, Gorenstein C-projective modules are exactly Gorenstein projective modules. From this point of view, the notion of Gorenstein C-projective modules is a generalization of Gorenstein projective modules.

Let $\mathscr{A}$ be an abelian category and $\mathcal{X}$ an additive full-subcategory of $\mathscr{A}$. Sather-Wagstaff, Sharif and White [9] introduced the definition of Gorenstein category $\mathcal{G}(\mathcal{X})$ on $\mathcal{X}$. $\mathcal{G}(\mathcal{X}) = \{A \in \mathscr{A} \mid$ there exists a both $\mathrm{Hom}_{\mathscr{A}}(\mathbb{X}, -)$-exact and $\mathrm{Hom}_{\mathscr{A}}(-, \mathbb{X})$-exact exact sequence $\mathbb{X} : \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$ in $\mathscr{A}$ with all X_i and X^i in $\mathcal{X}$ and $A \cong \mathrm{Im}(X_0 \to X^0)\}$. Especially, if $\mathscr{A} = R\text{-Mod}$, $\mathcal{X} = \mathcal{P}(R)$, then $\mathcal{G}(\mathcal{P}(R))$ is nothing but the subcategory of Gorenstein projective modules. Dually, $\mathcal{G}(\mathcal{I}(R))$ is nothing but the subcategory of Gorenstein injective modules. Let $\mathcal{G}^0(\mathcal{X}) = \mathcal{X}$, $\mathcal{G}^1(\mathcal{X}) = \mathcal{G}(\mathcal{X})$, for arbitrary $n \geq 1$, $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}^n(\mathcal{G}(\mathcal{X}))$. They proved that when $\mathcal{X}$ is self-orthogonal, $\mathcal{G}^n(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ for arbitrary $n \geq 1$; and they proposed the question whether $\mathcal{G}^n(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ holds for an arbitrary subcategory $\mathcal{X}$. See [9, 4.10 and 5.8]. Huang [10] proved that the answer to this question is affirmative. Kong and Zhang [11] gave a slight generalization of this stability by a different method. Recently, stability of other Gorenstein classes of modules was extensively studied by many authors, see for example, [12–15]. Inspired by these results, we consider the following questions:

Question 1 Given an exact sequence of Gorenstein C-projective R-modules $\mathbf{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ such that the complex $\mathrm{Hom}_R(\mathbf{G}, \mathrm{C} \otimes_R P)$ is exact for each $P \in \mathcal{P}(R)$, is the module $\mathrm{Im}(G_0 \to G^0)$ Gorenstein C-projective?

The same question of Gorenstein C-injective R-modules can be considered dually.

Question 2 Given an exact sequence of Gorenstein C-flat R-modules $\mathbf{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ such that the complex $\mathrm{Hom}_R(C, E) \otimes_R \mathbf{G}$ is exact for each $E \in \mathcal{I}(R)$, is the module $\mathrm{Im}(G_0 \to G^0)$ Gorenstein C-flat?

In this paper, we give an affirmative answer to Question 1 (see Theorem 3.5 and Theorem 3.9), and a partial positive answer to Question 2 (see Theorem 3.12). It is shown that Gorenstein C-projective (injective, flat) modules have a nice stability.

2 Preliminaries

Throughout this paper, R is a commutative ring with identity and all modules are unitary. R-Mod denotes the category of R-modules. $\mathcal{P}(R)$ (resp., $\mathcal{I}(R)$, $\mathcal{F}(R)$) denotes projective (resp., injective, flat) R-modules.

In this section, we mainly recall some necessary notions and definitions.

Definition 2.1 [14] Let $\mathscr{X}$, $\mathscr{Y}$ be the class of R-modules, M be an R-module.

(1) An $\mathscr{X}$-resolution of R-module M is an exact sequence

$$\mathbf{X} : \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_i \in \mathscr{X}$ $(i \geq 0)$.

(2) A $\mathscr{Y}$-coresolution of R-module M is an exact sequence

$$\mathbf{Y} : 0 \longrightarrow M \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow \cdots$$

with $Y_i \in \mathscr{Y}$ $(i \geq 0)$.

Definition 2.2 [5] Let $\mathcal{X}$ be a class of R-modules.

(1) $\mathcal{X}$ is called $\mathcal{P}_C(R)$-resolving if $\mathcal{P}_C(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

(2) $\mathcal{X}$ is called $\mathcal{I}_C(R)$-resolving if $\mathcal{I}_C(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

Recall from [6] that a module C is semidualizing if C admits a degreewise finite projective resolution, the natural homothety map $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}_R^{\geq 1}(C, C) = 0$. The rank one free modules and a dualizing module are semidualizing modules over a noetherian ring R. We refer the reader to [16–19].

In the remainder of the paper, let C be a fixed semidualizing R-module.

Recall from [18] that the R-modules in the following classes

$$\mathcal{P}_C(R) = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\},$$
$$\mathcal{F}_C(R) = \{C \otimes_R F \mid F \text{ is a flat } R\text{-module}\},$$
$$7\mathcal{I}_C(R) = \{\operatorname{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}$$

are called C-projective, C-flat, and C-injective, respectively.

Definition 2.3 An R-module M is said to be Gorenstein C-projective, if there exists an exact sequence of C-projective R-modules

$$\mathbf{X} : \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

with $M \cong \operatorname{Im}(C \otimes_R P_0 \to C \otimes_R P^0)$, such that the complex $\operatorname{Hom}_R(\mathbf{X}, C \otimes_R Q)$ is exact for each projective R-module Q. The complex $\mathbf{X}$ is called complete $\mathcal{P}_C$-resolution, then by symmetry, all the images, and hence also all the kernels, and cokernels of $\mathbf{X}$ are Gorenstein C-projective modules.

Dually, Gorenstein C-injective modules are defined.

An R-module L is said to be Gorenstein C-flat, if there exists an exact sequence of C-flat modules

$$\mathbf{X} : \cdots \to C \otimes_R F_1 \to C \otimes_R F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots$$

with $L \cong \operatorname{Im}(C \otimes_R F_0 \to C \otimes_R F^0)$ such that the complex $\operatorname{Hom}_R(C, I) \otimes_R \mathbf{X}$ is exact for each injective R-module I.

We use the symbol $\mathcal{GP}_C(R)$ (resp., $\mathcal{GI}_C(R)$, $\mathcal{GF}_C(R)$) to stand for the class of Gorenstein C-projective (resp., Gorenstein C-injective, Gorenstein C-flat) R-modules. It is obvious that $\mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R), \mathcal{I}_C(R) \subseteq \mathcal{GI}_C(R), \mathcal{F}_C(R) \subseteq \mathcal{GF}_C(R)$.

3 Stability of Gorenstein C-Projective (resp., C-Injective, C-Flat) Modules

We begin with the following:

Lemma 3.1 If $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ is a collection of complete $\mathcal{P}_C$-resolution, then $\amalg_\Lambda \mathbf{X}_\lambda$ is a complete $\mathcal{P}_C$-resolution. Thus, the class of Gorenstein C-projective R-modules is closed under direct sums.

Proof For each projective R-module Q, there is an isomorphism

$$\operatorname{Hom}_R(\amalg_\Lambda \mathbf{X}_\lambda, C \otimes_R Q) \cong \Pi_\Lambda \operatorname{Hom}_R(\mathbf{X}_\lambda, C \otimes_R Q).$$

So if the complex $\operatorname{Hom}_R(\mathbf{X}_\lambda, C \otimes_R Q)$ is exact for every λ, then the complex $\operatorname{Hom}_R(\amalg_\Lambda \mathbf{X}_\lambda, C \otimes_R Q)$ is also exact. It follows that a direct sum of Gorenstein C-projective R-modules is Gorenstein C-projective.

Lemma 3.2 Let $0 \to M' \to M \to M'' \to 0$ be the exact sequence of R-modules and M'' be a Gorenstein C-projective R-module. Then M' is Gorenstein C-projective if and only if M is Gorenstein C-projective. Namely, the class of Gorenstein C-projective modules is closed under extensions and direct summands.

Proof Firstly, assume that M' and M'' are Gorenstein C-projective modules with complete $\mathcal{P}_C$-resolution $\mathbf{X}'$ and $\mathbf{X}''$, respectively. By Horseshoe lemma [20, Lemma 6.20] and [6, Lemma 1.13], we can construct a complex

$$\mathbf{X} : \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots,$$

where each P_i and P^i is projective, and a degreewise split exact sequence of complexes

$$\mathbf{X} : 0 \to X' \to X \to X'' \to 0$$

such that $M \cong \operatorname{Im}(C \otimes_R P_0 \to C \otimes_R P^0)$. To show that M is Gorenstein C-projective, it suffices to show complex $\operatorname{Hom}_R(\mathbf{X}, C \otimes_R Q)$ is exact for every projective R-module Q. The sequence

$$\mathbf{X} : 0 \to \operatorname{Hom}_R(X'', C \otimes_R Q) \to \operatorname{Hom}_R(X, C \otimes_R Q) \to \operatorname{Hom}_R(X', C \otimes_R Q) \to 0$$

is an exact sequence of complex. Since the outer two complexes are exact, the associated long exact sequence in homology shows that the middle one is also exact.

Next, assume that M and M'' are Gorenstein C-projective modules with complete $\mathcal{P}_C$-resolution $\mathbf{X}$ and $\mathbf{X}''$, respectively. Comparison lemma provides a morphism of chain complexes $\varphi : \mathbf{X} \to \mathbf{X}''$ induced by $\pi : M \to M''$ on the degree 0. By adding complexes of the form $0 \to C \otimes_R P_i'' \to C \otimes_R P_i'' \to 0$ and $0 \to C \otimes_R (P^i)'' \to C \otimes_R (P^i)'' \to 0$ to $\mathbf{X}$, we assume φ is surjective. Since $\mathcal{P}_C(R)$ is closed under kernels of epimorphisms, thus the complex $\mathbf{X}' = \mathrm{Ker}(\varphi)$ has the form
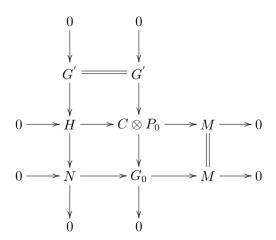
$$\mathbf{X}' : \cdots \to C \otimes_R P_1' \to C \otimes_R P_0' \to C \otimes_R (P^0)' \to C \otimes_R (P^1)' \to \cdots$$

with P_i' and $(P^i)'$ projective. The exact sequence $\mathbf{X} : 0 \to X' \to X \to X'' \to 0$ is degreewise split by [6, Lemma 1.13], so an argument similar to that of the previous paragraph implies that $\mathbf{X}'$ is a complete $\mathcal{P}_C$-resolution and M' is Gorenstein C-projective. Note that $\mathcal{P}_C(R) \subseteq \mathcal{GP}_C(R)$, it follows from Eilenberg's Swindle [5, 1.4] that $\mathcal{GP}_C(R)$ is closed under direct summands, as desired.

Next is a key lemma, which plays an important part in solving Question 1 from the introduction.
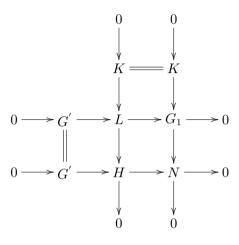
Lemma 3.3 Let M be an R-module. Then M has a $\mathcal{P}_C(R)$-resolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact if and only if M has a $\mathcal{GP}_C(R)$-resolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact.

Proof It is enough to show the "if" part. Let M has a $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact $\mathcal{GP}_C(R)$-resolution $\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ with $G_i \in \mathcal{GP}_C(R)$, $i \geq 0$. Set $N = \mathrm{Im}(G_1 \to G_0)$, then $0 \to N \to G_0 \to M \to 0$ be an exact sequence which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact, where $G_0 \in \mathcal{GP}_C(R)$ and N has a $\mathcal{GP}_C(R)$-resolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Then we have the following pullback diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
G' & = = = & G' \\
\downarrow & & \downarrow \\
0 \longrightarrow H \longrightarrow & C \otimes P_0 & \longrightarrow M \longrightarrow 0 \\
\downarrow & & \downarrow \quad \| \\
0 \longrightarrow N \longrightarrow & G_0 & \longrightarrow M \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

with $P_0 \in \mathcal{P}(R)$, $G' \in \mathcal{GP}_C(R)$ and all rows and columns $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Note that there exists an exact sequence $0 \to K \to G_1 \to N \to 0$, which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact, where $G_1 \in \mathcal{GP}_C(R)$ and K has a $\mathcal{GP}_C(R)$-resolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact.

Consider the following pullback diagram

$$
\begin{array}{ccc}
& 0 & & 0 \\
& \downarrow & & \downarrow \\
& K & = & K \\
& \downarrow & & \downarrow \\
0 \longrightarrow G' \longrightarrow L \longrightarrow G_1 \longrightarrow 0 \\
\| & & \downarrow & & \downarrow \\
0 \longrightarrow G' \longrightarrow H \longrightarrow N \longrightarrow 0 \\
& \downarrow & & \downarrow \\
& 0 & & 0
\end{array}
$$

The exactness of the middle row implies $L \in \mathcal{GP}_C(R)$ by Lemma 3.2. Since the short exact sequence

$$0 \to K \to L \to H \to 0$$

is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact by Snake lemma, and K has a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact $\mathcal{GP}_C(R)$-resolution, thus H has a $\mathcal{GP}_C(R)$-resolution which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact. Note that

$$0 \to H \to C \otimes P_0 \to M \to 0$$

is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, H and M have the same properties, so by repeating the preceding process, we have that M has a $\mathcal{P}_C(R)$-resolution which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, as required.
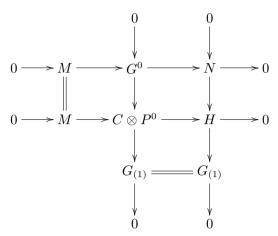
Dually, we have the following lemma.

Lemma 3.4   Let M be an R-module. Then M has a $\mathcal{P}_C(R)$-coresolution which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact if and only if M has a $\mathcal{GP}_C(R)$-coresolution which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact.

Proof   It is enough to show the "if" part. Let M has a $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact $\mathcal{GP}_C(R)$-coresolution

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \cdots$$

with $G^i \in \mathcal{GP}_C(R)$, $i \geq 0$. Set $N = \operatorname{Im}(G^0 \to G^1)$, then $0 \to M \to G^0 \to N \to 0$ be an exact sequence which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact, where $G^0 \in \mathcal{GP}_C(R)$ and N has a $\mathcal{GP}_C(R)$-coresolution which is $\operatorname{Hom}_R(-, \mathcal{P}_C(R))$-exact. Then we have the following pushout
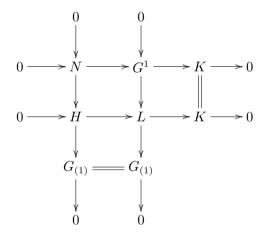
diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & N & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C \otimes P^0 & \longrightarrow & H & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & G_{(1)} & = \!\!= & G_{(1)} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

with $P^0 \in \mathcal{P}(R)$, $G_{(1)} \in \mathcal{GP}_C(R)$ and all rows and columns $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Note that there exists an exact sequence $0 \to N \to G^1 \to K \to 0$, which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact, where $G^1 \in \mathcal{GP}_C(R)$ and K has a $\mathcal{GP}_C(R)$-coresolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Consider the following pushout diagram.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & G^1 & \longrightarrow & K & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & G_{(1)} & = \!\!= & G_{(1)} & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

The exactness of the middle column implies $L \in \mathcal{GP}_C(R)$. Since the short exact sequence $0 \to H \to L \to K \to 0$ is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact by Snake lemma, and K has a $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact $\mathcal{GP}_C(R)$-coresolution, thus H has a $\mathcal{GP}_C(R)$-coresolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact. Note that $0 \to M \to C \otimes P^0 \to H \to 0$ is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact, H and M have the same properties, so by repeating the preceding process, we have that M has a $\mathcal{P}_C(R)$-coresolution which is $\mathrm{Hom}_R(-, \mathcal{P}_C(R))$-exact.

Let $\mathcal{G}^2\mathcal{P}_C(R)$ be the class of R-modules M for which there exists an exact sequence of Gorenstein C-projective R-modules

$$\mathbf{G} : \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_R(\mathbf{G}, \mathrm{C} \otimes_R P)$ is exact for each $P \in \mathcal{P}(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)$. It is obvious that $\mathcal{GP}_C(R) \subseteq \mathcal{G}^2\mathcal{P}_C(R)$, as a consequence of Lemmas 3.3 and 3.4, we have the following result.

Theorem 3.5 $\mathcal{GP}_C(R) = \mathcal{G}^2\mathcal{P}_C(R)$.

Let $\mathcal{G}(\mathcal{GP}_C(R)) = \{M \in R\text{-Mod} \mid \text{there exists a } \mathrm{Hom}_R(-, H)\text{-exact exact sequence}$

$$\mathbf{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in R-Mod with all G_i and G^i in $\mathcal{GP}_C(R)$, $H \in \mathcal{GP}_C(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)\}$. Then $\mathcal{GP}_C(R) \subseteq \mathcal{G}(\mathcal{GP}_C(R)) \subseteq \mathcal{G}^2\mathcal{P}_C(R)$, by Theorem 3.5, we have $\mathcal{GP}_C(R) = \mathcal{G}(\mathcal{GP}_C(R)) = \mathcal{G}^2\mathcal{P}_C(R)$.

The stability of Gorenstein C-injective R-modules can be given dually.

Lemma 3.6   Let $0 \to M' \to M \to M'' \to 0$ be the exact sequence of R-modules and M'' be a Gorenstein C-injective R-module. Then M' is Gorenstein C-injective if and only if M is Gorenstein C-injective.

Lemma 3.7   Let M be an R-module. Then M has a $\mathcal{I}_C(R)$-resolution which is $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact if and only if M has a $\mathcal{GI}_C(R)$-resolution which is $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact.

Lemma 3.8   Let M be an R-module. Then M has a $\mathcal{I}_C(R)$-coresolution which is $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact if and only if M has a $\mathcal{GI}_C(R)$-coresolution which is $\mathrm{Hom}_R(\mathcal{I}_C(R), -)$-exact.

Let $\mathcal{G}^2\mathcal{I}_C(R)$ be the class of R-modules M for which there exists an exact sequence of Gorenstein C-injective R-modules

$$\mathbf{G} : \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_R(\mathrm{Hom}_R(C, E), \mathbf{G})$ is exact for each $E \in \mathcal{I}(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)$. It is obvious that $\mathcal{GI}_C(R) \subseteq \mathcal{G}^2\mathcal{I}_C(R)$, as a consequence of Lemmas 3.7 and 3.8, we have the following result.

Theorem 3.9 $\mathcal{GI}_C(R) = \mathcal{G}^2\mathcal{I}_C(R)$.

Let $\mathcal{G}(\mathcal{GI}_C(R)) = \{M \in R\text{-Mod} \mid \text{there exists a } \mathrm{Hom}_R(H, -)\text{-exact exact sequence}$

$$\mathbf{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in R-Mod with all G_i and G^i in $\mathcal{GI}_C(R)$, $H \in \mathcal{GI}_C(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)\}$. Then $\mathcal{GI}_C(R) \subseteq \mathcal{G}(\mathcal{GI}_C(R)) \subseteq \mathcal{G}^2\mathcal{I}_C(R)$, by Theorem 3.9, we have $\mathcal{GI}_C(R) = \mathcal{G}(\mathcal{GI}_C(R)) = \mathcal{G}^2\mathcal{I}_C(R)$.
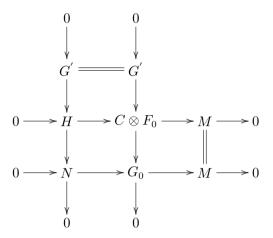
Theorem 3.5 and Theorem 3.9 give an affirmative answer to Question 1.

Finally, we discuss the stability of Gorenstein C-flat modules.

Lemma 3.10   Let M be an R-module. If $\mathcal{GF}_C(R)$ is closed under extensions, then M has an $\mathcal{F}_C(R)$-resolution which is $\mathcal{I}_C(R) \otimes_R$-exact if and only if M has a $\mathcal{GF}_C(R)$-resolution which is $\mathcal{I}_C(R) \otimes_R$-exact.

Proof   It is enough to show the "if" part. Let $0 \to N \to G_0 \to M \to 0$ be an $\mathcal{I}_C(R) \otimes_R$-exact exact sequence with $G_0 \in \mathcal{GF}_C(R)$ and N having a $\mathcal{GF}_C(R)$-resolution,

which is $\mathcal{I}_C(R)\otimes_R$-exact. Then we have the following pullback diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G^{'} & =\!\!=\!\!= & G^{'} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H & \longrightarrow & C\otimes F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with $F_0 \in \mathcal{F}(R)$, $G^{'} \in \mathcal{GF}_C(R)$. Since the bottom row is $\mathcal{I}_C(R)\otimes_R$-exact, so is the middle row. Note that there is an $\mathcal{I}_C(R)\otimes_R$-exact exact sequence $0 \to K \to G_1 \to N \to 0$ with $G_1 \in \mathcal{GF}_C(R)$ and K having a $\mathcal{GF}_C(R)$-resolution, which is $\mathcal{I}_C(R)\otimes_R$-exact. Consider the following pullback diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & =\!\!=\!\!= & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G^{'} & \longrightarrow & L & \longrightarrow & G_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^{'} & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathcal{GF}_C(R)$ is closed under extensions, $L \in \mathcal{GF}_C(R)$. Thus, H has a $\mathcal{GF}_C(R)$-resolution, which is $\mathcal{I}_C(R)\otimes_R$-exact. Note that $0 \to H \to C\otimes F_0 \to M \to 0$ is $\mathcal{I}_C(R)\otimes_R$-exact. By repeating the the preceding process, we have that M has an $\mathcal{F}_C(R)$-resolution which is $\mathcal{I}_C(R)\otimes_R$-exact.

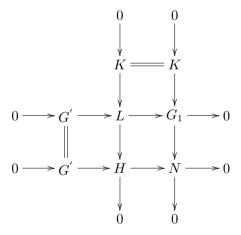Dually, we can prove the following lemma.

Lemma 3.11   Let M be an R-module. If $\mathcal{GF}_C(R)$ is closed under extensions, then M has an $\mathcal{F}_C(R)$-coresolution which is $\mathcal{I}_C(R)\otimes_R$-exact if and only if M has a $\mathcal{GF}_C(R)$-coresolution which is $\mathcal{I}_C(R)\otimes_R$-exact.

Let $\mathcal{G}^2\mathcal{F}_C(R)$ be the class of R-modules M for which there exists an exact sequence of Gorenstein C-flat R-modules

$$\mathbf{G} : \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_R(C, E) \otimes_R \mathbf{G}$ is exact for each $E \in \mathcal{I}(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)$. It is obvious that $\mathcal{GF}_C(R) \subseteq \mathcal{G}^2\mathcal{F}_C(R)$, as a consequence of Lemmas 3.10 and 3.11, we have the following result.

Theorem 3.12 If $\mathcal{GF}_C(R)$ is closed under extensions, then $\mathcal{GF}_C(R) = \mathcal{G}^2\mathcal{F}_C(R)$.

Let $\mathcal{G}(\mathcal{GF}_C(R)) = \{M \in R\text{-Mod} \mid \text{there exists a } H \otimes_R\text{-exact exact sequence}$

$$\mathbf{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in R-Mod with all G_i and G^i in $\mathcal{GF}_C(R)$, $H \in \mathcal{GI}_C(R)$ and $M \cong \mathrm{Im}(G_0 \to G^0)\}$. It is obvious that $\mathcal{GF}_C(R) \subseteq \mathcal{G}(\mathcal{GF}_C(R)) \subseteq \mathcal{G}^2\mathcal{F}_C(R)$, by Theorem 3.12, we have

$$\mathcal{GF}_C(R) = \mathcal{G}(\mathcal{GF}_C(R)) = \mathcal{G}^2\mathcal{F}_C(R).$$

Theorem 3.12 gives a partial positive answer to Question 2.

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关于半对偶化模的Gorenstein模的稳定性

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摘要: 本文研究了相对于半对偶化模C的Gorenstein模(即Gorenstein C-投射模, Gorenstein C-内射模和Gorenstein C-平坦模)的稳定性的问题. 利用同调的方法, 获得了Gorenstein C-投射(C-内射, C-平坦)模具有很好的稳定性的结果, 推广了Gorenstein投射(内射, 平坦)模具有很好的稳定性的结果.

关键词: Gorenstein C-投射(内射, 平坦) 模; 稳定性

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