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A CLASS OF PROJECTIVELY FLAT SPHERICALLY SYMMETRIC FINSLER METRICS

CHEN Ya-li[1], SONG Wei-dong[2]

(*1.School of Environmental Science and Engineering, Anhui Normal University, Wuhu 241000, China*)

(*2.School of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, China*)

Abstract: In this paper, we investigate the construction of projectively flat Finsler metrics. By analysing the solution of the spherically symmetric projectively flat equation, we construct new examples of projectively flat Finsler metrics, and obtain the projective factor and flag curvature of spherically symmetric Finsler metrics to be projectively flat.

Keywords: projectively flat; Finsler metric; spherically symmetric; projective factor; flag curvature

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1 Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metrics on an open domain in $\mathbb{R}^m$. Hilbert's 4th problem is to characterize the distance functions on an open subset in $\mathbb{R}^m$ such that straight lines are geodesics [5]. Regular distance functions with straight geodesics are projectively flat Finsler metrics. A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^m$ is projectively flat if and only if it satisfies the following equation

$$F_{x^i y^j} y^i = F_{x^j}. \tag{1.1}$$

In Finsler geometry, the flag curvature $\mathbf{K}(P, y)$ is an analogue of the sectional curvature in Riemannian geometry. It is known that every projective Finsler metric is of scalar curvature, namely, the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(y)$ is a scalar function of tangent vectors $\mathbf{y}$. Shen discussed the classification problem on projective Finsler metrics of constant flag curvature [14]. The second author provided the projective factor of a class of projectively flat general (α, β)-metrics [12] and studied a necessary and sufficient condition for a class of Finsler metric to be projectively flat [13]. Li proved the locally projectively flat Finsler metrics with constant flag curvature $\mathbf{K}$ are totally determined by their behaviors at the

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Biography: Chen Yali (1990–), femal, born at Wuhu, Anhui, master, major in differential geometry and its applications.

origin by solving some nonlinear PDEs. The classifications when $\mathbf{K}=0$, $\mathbf{K}=-1$, $\mathbf{K}=1$ are given in an algebraic way [15].

For a Finsler metric $F=F(x,y)$ on a manifold M, the geodesics $c=c(t)$ of F in local coordinates (x^i) are characterized by

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+2G^i(x,\frac{\mathrm{d}x}{\mathrm{d}t})=0,$$

where $(x^i(t))$ are the coordinates of $c(t)$ and $G^i=G^i(x,y)$ are defined by

$$G^i=\frac{g^{il}}{4}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\},$$

where $g_{ij}=\frac{1}{2}[F^2]_{y^iy^i}$ and $(g^{ij})=(g_{ij})^{-1}$. G^i are called the spray coefficients. The Riemann curvature is a family of linear maps

$$\mathbf{R}_y=R^i{}_k\frac{\partial}{\partial x^i}\bigotimes dx^k:T_xM\to T_xM$$

defined by

$$R^i{}_k=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a tangent plane $P\subset T_pM$ and a non-zero vector $\mathbf{y}\in T_pM$, the flag curvature $\mathbf{K}(P,y)$ is defined by

$$\mathbf{K}(P,y):=\frac{g_{\mathbf{y}}(\mathbf{u},\mathbf{R_y}(\mathbf{u}))}{g_{\mathbf{y}}(\mathbf{y},\mathbf{y})g_{\mathbf{y}}(\mathbf{u},\mathbf{u})-g_{\mathbf{y}}(\mathbf{y},\mathbf{u})^2},$$

where $P=\mathrm{span}\{\mathbf{y},\mathbf{u}\}$. It is known that if F is projectively flat, the spray coefficients of F are in the form $G^i=Py^i$ where

$$P=\frac{F_{x^k}y^k}{2F},$$

then F is of scalar curvature with flag curvature

$$\mathbf{K}=\frac{P^2-P_{x^k}y^k}{F^2}.$$

On the other hand, the study of spherically symmetric Finsler metrics attracted a lot of attention. Many known Finsler metrics are spherically symmetric [1, 4, 7, 14, 15, 17]. A Finsler metric F is said to be spherically symmetric (orthogonally invariant in an alternative terminology in [6]) if F satisfies

$$F(Ax,Ay)=F(x,y) \tag{1.2}$$

for all $A\in O(m)$, equivalently, if the orthogonal group $O(m)$ acts as isometrics of F. Such metrics were first introduced by Rutz [16].

It was pointed out in [6] that a Finsler metric F on $\mathbb{B}^m(\mu)$ is a spherically symmetric if and only if there is a function $\phi:[0,\mu)\times\mathbb{R}\to\mathbb{R}$ such that

$$F(x,y)=\mid y\mid\phi(\mid x\mid,\frac{\langle x,y\rangle}{\mid y\mid}), \tag{1.3}$$

where $(x,y)\in T\mathbb{R}^m(\mu)\backslash\{0\}$. The spherically symmetric Finsler metric of form (1.3) can be rewritten as the following form [8]

$$F=\mid y\mid\phi(\frac{\mid x\mid^2}{2},\frac{\langle x,y\rangle}{\mid y\mid}).$$

Spherically symmetric Finsler metrics are the simplest and most important general (α,β)-metrics [4]. Mo, Zhou and Zhu classified the projective spherically symmetric Finsler metrics with constant flag curvature in [2, 9, 10]. A lot of spherically symmetric Finsler metrics with nice curvature properties were investigated by Mo, Huang and et al. [3, 6–11].

An important example of projectively flat Finsler metric was given by Berwald. It can be written as

$$F=\frac{(\sqrt{\mid y\mid^2-(\mid x\mid^2\mid y\mid^2-\langle x,y\rangle^2)}+\langle x,y\rangle)^2}{(1-\mid x\mid^2)^2\sqrt{\mid y\mid^2-(\mid x\mid^2\mid y\mid^2-\langle x,y\rangle^2)}}$$

on the unit ball $\subset\mathbb{R}^m$, where $y\in T_x\mathbb{B}^m\subset\mathbb{R}^m$. It could also be expressed as

$$F=\mid y\mid[g(t,s)(f_0(t)+f_2(t)s^2)+h(t)s],$$

where

$$g(t,s)=\frac{1}{\sqrt{1-2t+s^2}},\quad f_0(t)=\frac{1}{1-2t},\quad f_2(t)=\frac{2}{(1-2t)^2},$$
$$h(t)=\frac{2s}{1-2t},\quad t=\frac{\mid x\mid^2}{2},\quad s=\frac{\langle x,y\rangle}{\mid y\mid}.$$

Inspired by the Berwald metric, we try to find the solution of the projectively flat eq.(1.1) in the following forms

$$F=\mid y\mid\phi(\frac{\mid x\mid^2}{2},\frac{\langle x,y\rangle}{\mid y\mid})=\mid y\mid\phi(t,s),$$

where

$$\phi(t,s)=\sum_{i=0}^{n}\phi_i(t)s^i+(1-2t+s^2)^{-\frac{1}{r}}\sum_{j=0}^{l}f_j(t)s^j.$$

Through caculations, we have the following conclusions.

Theorem 1.1 Let $\phi(t,s)$ be a function defined by

$$\begin{aligned}\phi(t,s)=&\phi_0(t)+\phi_1(t)s+\frac{1}{2}\phi_0'(t)s^2+\sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}\\&+b(1-2t+s^2)^{-\frac{1}{2}}[f_0(t)+(\frac{f_0(t)}{1-2t}+\frac{1}{2}f_0'(t))s^2]\end{aligned}$$

and $f_0(t)$ is a differentiable function which satisfies

$$f_0(t)=C_1+\frac{C_2}{t-\frac{1}{2}},$$

where b, C_1, C_2 are constants and ϕ_1 is an any continuous function, ϕ_0 is a polynomial function of N degree where $N \leqslant n$, $\phi_0^{(j)}$ denotes the j-order derivative for $\phi_0(t)$, $\phi(t,s)$ needs to satisfy $\phi - s\phi_s > 0$, when $m = 2$. Moreover, the additional equality holds

$$\phi - s\phi_s + (t - s^2)\phi_{ss} > 0,$$

when $m \geq 3$. Then the following spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$

$$F = \mid y \mid \phi(\frac{\mid x \mid^2}{2}, \frac{\langle x, y\rangle}{\mid y \mid})$$

is projectively flat.

Remark 1 Let us take a look at a special case $b = 1$, $C_1 = 0$, $C_2 = -\frac{1}{2}$, setting $\phi_0(t) = 0$, $\phi_1(t) = \frac{2}{1-2t}$, we obtain the Berwald metric.

Theorem 1.2 Let $\phi(t,s)$ be a function defined by

$$\begin{aligned}\phi(t,s) =& \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}\\ &+ b\frac{1}{1-2t+s^2}(f_0(t) - \frac{3}{5}f_0'(t)s^2)\end{aligned}$$

and $f_0(t)$ is a differentiable function which satisfies $f_0(t) = C_1(-1+2t)$, where b, C_1 are constants and ϕ_1 is an any continuous function, ϕ_0 is a polynomial function of N degree where $N \leqslant n$, $\phi_0^{(j)}$ denotes the j-order derivative for $\phi_0(t)$, $\phi(t,s)$ needs to satisfy $\phi - s\phi_s > 0$, when $m = 2$. Moreover, the additional equality holds

$$\phi - s\phi_s + (t - s^2)\phi_{ss} > 0,$$

when $m \geq 3$. Then the following spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$

$$F = \mid y \mid \phi(\frac{\mid x \mid^2}{2}, \frac{\langle x, y\rangle}{\mid y \mid})$$

is projectively flat.

Theorem 1.3 Let $\phi(t,s)$ be a function defined by

$$\begin{aligned}\phi(t,s) =& \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j} + b\frac{1}{1-2t+s^2}\\ &[f_0(t) + (\frac{2f_0(t)}{1-2t} + \frac{1}{2}f_0'(t))s^2 + (\frac{2}{3}\frac{1}{(1-2t)^2}f_0(t) + \frac{1}{3}\frac{1}{1-2t}f_0'(t) - \frac{1}{2}f_0''(t))s^4]\end{aligned}$$

and $f_0(t)$ is a differentiable function which satisfies $f_0(t) = C_1(t - \frac{1}{2}) + C_2(t - \frac{1}{2})^2$, where b, C_1, C_2 are constants and ϕ_1 is an any continuous function, ϕ_0 is a polynomial function of N degree where $N \leqslant n$, $\phi_0^{(j)}$ denotes the j-order derivative for $\phi_0(t)$, $\phi(t,s)$ needs to satisfy $\phi - s\phi_s > 0$, when $m = 2$. Moreover, the additional equality holds

$$\phi - s\phi_s + (t - s^2)\phi_{ss} > 0,$$

when $m \geq 3$. Then the following spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$

$$F =| y | \phi(\frac{| x |^2}{2}, \frac{\langle x, y\rangle}{| y |})$$

is projectively flat.

Theorem 1.4 Let $F =| y | \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$. If F is projectively flat, its projective factor P is given by

$$P =| y | \frac{\phi_s + s\phi_t}{2\phi}$$

and its flag curvature K is given by

$$\mathbf{K} = \frac{3}{4}\frac{(\phi_s + s\phi_t)^2}{\phi^4} - \frac{\phi_t}{\phi^3} - \frac{\phi_{st} + s\phi_{tt}}{2\phi^3}s.$$

2 The Solutions of the Hamel Equation

In this section, we will construct a lot of projectively flat Finsler metrics which contains the Berwald metric. From [8], we know that

Lemma 2.1 $F =| y | \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ is a solution of (1.1) if and only if ϕ satisfies

$$s\phi_{ts} + \phi_{ss} - \phi_t = 0, \tag{2.1}$$

where $t = \frac{|x|^2}{2}$ and $s = \frac{\langle x,y\rangle}{|y|}$.

Consider the spherically symmetric Finsler metric $F =| y | \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ on $\mathbb{B}^m(\mu)$, where $\phi = \phi(t, s)$ is given by $\phi(t, s) = \sum\limits_{j=0}^{l} \phi_j(t)s^j$. By a direct caculation, we get

$$\phi_t(t, s) = \sum_{j=0}^{l} \phi_j'(t)s^j, \tag{2.2}$$

$$\phi_{ts}(t, s) = \sum_{j=0}^{l} j\phi_j'(t)s^{j-1}, \tag{2.3}$$

$$\phi_{ss}(t, s) = \sum_{j=0}^{l} j(j-1)\phi_j(t)s^{j-2}. \tag{2.4}$$

Plugging (2.2), (2.3), (2.4) into (2.1), the following equation is deduced,

$$\sum_{j=0}^{l}(j-1)\phi_j'(t)s^j + \sum_{j=0}^{l-2}(j+2)(j+1)\phi_{j+2}s^j = 0. \tag{2.5}$$

It is equivalent to

$$\sum_{j=0}^{l-2}[(j-1)\phi_j'(t) + (j+2)(j+1)\phi_{j+2}(t)]s^j + \sum_{j=l-1}^{l}(j-1)\phi_j'(t)s^j = 0. \tag{2.6}$$

By (2.6), $F =\mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ is projectively flat if and only if

$$\begin{aligned}&(j-1)\phi_j'(t)+(j+2)(j+1)\phi_{j+2}(t)=0, \qquad && j=0,1,2\cdots l-2,\\ &(j-1)\phi_j'(t)=0, && j=l-1,l.\end{aligned} \tag{2.7}$$

When $j = 0$, from the first equation of (2.7), we get

$$\phi_0'(t)=2\phi_2(t). \tag{2.8}$$

Similarity, taking $j = 1$ and $j = 2$, we obtain

$$\phi_3(t)=0, \quad \phi_2'(t)+12\phi_4(t)=0. \tag{2.9}$$

If $k = j + 2$, the first equation of (2.7) is equivalent to

$$k(k-1)\phi_k(t)+(k-3)\phi_{k-2}'(t)=0. \tag{2.10}$$

It is easy to see the recurrence fomula on $\phi_k(t)$ and $\phi_k'(t)$,

$$\phi_k(t)=-\frac{k-3}{k(k-1)}\phi_{k-2}'(t). \tag{2.11}$$

If $k =$ odd, $k \geq 3$, then by (2.11),

$$\phi_k(t)=(-1)^{\frac{k-3}{2}}\frac{(k-3)(k-5)\cdots 2}{k(k-1)\cdots 4}\phi_3^{(\frac{k-3}{2})}(t)=0. \tag{2.12}$$

If $k =$ even, $k \geq 4$, we have

$$\phi_k(t)=(-1)^{\frac{k-4}{2}}\frac{24(k-3)!!}{k!}\phi_4^{(\frac{k-4}{2})}(t)=(-1)^{\frac{k-2}{2}}\frac{(k-3)!!}{k!}\phi_0^{(\frac{k}{2})}(t). \tag{2.13}$$

Case 1 $k =$ odd ≥ 5, setting $l = 2n + 1$, by the second equation of (2.7),

$$\phi_{2n+1}(t)=0, \quad \phi_{2n}(t)=\text{const.}, \tag{2.14}$$

then it follows from (2.1), (2.12), (2.13), (2.14),

$$\begin{aligned}\phi(t,s)&=\phi_0(t)+\phi_1(t)s+\phi_2(t)s^2+\cdots+\phi_{2n-1}(t)s^{2n-1}+\phi_{2n}(t)s^{2n}+\phi_{2n+1}(t)s^{2n+1}\\ &=\phi_0(t)+\phi_1(t)s+\frac{1}{2}\phi_0'(t)s^2+\sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}.\end{aligned} \tag{2.15}$$

Case 2 $k =$ even ≥ 4, setting $l = 2n + 2$, by the second equation of (2.7),

$$\phi_{2n+2}(t)=\text{const.}, \quad \phi_{2n+1}(t)=0, \tag{2.16}$$

then it follows from (2.1), (2.12), (2.13), (2.16),

$$\phi(t,s)=\phi_0(t)+\phi_1(t)s+\frac{1}{2}\phi_0'(t)s^2+\sum_{j=1}^{n}(-1)^{j}\frac{(2j-1)!!}{(2j+2)!}\phi_0^{(j+1)}(t)s^{2j+2}. \tag{2.17}$$

The case $l \in \{1,2,3\}$ is similar. Through the above analysis, we obtain the following.

Proposition 2.1 $F =| y | \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ in the form $F =| y | \sum\limits_{j=0}^{l} \phi_j(t)s^j$ is a solution of the projectively flat eq.(2.1) if and only if $\phi(t,s)$ satisfies

$$\begin{aligned}\phi(t,s) &= \phi_0(t) + \phi_1(t)s + \phi_2(t)s^2 + \cdots + \phi_l(t)s^l \\ &= \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}\end{aligned}$$

and $\phi_0^{(n)} = \text{const.}$.

Consider the solution of (2.1) where $\phi = \phi(t,s)$ is given by

$$\phi(t,s) = (1-2t+s^2)^{-\frac{1}{r}}\sum_{j=0}^{l} f_j(t)s^j. \tag{2.18}$$

Suppose that $g = (1-2t+s^2)^{-\frac{1}{r}}$, thus

$$g_t = \frac{2}{r}g^{r+1}, \quad g_s = -\frac{2}{r}g^{r+1}s. \tag{2.19}$$

Differentiating (2.18), by using (2.19), we get

$$\phi_t(t,s) = \frac{2}{r}g^{r+1}(\sum_{j=0}^{l} f_j(t)s^j) + g(\sum_{j=0}^{l} f_j'(t)s^j), \tag{2.20}$$

$$\phi_s(t,s) = -\frac{2}{r}g^{r+1}s(\sum_{j=0}^{l} f_j(t)s^j) + g(\sum_{j=0}^{l} jf_j(t)s^{j-1}), \tag{2.21}$$

$$\begin{aligned}\phi_{ts}(t,s) &= -\frac{2}{r}g^{r+1}(\sum_{j=0}^{l} f_j'(t)s^{j+1}) + g(\sum_{j=0}^{l} jf_j'(t)s^{j-1}) \\ &\quad -\frac{4}{r^2}(r+1)g^{2r+1}(\sum_{j=0}^{l} f_j(t)s^{j+1}) + \frac{2}{r}g^{r+1}(\sum_{j=0}^{l} jf_j(t)s^{j-1}),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\phi_{ss}(t,s) &= \frac{4}{r^2}(r+1)g^{2r+1}(\sum_{j=0}^{l} f_j(t)s^{j+2}) - \frac{2}{r}g^{r+1}[\sum_{j=0}^{l}(j+1)f_j(t)s^j] \\ &\quad -\frac{2}{r}g^{r+1}(\sum_{j=0}^{l} jf_j(t)s^j) + g[\sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2}].\end{aligned} \tag{2.23}$$

Plugging (2.20), (2.22), (2.23) into (2.1), we get the following

$$-\frac{2}{r}g^r(\sum_{j=0}^{l} f_j'(t)s^{j+2}) - \frac{2}{r}g^r[\sum_{j=0}^{l}(j+2)f_j(t)s^j] + \sum_{j=0}^{l}(j-1)f_j'(t)s^j + \sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2} = 0. \tag{2.24}$$

Multiplying g^{-r} on the both sides of (2.24), then

$$
\begin{aligned}
0=&-\frac{2}{r}[\sum_{j=0}^{l}(j+2)f_j(t)s^j]-\frac{2}{r}(\sum_{j=0}^{l}f_j'(t)s^{j+2})+g^{-r}[\sum_{j=0}^{l}(j-1)f_j'(t)s^j]\\
&+g^{-r}[\sum_{j=0}^{l}j(j-1)f_j(t)s^{j-2}]\\
=&-\frac{2}{r}[\sum_{j=0}^{l}(j+2)f_j(t)s^j]+\sum_{j=0}^{l}(j-1-\frac{2}{r})f_j'(t)s^{j+2}+\sum_{j=0}^{l}(j-1)(1-2t)f_j'(t)s^j\\
&+\sum_{j=0}^{l}j(j-1)(1-2t)f_j(t)s^{j-2}+\sum_{j=0}^{l}j(j-1)f_j(t)s^j\\
=&\sum_{j=0}^{l}[j^2-(1+\frac{2}{r})j-\frac{4}{r}]f_j(t)s^j+\sum_{j=2}^{l+2}(j-3-\frac{2}{r})f_{j-2}'(t)s^j+\sum_{j=0}^{l}(j-1)(1-2t)f_j'(t)s^j\\
&+\sum_{j=0}^{l-2}(j+2)(j+1)(1-2t)f_{j+2}(t)s^j.
\end{aligned}
\tag{2.25}
$$

From (2.25), we obtain the following equations

$$
\begin{aligned}
&[j^2-(1+\frac{2}{r})j-\frac{4}{r}]f_j(t)+(j-3-\frac{2}{r})f_{j-2}'(t)+(j-1)(1-2t)f_j'(t)\\
&+(j+2)(j+1)(1-2t)f_{j+2}(t)=0,\quad j=2,\cdots,l-2,
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
&(j^2-j-\frac{2j}{r}-\frac{4}{r})f_j(t)+(j-1)(1-2t)f_j'(t)+(j+2)(j+1)(1-2t)f_{j+2}(t)=0,\\
&j=0,1,
\end{aligned}
\tag{2.27}
$$

$$
\begin{aligned}
&(j^2-j-\frac{2j}{r}-\frac{4}{r})f_j(t)+(j-3-\frac{2}{r})f_{j-2}'(t)+(j-1)(1-2t)f_j'(t)=0,\\
&j=l-1,l,
\end{aligned}
\tag{2.28}
$$

$$
(j-3-\frac{2}{r})f_{j-2}'(t)=0,\quad j=l+1,l+2.
\tag{2.29}
$$

Let us take a look at a special case $l=4$, $f_1(t)=f_3(t)=0$, then

$$
-\frac{4}{r}f_0(t)-(1-2t)f_0'(t)+2(1-2t)f_2(t)=0,
\tag{2.30}
$$

$$
(12-\frac{12}{r})f_4(t)+(1-\frac{2}{r})f_2'(t)+3(1-2t)f_4'(t)=0,
\tag{2.31}
$$

$$
(2-\frac{8}{r})f_2(t)-(1+\frac{2}{r})f_0'(t)+(1-2t)f_2'(t)+12(1-2t)f_4(t)=0,
\tag{2.32}
$$

$$
f_4'(t)=0.
\tag{2.33}
$$

Case 1 $r=2$. In this case, by (2.30)–(2.33), we can get the following equations

$$
\begin{cases}
-2f_2(t)-2f_0'(t)+(1-2t)f_2'(t)=0,\\
-2f_0(t)+(2t-1)f_0'(t)+2(1-2t)f_2(t)=0,\\
6f_4(t)+3(1-2t)f_4'(t)=0,\\
f_4'(t)=0.
\end{cases}
\tag{2.34}
$$

Substituting the fourth equation of (2.34) into the third equation of it, we have $f_4(t) = 0$. From (2.30), we obtain

$$f_2(t) = \frac{f_0(t)}{1-2t} + \frac{1}{2}f_0'(t). \tag{2.35}$$

Differentiating (2.35), we get

$$f_2'(t) = 2\frac{f_0(t)}{(1-2t)^2} + \frac{f_0'(t)}{1-2t} + \frac{1}{2}f_0''(t). \tag{2.36}$$

Substituting (2.35), (2.36) into the first equation of (2.34), we obtain that $f_0(t)$ satisfies

$$-2f_0'(t) + \frac{1}{2}(1-2t)f_0''(t) = 0. \tag{2.37}$$

Solving (2.37), we have

$$f_0(t) = C_1 + \frac{C_2}{t-\frac{1}{2}}, \tag{2.38}$$

where C_1, C_2 are constants. Thus we have the following proposition.

Proposition 2.2 $\phi(t,s) = \sum\limits_{i=0}^{n}\phi_i(t)s^i + (1-2t+s^2)^{-\frac{1}{r}}\sum\limits_{j=0}^{l}f_j(t)s^j$ is a solution of the projectively flat eq.(2.2) if and only if

$$\begin{aligned}\phi(t,s) =&\phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}\\ &+ b(1-2t+s^2)^{-\frac{1}{2}}[f_0(t) + (\frac{f_0(t)}{1-2t} + \frac{1}{2}f_0'(t))s^2]\end{aligned}$$

and $f_0(t) = C_1 + \frac{C_2}{t-\frac{1}{2}}$, where b, C_1, C_2 are constants.

Case 2 $r \neq 2$, $f_4(t) = 0$. From (2.31), we know

$$f_2'(t) = 0. \tag{2.39}$$

Plugging (2.39) into (2.32), we obtain

$$(2-\frac{8}{r})f_2(t) = (1+\frac{2}{r})f_0'(t). \tag{2.40}$$

If $r = 4$, $f_0'(t) = 0$, from (2.30),

$$f_2(t) = \frac{2}{r}\frac{1}{1-2t}f_0(t). \tag{2.41}$$

Thus $f_0(t)$ and $f_2(t)$ can't be constants at the same time, so in this case, $r \neq 4$, together with (2.39), (2.40), (2.41), we know that $f_2(t)$ needs to satisfy the following

$$\begin{cases} f_2(t) = \frac{2}{r}\frac{1}{1-2t}f_0(t) = \frac{(1+\frac{2}{r})f_0'(t)}{2-\frac{8}{r}},\\ f_2'(t) = 0.\end{cases} \tag{2.42}$$

Through (2.42), we get that $f_0(t)$ needs to satisfy

$$\begin{cases} \frac{(1+\frac{2}{r})f_0'(t)}{2-\frac{8}{r}} = \frac{2}{r}\frac{1}{1-2t}f_0(t), \\ f_0''(t) = 0. \end{cases} \tag{2.43}$$

From the first equation of (2.43),

$$f_0(t) = C_1(-1+2t)^{\frac{4}{3r}-\frac{1}{3}}, \tag{2.44}$$

where C_1 is a constant. But the $f_0(t)$ in (2.44) doesn't satisfy the second equation of (2.43) only if $r = 1$, thus we can get the following proposition.

Proposition 2.3 $\phi(t,s) = \sum\limits_{i=0}^{n} \phi_i(t)s^i + (1-2t+s^2)^{-\frac{1}{r}} \sum\limits_{j=0}^{l} f_j(t)s^j$ is a solution of the projectively flat eq.(2.2) if and only if

$$\begin{aligned} \phi(t,s) &= \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j} \\ &\quad + b\frac{1}{1-2t+s^2}(f_0(t) - \frac{3}{5}f_0'(t)s^2) \end{aligned}$$

and $f_0(t) = C_1(-1+2t)$, where b, C_1 are constants.

Case 3 $r \neq 2$, $f_4(t) \neq 0$. In this case, from the first equation of (2.30),

$$f_2(t) = \frac{2}{r}\frac{1}{1-2t}f_0(t) + \frac{1}{2}f_0'(t). \tag{2.45}$$

Differentiating (2.45), we have

$$f_2'(t) = \frac{4}{r}\frac{1}{(1-2t)^2}f_0(t) + \frac{2}{r}\frac{1}{1-2t}f_0'(t) + \frac{1}{2}f_0''(t), \tag{2.46}$$

$$f_2''(t) = \frac{16}{r}\frac{1}{(1-2t)^3}f_0(t) + \frac{8}{r}\frac{1}{(1-2t)^2}f_0'(t) + \frac{2}{r}\frac{1}{1-2t}f_0''(t) + \frac{1}{2}f_0'''(t). \tag{2.47}$$

From (2.32), we get

$$f_4(t) = \frac{1}{6r}\frac{1}{(1-2t)^2}f_0(t)(\frac{8}{r}-4) + \frac{1}{3r}\frac{1}{1-2t}f_0'(t) - \frac{1}{2}f_0''(t). \tag{2.48}$$

Differentiating (2.48), we obtain

$$f_4'(t) = \frac{2}{3r}(\frac{8}{r}-4)\frac{f_0(t)}{(1-2t)^3} + \frac{4}{3r^2}\frac{1}{(1-2t)^2}f_0'(t) + \frac{1}{3r}\frac{1}{1-2t}f_0''(t) - \frac{1}{2}f_0'''(t). \tag{2.49}$$

Plugging (2.33) into (2.31), we have

$$(12-\frac{12}{r})f_4(t) + (1-\frac{2}{r})f_2'(t) = 0. \tag{2.50}$$

Thus from (2.50), no matter $r = 1$ or not,

$$f_2''(t) = 0. \tag{2.51}$$

Combining the fourth equation of (2.33) and (2.51), we obtain that $f_0(t)$ satisfies

$$\begin{aligned}&\frac{16}{r}\frac{1}{(1-2t)^3}f_0(t)+\frac{8}{r}\frac{1}{(1-2t)^2}f_0'(t)+\frac{2}{r}\frac{1}{1-2t}f_0''(t)+\frac{1}{2}f_0'''(t)=0,\\&\frac{2}{3r}(\frac{8}{r}-4)\frac{f_0(t)}{(1-2t)^3}+\frac{4}{3r^2}\frac{1}{(1-2t)^2}f_0'(t)+\frac{1}{3r}\frac{1}{1-2t}f_0''(t)-\frac{1}{2}f_0'''(t)=0.\end{aligned}\tag{2.52}$$

Solving the first equation of (2.52), we get

$$f_0(t)=C_1(t-\frac{1}{2})+C_2(t-\frac{1}{2})^2+C_3(t-\frac{1}{2})^{\frac{2}{r}}.\tag{2.53}$$

Solving the second equation of (2.52), we know

$$f_0(t)=C_4(t-\frac{1}{2})^2+C_5(t-\frac{1}{2})^{\frac{3r-1+\sqrt{9r^2-18r+25}}{6r}}+C_6(t-\frac{1}{2})^{\frac{3r-1-\sqrt{9r^2-18r+25}}{6r}}.\tag{2.54}$$

If $r=1$, $C_3=C_6=0$, two equations of (2.52) have the same solutions. Thus we have the following proposition.

Proposition 2.4 $\phi(t,s)=\sum\limits_{i=0}^{n}\phi_i(t)s^i+(1-2t+s^2)^{-\frac{1}{r}}\sum\limits_{j=0}^{l}f_j(t)s^j$ is a solution of the projectively flat eq.(2.2) if and only if

$$\begin{aligned}\phi(t,s)=&\phi_0(t)+\phi_1(t)s+\frac{1}{2}\phi_0'(t)s^2+\sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}+b\frac{1}{1-2t+s^2}\\&[f_0(t)+(\frac{2f_0(t)}{1-2t}+\frac{1}{2}f_0'(t))s^2+(\frac{2}{3}\frac{1}{(1-2t)^2}f_0(t)+\frac{1}{3}\frac{1}{1-2t}f_0'(t)-\frac{1}{2}f_0''(t))s^4]\end{aligned}$$

and $f_0(t)=C_1(t-\frac{1}{2})+C_2(t-\frac{1}{2})^2$, where b, C_1, C_2 are constants.

3 Proof of Theorems

$\phi(t,s)$ in Propositions 2.2, 2.3, 2.4 can't ensure that $F=|y|\phi(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})$ is a Finsler metric. In order to obtain projectively flat Finsler metric, $\phi(t,s)$ in Propositions 2.2–2.4 needs to satisfy the necessary and sufficient condition for $F=\alpha\phi(\|\beta_x\|_\alpha,\frac{\beta}{\alpha})$ to be a Finsler metric for any α and β with $\|\beta_x\|_\alpha<b_0$ given by Yu and Zhu [4]. In particular, considering $F=|y|\phi(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})=|y|\phi(t,s)$, then F is a Finsler metric if and only if the positive function ϕ satisfies

$$\phi-s\phi_s>0,\quad \phi-s\phi_s+(t-s^2)\phi_{ss}>0,\tag{3.1}$$

when $m\geq 3$ or

$$\phi-s\phi_s+(t-s^2)\phi_{ss}>0,\tag{3.2}$$

when $m=2$.

Proof of Theorem 1.1 Combine Proposition 2.2, (3.1), (3.2) and the fundamental property of the projectively flat equation (2.1).

Proof of Theorem 1.2 Combine Proposition 2.3, (3.1), (3.2) and the fundamental property of the projectively flat equation (2.1).

Proof of Theorem 1.3 Combine Proposition 2.4, (3.1), (3.2) and the fundamental property of the projectively flat equation (2.1).

Proof of Theorem 1.4 Suppose that

$$r :=\mid y \mid, \quad r^i := r_i := \frac{y^i}{\mid y \mid}, \quad x_i := x^i, \quad s^i := s_i := x_i - sr_i. \tag{3.3}$$

Direct computations yield that

$$t_{x^i} = x^i = s_i + sr_i, \quad s_{x^i} = r_i, \quad r_{y^i} = r_i, \quad s_{y^i} = \frac{s_i}{r}, \tag{3.4}$$

where we use of (3.3). By (3.3), (3.4), we get the following lemma.

Lemma 3.1 Let $f = f(r,t,s)$ be a function on a domain $\mathrm{U} \subset \mathbb{R}^3$. Then

$$f_{x^i} = r_i(f_s + sf_t) + s_i f_t, \quad f_{y^i} = r_i f_r + s_i \frac{f_s}{r}. \tag{3.5}$$

Let $F =\mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|}) = r\phi(t,s)$ be a spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$. From (3.3), (3.4), (3.5), we have the following

$$F_r = \phi, \quad F_t = r\phi_t, \quad F_s = s\phi_s. \tag{3.6}$$

Note that s_i and r_i are positively homogeneous of degree 0 and 1. Hence

$$s_i y^i = 0, \quad r_i y^i = 0 \tag{3.7}$$

and we get

$$F_{x^i} = r[\phi_t s_i + (\phi_s + s\phi_t)r_i], \quad F_{y^i} = \phi r_i + \phi_s s_i. \tag{3.8}$$

Thus from (3.7), (3.8), we have $F_0 = F_{x^i} y^i = r^2(\phi_s + s\phi_t)$,

$$P = \frac{F_0}{2F} = \frac{(\phi_s + s\phi_t)r}{2\phi}. \tag{3.9}$$

Differentiating (3.9), we know

$$P_{x^k} = \frac{r}{2}[\frac{2\phi_t r_k + (\phi_{st} + s\phi_{tt})(s_k + sr_k)}{\phi} - \frac{(\phi_s + s\phi_t)(\phi_s r_k + \phi_t s_k + \phi_t s r_k)}{\phi^2}]. \tag{3.10}$$

From (3.7), (3.10), we obtain

$$P_{x^k} y^k = \frac{r^2}{2}[\frac{2\phi_t + (\phi_{st} + s\phi_{tt})s}{\phi} - \frac{(\phi_s + s\phi_t)^2}{\phi^2}]. \tag{3.11}$$

Thus using (3.9), (3.11), we have

$$\mathbf{K} = \frac{P^2 - P_{x^k} y^k}{F^2} = \frac{3}{4}\frac{(\phi_s + s\phi_t)^2}{\phi^4} - \frac{\phi_t}{\phi^3} - \frac{\phi_{st} + s\phi_{tt}}{2\phi^3}s.$$

Theorem 1.4 can be achieved.

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一类射影平坦的球对称的芬斯勒度量

陈亚力[1], 宋卫东[2]

(1.安徽师范大学环境科学与工程学院, 安徽 芜湖 241000)

(2.安徽师范大学数学计算机科学学院, 安徽 芜湖 241000)

摘要: 本文研究了射影平坦芬斯勒度量的构造问题. 通过分析射影平坦的球对称的芬斯勒度量的方程的解, 构造了一类新的射影平坦的芬斯勒度量, 并得到了射影平坦的球对称的芬斯勒度量的射影因子和旗曲率.

关键词: 射影平坦; 芬斯勒度量; 球对称; 射影因子; 旗曲率

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