Vol. 37 (2017)
No. 4

数 学 杂 志
J. of Math. (PRC)

Hom - 弱 Hopf 代数上的 Hom-smash 积

郑乃峰
(宁波大学理学院, 浙江 宁波 315211)

摘 要: 本文研究了在 Hom-Hopf 代数上引入 Hom - 弱 Hop 代数的问题. 通过建立弱左 H - 模 Hom - 代数的方法, 构造 Hom-smash 积, 证明 Hom-smash 积是 Hom - 代数, 且给出使之成为 Hom - 弱 Hopf 代数的充分条件, 推广了由 Bohm 等人定义的弱 Hop 代数.

关键词: Hom - 弱 Hopf 代数; 弱左 H - 模 Hom - 代数; Hom-smash 积
MR(2010) **主题分类号**: 16W30; 16E10 **中图分类号**: O153.3
文献标识码: A **文章编号**: 0255-7797(2017)04-0871-10

1 引言

代数形变理论现在已是代数学的重要分支之一. 近年来作为代数另一类形变代数 -Hom - 代数的引入, 引起了许多代数学者的关注. Hom - 代数的概念是由 Makhlouf 和 Silvestrov 于 2006 年在研究拟李代数时引入的 (见文献 [1]). Hom - 代数的引入实际上是推广了结合代数的概念, 把结合代数中的结合性法则作了形变, 将其变成了线性变换 α 结合性条件, 即 $\alpha(a)(bc) = (ab)\alpha(c)$. 随着 Hom - 代数研究的深入, 一些学者在文献 [2–5] 中又陆续引入了 Hom - 余结合余代数、Hom - 双代数和 Hom-Hopf 代数等, 并给出了一些重要的性质. 在文献 [6, 7] 中, 作者定义了 Hom-ω-smash 积和 Hom-ω-smash 余积, 并分别研究了它们的拟三角结构和辫化结构.

弱 Hopf 代数是由 Bohm 和 Nill 等人定义的 (见文 [8]), 作为 Hopf 代数 (见文 [9]) 的推广, 弱 Hopf 代数与 Hopf 代数有着相似的构成, 只是用更弱的条件去代替余乘法运算的保单位性和余单位运算的保乘法性. 因此, 弱 Hopf 代数的结构远比 Hopf 代数复杂.

综合上述讨论, 在弱 Hopf 代数上引入 Hom - 代数的结合性条件成为自然的问题, 这也是写这篇文章的动机. 在 Hom - 弱 Hopf 代数和模结构的基础上, 建立弱左 H - 模 Hom - 代数的结构并通过它构造 Hom-smash 积, 证明 Hom-smash 积是 Hom - 代数, 且给出使之成为 Hom - 弱 Hopf 代数的充分条件.

2 Hom - 弱 Hopf 代数

本文的所有工作都在域 k 上进行的. 所讨论的张量积和线性映射均指域 k 上的. 文中将使用 Sweedler 关于余代数余乘法的记号, 即对于 H 中的任意元 $h, \triangle(h) = \sum h_1 \otimes h_2$. 关于 Hom - 代数和 Hom - 余代数的概念请参阅文献 [1–3].

*收稿日期: 2015-7-31 接收日期: 2015-11-25
基金项目: 国家自然科学基金资助 (60873267); 宁波自然科学基金资助 (2011A610172).
作者简介: 郑乃峰 (1968–), 男, 浙江慈溪, 副教授, 主要研究方向: Hopf 代数及量子群.

定义 2.1 [10] 如果 (H,μ,η,α) 是一个 Hom - 代数, $(H,\triangle,\varepsilon,\alpha)$ 是一个 Hom - 余代数, 且代数和余代数结构满足下列相容性

$$\triangle(xy)=\triangle(x)\triangle(y), \tag{2.1}$$
$$\varepsilon((xy)\alpha(z))=\sum\varepsilon(xy_1)\varepsilon(y_2\alpha(z)), \tag{2.2}$$
$$\varepsilon(\alpha(x)(yz))=\sum\varepsilon(\alpha(x)y_2)\varepsilon(y_1z), \tag{2.3}$$
$$(\triangle\otimes\alpha)\triangle(1)=\sum 1_1\otimes 1_2 1_{1'}\otimes\alpha(1_{2'}), \tag{2.4}$$
$$(\alpha\otimes\triangle)\triangle(1)=\sum\alpha(1_1)\otimes 1_{1'}1_2\otimes 1_{2'}, \tag{2.5}$$

则称六元组 $(H,\mu,\eta,\triangle,\varepsilon,\alpha)$ 为一个 Hom - 弱双代数, 并简记为 (H,α).

定义 2.2 [10] 设 (H,α) 是一个 Hom - 弱双代数, $S:H\to H$ 是一个线性映射, 如果满足

$$S\circ\alpha=\alpha\circ S, \tag{2.6}$$
$$\sum x_1S(x_2)=\sum\varepsilon(1_1\alpha^2(x))1_2, \tag{2.7}$$
$$\sum S(x_1)x_2=\sum 1_1\varepsilon(\alpha^2(x)1_2), \tag{2.8}$$
$$\sum(S(x_{11})x_{12})S(\alpha^2(x_2))=S(\alpha^4(x)), \tag{2.9}$$

则称 (H,α) 是一个 Hom - 弱 Hopf 代数, 并称 S 是 Hom - 弱 Hopf 代数 (H,α) 的对极映射.

注 2.3 由 (2.7) 和 (2.8) 式, 容易得到 $\sum(x_{11}S(x_{12}))\alpha^2(x_2)=\alpha^4(x)$ 和 $\sum\alpha^2(x_1)(S(x_{21})x_{22})=\alpha^4(x)$. 对于 (2.9) 式的合理性证明如下

$$\begin{aligned}
\sum(S(x_{11})x_{12})S(\alpha^2(x_2)) &= \sum(1_1\varepsilon(\alpha^2(x_1)1_2))S(\alpha^2(x_2))\\
&= \sum(\alpha(1_1)\varepsilon((\alpha(x_1)1_{1'})\alpha(1_2)))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(x_1)(1_{1'}1_2)))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(x_1)1_{1'2})\varepsilon(1_{1'1}1_2))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha(1_{1'})1_2))S(\varepsilon(\alpha^2(x_1)1_{2'1})\alpha^{-1}(\alpha^2(x_2)1_{2'2}))\\
&= \sum(\alpha(1_1)\varepsilon(\alpha(1_{1'})1_2))S(\alpha^2(x)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(1_{1'}1_2))S(\alpha^2(x)\alpha^{-1}(1_{2'}))\\
&= \sum(\alpha(1_1)\varepsilon(1_{21}))S(\alpha^2(x)\alpha^{-1}(1_{22}))\\
&= \sum\alpha(1_1)S(\alpha^2(x)1_2)\\
&= \sum(1_1S(1_2))S(\alpha^3(x))\\
&= S(\alpha^4(x)).
\end{aligned}$$

注 2.4 Hom - 弱 Hopf 代数既不满足结合律也不满足余结合律, 但当扭曲映射 $\alpha=Id$ 时, 它就是弱 Hopf 代数. 但当余单位 ε 是代数映射时, Hom - 弱 Hopf 代数就是 Hom-Hopf 代数. 相对于 (余) 结合性, Hom - 弱 Hopf 代数也有 Hom-(余) 结合性, 即 $\mu\circ(\alpha\otimes\mu)=\mu\circ(\mu\otimes\alpha)$

和 $(\alpha\otimes\triangle)\circ\triangle=(\triangle\otimes\alpha)\circ\triangle$. 因此 Hom - 弱 Hopf 代数的非 (余) 结合性的程度是由扭曲映射 α 偏离恒等映射的距离决定的. 关于 Hom - 弱 Hopf 代数的相关性质请参阅文献 [10].

命题 2.5 设 (H,α) 是一个 Hom - 弱 Hopf 代数, 且满足条件 $\varepsilon\circ S=\varepsilon$, 则有如下结论

$$S\circ\sqcap^L=\sqcap^R\circ S,\ S\circ\sqcap^R=\sqcap^L\circ S. \tag{2.10}$$

证 对任意 $x\in H$, 有

$$\begin{aligned}S\circ\sqcap^L(x) &= \sum\varepsilon(1_1\alpha^2(x))S(1_2)\\ &= \sum\varepsilon(S(1_1\alpha^2(x)))S(1_2)\\ &= \sum\varepsilon(S(\alpha^2(x))S(1_1))S(1_2)\\ &= \sum 1_1\varepsilon(\alpha^2(S(x))1_2)\\ &= \sqcap^R\circ S(x).\end{aligned}$$

同理可证 $S\circ\sqcap^R=\sqcap^L\circ S$ 成立.

上面的命题说明 $S(H^L)\subset H^R$ 和 $S(H^R)\subset H^L$. 由于 $\alpha(1)=1$, 因此有 $\sum\alpha(1_1)\otimes\alpha(1_2)=\sum 1_1\otimes 1_2$, 所以有 $\alpha(H^L)\subset H^L$ 和 $\alpha(H^R)\subset H^R$.

命题 2.6 设 (H,α) 是一个 Hom - 弱 Hopf 代数, 则有如下结论

$$\sqcap^L(x)\sqcap^R(y)=\alpha(\sqcap^R(y))\alpha^{-1}(\sqcap^L(x)). \tag{2.11}$$

证 对任意 $x,y\in H$, 有

$$\begin{aligned}\sqcap^L(x)\sqcap^R(y) &= \sum\varepsilon(1_1\alpha^2(x))1_2 1_{1'}\varepsilon(\alpha^2(y)1_{2'})\\ &= \sum\varepsilon(1_{11}\alpha^2(x))1_{12}\varepsilon(\alpha^2(y)1_2)\\ &= \sum\varepsilon(\alpha(1_1)\alpha^2(x))1_{21}\varepsilon(\alpha^2(y)\alpha^{-1}(1_{22}))\\ &= \sum\varepsilon(\alpha(1_1)\alpha^2(x))1_{1'}1_2\varepsilon(\alpha^2(y)\alpha^{-1}(1_{2'}))\\ &= \sum\alpha(1_{1'})\varepsilon(\alpha^2(y)1_{2'})\varepsilon(1_1\alpha^2(x))\alpha^{-1}(1_2)\\ &= \alpha(\sqcap^R(y))\alpha^{-1}(\sqcap^L(x)).\end{aligned}$$

3 Hom - 弱 Hopf 代数上的 smash 积

关于 Hom - 模、Hom - 余模、Hom - 模代数和 Hom - 模余代数的相关概念可参阅文献 [5]. 下面, 给出弱左 H - 模 Hom - 代数的概念.

定义 3.1 设 (H,α) 是 Hom - 弱 Hopf 代数,(A,β) 是 Hom - 代数. 如果有一个线性映射 $\rho:H\otimes A\to A$, $\rho(h\otimes a)=h\cdot a$, 使得对任意 $h,g\in H$ 和 $a,b\in A$, 有下面条件成立

$$\alpha^2(h)\cdot(ab)=\sum(h_1\cdot a)(h_2\cdot b),\quad h\cdot 1_A=\sqcap^L(h)\cdot 1_A, \tag{3.1}$$

$$(hg)\cdot\beta(a)=\alpha(h)\cdot(g\cdot a),\quad \beta(h\cdot a)=\alpha(h)\cdot\beta(a),\quad 1_H\cdot a=\beta(a), \tag{3.2}$$

则称 Hom - 代数 (A,β) 是一个弱左 H - 模 Hom - 代数. 若 $\alpha=Id$ 和 $\beta=Id$, 则弱左 H - 模 Hom - 代数是文献 [11] 中的一个弱左 H - 模代数. 若 (H,α) 是 Hom-Hopf 代数, 则弱左 H - 模 Hom - 代数是文献 [5] 中的一个左 H - 模 Hom - 代数.

本节设 (H,α) 是 Hom - 弱 Hopf 代数, 其弱对极 S 是双射,(A,β) 是弱左 H - 模 Hom - 代数, 设 $\gamma=\beta\otimes\alpha$. 为方便, 分别记 1_H 为 1,1_A 为 $\widehat{1}$.

(A,β) 在 (H,α) 上的 Hom-smash 积 $(A\sharp H,\gamma)$ 是指带有下面乘法运算的向量空间 $A\otimes_{H^L}H$, 运算规定如下

$$(a\sharp h)(b\sharp g)=\sum a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b))\sharp\alpha^{-1}(h_2)g,\quad a,b\in A,\ h,g\in H,$$

其中 $a\sharp h$ 表示向量 $a\otimes h$ 在 $A\sharp H$ 中所在的类, 即

$$A\otimes_{H^L}H=\{a\otimes h|a\cdot x\otimes h=\beta(a)\otimes\alpha^{-1}(x)\alpha^{-1}(h)\ \text{或}\ a\otimes xh=\beta^{-1}(a)\cdot\alpha(x)\otimes\alpha(h),x\in H^L\}.$$

H 通过乘法构成左 H^L 模, A 通过下面作用构成右 H - 模

$$a\cdot x=S^{-1}(\alpha^{-1}(x))\cdot a=a(x\cdot\widehat{1}),\ a\in A,x\in H^L.$$

引理 3.2 (A,β) 在 (H,α) 上的 Hom-smash 积 $(A\sharp H,\gamma)$ 是一个带有单位元 $\widehat{1}\sharp 1$ 的 Hom - 代数.

证 对任意 $a,b,c\in A$ 和 $h,g,k\in H$, 有

$$\begin{aligned}
\gamma((a\sharp h)(b\sharp g)) &= \sum\beta(a)(\alpha^{-1}(h_1)\cdot b)\sharp h_2\alpha(g)=\gamma(a\sharp h)\gamma(b\sharp g),\\
(a\sharp h)(\widehat{1}\sharp 1) &= \sum a(\alpha^{-2}(h_1)\cdot\widehat{1})\sharp h_2=\sum a(\sqcap^L(\alpha^{-2}(h_1))\cdot\widehat{1})\sharp h_2\\
&= \sum a\cdot\sqcap^L(\alpha^{-2}(h_1))\sharp h_2=\sum\beta(a)\sharp\alpha^{-1}(\sqcap^L(\alpha^{-2}(h_1)))\alpha^{-1}(h_2)\\
&= \beta(a)\sharp\alpha(h),\\
(\widehat{1}\sharp 1)(a\sharp h) &= \sum\alpha^{-1}(1_1)\cdot a\sharp\alpha^{-1}(1_2)h=\sum 1_1\cdot a\sharp 1_2h\\
&= \sum\beta^{-1}(1_1\cdot a)\cdot\alpha(1_2)\sharp\alpha(h)=\sum S^{-1}(1_2)\cdot(\alpha^{-1}(1_1)\cdot\beta^{-1}(a))\sharp\alpha(h)\\
&= \sum\alpha^{-1}(S^{-1}(1_2)1_1)\cdot a\sharp\alpha(h)=\sum\alpha^{-1}(S^{-1}(\sqcap^R(1)))\cdot a\sharp\alpha(h)\\
&= \beta(a)\sharp\alpha(h),\\
\gamma(a\sharp h)((b\sharp g)(c\sharp k)) &= \sum(\beta(a)\sharp\alpha(h))(b(\alpha^{-2}(g_1)\cdot\beta^{-1}(c))\sharp\alpha^{-1}(g_2)k)\\
&= \sum\beta(a)(\alpha^{-1}(h_1)\cdot(\beta^{-1}(b)(\alpha^{-3}(g_1)\cdot\beta^{-2}(c))))\sharp h_2(\alpha^{-1}(g_2)k)\\
&= \sum\beta(a)((\alpha^{-3}(h_{11})\cdot\beta^{-1}(b))(\alpha^{-3}(h_{12})\cdot(\alpha^{-3}(g_1)\cdot\beta^{-2}(c))))\sharp h_2(\alpha^{-1}(g_2)k)\\
&= \sum\beta(a)((\alpha^{-2}(h_1)\cdot\beta^{-1}(b))(\alpha^{-3}(\alpha^{-1}(h_{21})g_1)\cdot\beta^{-1}(c)))\sharp\alpha^{-1}(h_{22})(\alpha^{-1}(g_2)k)\\
&= \sum(a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b)))(\alpha^{-2}(\alpha^{-1}(h_{21})g_1)\cdot c)\sharp\alpha^{-1}(\alpha^{-1}(h_{22})g_2)\alpha(k)\\
&= \sum(a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b))\sharp\alpha^{-1}(h_2)g)(\beta(c)\sharp\alpha(k))\\
&= ((a\sharp h)(b\sharp g))\gamma(c\sharp k).
\end{aligned}$$

这证明 Hom-smash 积 $(A\sharp H,\gamma)$ 是一个带有单位元 $\widehat{1}\sharp 1$ 的 Hom - 代数.

注 3.3 若 $\alpha=Id$ 和 $\beta=Id$, 则 Hom-smash 积 $(A\sharp H,\gamma)$ 是文献 [12] 中的 smash 积; 若 (H,α) 是 Hom-Hopf 代数, 则 Hom-smash 积 $(A\sharp H,\gamma)$ 是文献 [6] 中的 Hom-smash 积.

引理 3.4 设 $(A\sharp H,\gamma)$ 是一个 Hom-smash 积, 则对任意 $a,b\in A$ 和 $h,g\in H$, 下面关系式成立

(1) $(a\sharp 1)(b\sharp 1)=ab\sharp 1$;

(2) $(\widehat{1}\sharp h)(\widehat{1}\sharp g)=\widehat{1}\sharp hg$;

(3) $(a\sharp 1)(\widehat{1}\sharp g)=\beta(a)\sharp\alpha(g)$;

(4) $(\widehat{1}\sharp h)(b\sharp 1)=\sum\alpha^{-1}(h_1)\cdot b\sharp h_2$.

证 对任意 $a,b\in A$ 和 $h,g\in H$, 有

$$\begin{aligned}
(a\sharp 1)(b\sharp 1) &= \sum a(\alpha^{-2}(1_1)\cdot\beta^{-1}(b))\sharp 1_2=\sum a(S^{-1}(\alpha^{-1}(S(\alpha^{-1}(1_1))))\cdot\beta^{-1}(b))\sharp 1_2\\
&= \sum a(\beta^{-1}(b)(S(\alpha^{-1}(1_1))\cdot\widehat{1}))\sharp 1_2=\sum\beta^{-1}(ab)(S(1_1)\cdot\widehat{1})\sharp 1_2\\
&= \sum\beta^{-1}(ab)\cdot S(1_1)\sharp 1_2=\sum ab\sharp\alpha^{-1}(S(1_1))\alpha^{-1}(1_2)=ab\sharp 1,\\
(\widehat{1}\sharp h)(\widehat{1}\sharp g) &= \sum\widehat{1}(\alpha^{-2}(h_1)\cdot\widehat{1})\sharp\alpha^{-1}(h_2)g=\sum\widehat{1}(\sqcap^L(\alpha^{-2}(h_1))\cdot\widehat{1})\sharp\alpha^{-1}(h_2)g\\
&= \sum\widehat{1}\cdot\sqcap^L(\alpha^{-2}(h_1))\sharp\alpha^{-1}(h_2)g=\sum\widehat{1}\sharp\alpha^{-1}(\sqcap^L(\alpha^{-2}(h_1))(\alpha^{-1}(h_2)g))\\
&= \sum\widehat{1}\sharp\alpha^{-1}(\alpha^{-1}(\sqcap^L(\alpha^{-2}(h_1))h_2)\alpha(g))=\widehat{1}\sharp hg,\\
(a\sharp 1)(\widehat{1}\sharp g) &= \sum a(\alpha^{-2}(1_1)\cdot\widehat{1})\sharp\alpha^{-1}(1_2)g=\sum a(\sqcap^L(\alpha^{-2}(1_1))\cdot\widehat{1})\sharp\alpha^{-1}(1_2)g\\
&= \sum a\cdot\sqcap^L(\alpha^{-2}(1_1))\sharp\alpha^{-1}(1_2)g=\sum\beta(a)\sharp\alpha^{-1}(\sqcap^L(\alpha^{-2}(1_1))(\alpha^{-1}(1_2)g))\\
&= \sum\beta(a)\sharp\alpha^{-1}(\alpha^{-1}(\sqcap^L(\alpha^{-2}(1_1))1_2)\alpha(g))=\beta(a)\sharp\alpha(g),\\
(\widehat{1}\sharp h)(b\sharp 1) &= \sum\widehat{1}(\alpha^{-2}(h_1)\cdot\beta^{-1}(b))\sharp\alpha^{-1}(h_2)1\\
&= \sum\alpha^{-1}(h_1)\cdot b\sharp h_2.
\end{aligned}$$

定理 3.5 设 (H,α) 是 Hom - 弱 Hopf 代数, 其弱对极为 $S,(A,\beta)$ 是 Hom - 弱双代数. 如果 (A,β) 是弱左 H - 模 Hom - 代数, 并且对于任意 $h\in H,a\in A$, 下面条件成立

$$\begin{aligned}
&\varepsilon_A(h\cdot a)=\varepsilon_H(h)\varepsilon_A(a),\ \varepsilon_H\circ S=\varepsilon_H,\\
&\triangle_A(h\cdot a)=\sum h_1\cdot a_1\otimes h_2\cdot a_2,\ \sum h_1\otimes h_2\cdot a=\sum h_2\otimes h_1\cdot a,
\end{aligned}$$

则 Hom-smash 积 $(A\sharp H,\gamma)$ 是 Hom - 弱双代数, 其中 $(A\sharp H,\gamma)$ 的余代数结构是张量积 Hom - 余代数, 其余乘和余单位定义为

$$\triangle_{A\sharp H}(a\sharp h)=\sum(a_1\sharp h_1)\otimes(a_2\sharp h_2),\varepsilon_{A\sharp H}(a\sharp h)=\varepsilon_A(a)\varepsilon_H(h).$$

若此时 (A,β) 又是 Hom - 弱 Hopf 代数, 其弱对极为 S_A, 满足性质 $H^L\cdot\widehat{1}\subseteq Z(A)$, 则 Hom-smash 积 $(A\sharp H,\gamma)$ 也是 Hom - 弱 Hopf 代数, 其弱对极为

$$S_{A\sharp H}(a\sharp h)=\sum S(\alpha^{-2}(h_2))\cdot S_A(\beta^{-1}(a))\sharp S(\alpha^{-1}(h_1)).$$

证 显然, Hom-smash 积 $(A\sharp H,\gamma)$ 是 Hom - 代数和 Hom - 余代数. 假设定理条件成立,

证明 Hom-smash 积 $(A\sharp H,\gamma)$ 满足定义 2.1 的 (2.1)–(2.5) 项. 对任意 $a,b\in A$ 和 $h,g\in H$, 有

$$\begin{aligned}
&\triangle_{A\sharp H}((a\sharp h)(b\sharp g))\\
=\ &\sum\triangle_{A\sharp H}(a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b))\sharp\alpha^{-1}(h_2)g)\\
=\ &\sum a_1(\alpha^{-2}(h_{11})\cdot\beta^{-1}(b_1))\sharp\alpha^{-1}(h_{21})g_1\otimes a_2(\alpha^{-2}(h_{12})\cdot\beta^{-1}(b_2))\sharp\alpha^{-1}(h_{22})g_2\\
=\ &\sum a_1(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1))\sharp\alpha^{-2}(h_{212})g_1\otimes a_2(\alpha^{-3}(h_{211})\cdot\beta^{-1}(b_2))\sharp\alpha^{-1}(h_{22})g_2\\
=\ &\sum a_1(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1))\sharp\alpha^{-2}(h_{211})g_1\otimes a_2(\alpha^{-3}(h_{212})\cdot\beta^{-1}(b_2))\sharp\alpha^{-1}(h_{22})g_2\\
=\ &\sum a_1(\alpha^{-2}(h_{11})\cdot\beta^{-1}(b_1))\sharp\alpha^{-1}(h_{12})g_1\otimes a_2(\alpha^{-2}(h_{21})\cdot\beta^{-1}(b_2))\sharp\alpha^{-1}(h_{22})g_2\\
=\ &\sum(a_1\sharp h_1)(b_1\sharp g_1)\otimes(a_2\sharp h_2)(b_2\sharp g_2)\\
=\ &\sum((a_1\sharp h_1)\otimes(a_2\sharp h_2))((b_1\sharp g_1)\otimes(b_2\sharp g_2))\\
=\ &\triangle_{A\sharp H}(a\sharp h)\triangle_{A\sharp H}(b\sharp g).
\end{aligned}$$

运用定理中的条件, 可得如下关于余单位的弱乘运算

$$\begin{aligned}
&\varepsilon_{A\sharp H}(((a\sharp h)(b\sharp g))\gamma(c\sharp k))\\
=\ &\sum\varepsilon_{A\sharp H}((a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b)))(\alpha^{-2}(\alpha^{-1}(h_{21})g_1)\cdot c)\sharp\alpha^{-1}(\alpha^{-1}(h_{22})g_2)\alpha(k))\\
=\ &\sum\varepsilon_A((a(\alpha^{-2}(h_1)\cdot\beta^{-1}(b)))(\alpha^{-2}(\alpha^{-1}(h_{21})g_1)\cdot c))\varepsilon_H(\alpha^{-1}(\alpha^{-1}(h_{22})g_2)\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-2}(h_{11})\cdot\beta^{-1}(b_1)))\varepsilon_A((\alpha^{-2}(h_{12})\cdot\beta^{-1}(b_2))(\alpha^{-2}(\alpha^{-1}(h_{21})g_1)\cdot c))\\
&\varepsilon_H(\alpha^{-2}(h_{22})\alpha^{-1}(g_{21}))\varepsilon_H(\alpha^{-1}(g_{22})\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1)))\varepsilon_A((\alpha^{-3}(h_{211})\cdot\beta^{-1}(b_2))(\alpha^{-3}(h_{212})\cdot(\alpha^{-2}(g_1)\\
&\cdot\beta^{-1}(c))))\varepsilon_H(\alpha^{-2}(h_{22})\alpha^{-1}(g_{21}))\varepsilon_H(\alpha^{-1}(g_{22})\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1)))\varepsilon_A(\alpha^{-1}(h_{21})\cdot(\beta^{-1}(b_2)(\alpha^{-2}(g_1)\cdot\beta^{-1}(c))))\\
&\varepsilon_H(\alpha^{-2}(h_{22})\alpha^{-1}(g_{21}))\varepsilon_H(\alpha^{-1}(g_{22})\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1)))\varepsilon_H(\alpha^{-1}(h_{21}))\varepsilon_A(\beta^{-1}(b_2(\alpha^{-2}(g_{11})\cdot c)))\\
&\varepsilon_H(\alpha^{-2}(h_{22})\alpha^{-1}(g_{12}))\varepsilon_H(g_2\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1)))\varepsilon_A(b_2(\alpha^{-2}(g_{12})\cdot c))\\
&\varepsilon_H(\alpha^{-1}(h_2)\alpha^{-1}(g_{11}))\varepsilon_H(g_2\alpha(k))\\
=\ &\sum\varepsilon_A(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1)))\varepsilon_A(b_2(\alpha^{-2}(g_{21})\cdot c))\\
&\varepsilon_H(\alpha^{-1}(h_2)g_1)\varepsilon_H(\alpha^{-1}(g_{22})\alpha(k))\\
=\ &\sum\varepsilon_{A\sharp H}(a(\alpha^{-1}(h_1)\cdot\beta^{-1}(b_1))\sharp\alpha^{-1}(h_2)g_1)\varepsilon_{A\sharp H}(b_2(\alpha^{-2}(g_{21})\cdot c)\sharp\alpha^{-1}(g_{22})\alpha(k))\\
=\ &\sum\varepsilon_{A\sharp H}((a\sharp h)(b_1\sharp g_1))\varepsilon_{A\sharp H}((b_2\sharp g_2)\gamma(c\sharp k)).
\end{aligned}$$

同理可得

$$\varepsilon_{A\sharp H}(\gamma(a\sharp h)((b\sharp g)(c\sharp k)))=\sum\varepsilon_{A\sharp H}(\gamma(a\sharp h)(b_2\sharp g_2))\varepsilon_{A\sharp H}((b_1\sharp g_1)(c\sharp k)).$$

由于对于任意的 $a\in A, x\in H^L$, 有 $a\cdot x=S^{-1}(\alpha^{-1}(x))\cdot a=a(x\cdot\widehat{1})$, 因此有 $\alpha(x)\cdot\widehat{1}=\beta(x\cdot\widehat{1})=\widehat{1}\cdot x$, 又由于 $\varepsilon_H\circ S=\varepsilon_H$ 成立, 因此由命题 2.5 可知 $S(H^L)\subset H^R$ 和 $S(H^R)\subset H^L$ 成立, 所以有

$$
\begin{aligned}
& \sum(\widehat{1}_1\sharp 1_1)\otimes(\widehat{1}_2\sharp 1_2)(\widehat{1}_{1'}\sharp 1_{1'})\otimes\gamma(\widehat{1}_{2'}\sharp 1_{2'})\\
= & \sum(\widehat{1}_1\sharp 1_1)\otimes(\widehat{1}_2(\alpha^{-2}(1_{21})\cdot\beta^{-1}(\widehat{1}_{1'}))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{11}))\otimes(\widehat{1}_2(\alpha^{-2}(1_{12})\cdot\beta^{-1}(\widehat{1}_{1'}))\sharp 1_2 1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{12}))\otimes(\widehat{1}_2(\alpha^{-2}(1_{11})\cdot\beta^{-1}(\widehat{1}_{1'}))\sharp 1_2 1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{21}))\otimes(\widehat{1}_2(\alpha^{-1}(1_1)\cdot\beta^{-1}(\widehat{1}_{1'}))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{21}))\otimes(\widehat{1}_2(\beta^{-1}(\widehat{1}_{1'})\cdot S(1_1))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{21}))\otimes(\widehat{1}_2(\beta^{-1}(\widehat{1}_{1'})(S(1_1)\cdot\widehat{1}))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_1\sharp\alpha^{-1}(1_{21}))\otimes(\beta^{-1}(\widehat{1}_2\widehat{1}_{1'})(\alpha(S(1_1))\cdot\widehat{1})\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_{2'})\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp\alpha^{-1}(1_{21}))\otimes(\beta^{-1}(\widehat{1}_{12})(\widehat{1}\cdot S(1_1))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp\alpha^{-1}(1_{21}))\otimes(\beta^{-1}(\widehat{1}_{12})(\alpha^{-1}(1_1)\cdot\widehat{1})\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp\alpha^{-1}(1_{12}))\otimes(\beta^{-1}(\widehat{1}_{12})(\alpha^{-2}(1_{11})\cdot\widehat{1})\sharp 1_2 1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp\alpha^{-1}(1_{11}))\otimes(\beta^{-1}(\widehat{1}_{12})(\alpha^{-2}(1_{12})\cdot\widehat{1})\sharp 1_2 1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\beta^{-1}(\widehat{1}_{12})(\alpha^{-2}(1_{21})\cdot\widehat{1})\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\beta^{-1}(\widehat{1}_{12})(\sqcap^L(\alpha^{-2}(1_{21}))\cdot\widehat{1})\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\beta^{-1}(\widehat{1}_{12})\cdot\sqcap^L(\alpha^{-2}(1_{21}))\sharp\alpha^{-1}(1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\widehat{1}_{12}\sharp\alpha^{-1}(\sqcap^L(\alpha^{-2}(1_{21}))(\alpha^{-1}(1_{22})1_{1'})))\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\widehat{1}_{12}\sharp\alpha^{-2}(\sqcap^L(\alpha^{-2}(1_{21}))1_{22})1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_1)\otimes(\widehat{1}_{12}\sharp 1_2 1_{1'})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_{2'}))\\
= & \sum(\widehat{1}_{11}\sharp 1_{11})\otimes(\widehat{1}_{12}\sharp 1_{12})\otimes(\beta(\widehat{1}_2)\sharp\alpha(1_2))\\
= & (\triangle_{A\sharp H}\otimes\gamma)\triangle_{A\sharp H}(\widehat{1}\sharp 1).
\end{aligned}
$$

同理可得

$$(\gamma\otimes\triangle_{A\sharp H})\triangle_{A\sharp H}(\widehat{1}\sharp 1)=\sum\gamma(\widehat{1}_1\sharp 1_1)\otimes(\widehat{1}_{1'}\sharp 1_{1'})(\widehat{1}_2\sharp 1_2)\otimes(\widehat{1}_{2'}\sharp 1_{2'}).$$

因此 Hom-smash 积 $(A\sharp H,\gamma)$ 是 Hom - 弱双代数.

最后, 设 (H,α) 和 (A,β) 是 Hom - 弱 Hopf 代数且 $H^L\cdot\widehat{1}\subseteq Z(A)$. 由于

$$S_{A\sharp H}\circ\gamma=\gamma\circ S_{A\sharp H},$$

直接验证可得. 现需证明 $S_{A\sharp H}$ 满足定义 2.2 中的 (2.6)–(2.8) 项. 设任意 $a\in A$ 和 $h\in H$, 有

$$\begin{aligned}
&(I*S_{A\sharp H})(a\sharp h)=\sum(a_1\sharp h_1)S_{A\sharp H}(a_2\sharp h_2)\\
=\ &\sum(a_1\sharp h_1)(S(\alpha^{-2}(h_{22}))\cdot S_A(\beta^{-1}(a_2))\sharp S(\alpha^{-1}(h_{21})))\\
=\ &\sum a_1(\alpha^{-2}(h_{11})\cdot(S(\alpha^{-3}(h_{22}))\cdot S_A(\beta^{-2}(a_2))))\sharp\alpha^{-1}(h_{12}S(h_{21}))\\
=\ &\sum a_1(\alpha^{-3}(h_{11}S(h_{22}))\cdot S_A(\beta^{-1}(a_2)))\sharp\alpha^{-1}(h_{12}S(h_{21}))\\
=\ &\sum a_1(\alpha^{-3}((h_1S(h_2))_1)\cdot S_A(\beta^{-1}(a_2)))\sharp\alpha^{-1}((h_1S(h_2))_2)\\
=\ &\sum(a_1\sharp\alpha^{-1}(h_1S(h_2)))(S_A(a_2)\sharp 1)=\sum(a_1\sharp\alpha^{-1}(\sqcap^L(h)))(S_A(a_2)\sharp 1)\\
=\ &\sum(\beta^{-1}(a_1)\cdot\alpha^{-1}(\sqcap^L(h))\sharp 1)(S_A(a_2)\sharp 1)=\sum(\beta^{-1}(a_1)\cdot\alpha^{-1}(\sqcap^L(h)))S_A(a_2)\sharp 1\\
=\ &\sum(\beta^{-1}(a_1)(\alpha^{-1}(\sqcap^L(h))\cdot\widehat{1}))S_A(a_2)\sharp 1=\sum a_1((\alpha^{-1}(\sqcap^L(h))\cdot\widehat{1})\beta^{-1}(S_A(a_2)))\sharp 1\\
=\ &\sum a_1(\beta^{-1}(S_A(a_2))(\alpha^{-1}(\sqcap^L(h))\cdot\widehat{1}))\sharp 1=\sum\beta^{-1}(a_1S_A(a_2))(\sqcap^L(h)\cdot\widehat{1})\sharp 1\\
=\ &\sum\beta^{-1}(\sqcap^L(a))\cdot\sqcap^L(h)\sharp 1=\sqcap^L(a)\sharp\sqcap^L(h).
\end{aligned}$$

同时有

$$\begin{aligned}
\sqcap^L(a\sharp h)&=\sum\varepsilon_{A\sharp H}((\widehat{1}_1\sharp 1_1)\gamma^2(a\sharp h))\widehat{1}_2\sharp 1_2\\
&=\sum\varepsilon_{A\sharp H}((\widehat{1}_1\sharp 1_1)(\beta^2(a)\sharp\alpha^2(h)))\widehat{1}_2\sharp 1_2\\
&=\sum\varepsilon_{A\sharp H}(\widehat{1}_1(\alpha^{-2}(1_{11})\cdot\beta(a))\sharp\alpha^{-1}(1_{12})\alpha^2(h))\widehat{1}_2\sharp 1_2\\
&=\sum\varepsilon_{A\sharp H}(\widehat{1}_1(\alpha^{-1}(1_1)\cdot\beta(a))\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\widehat{1}_1(\beta(a)(S(1_1)\cdot\widehat{1}))\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\beta^{-1}(\widehat{1}_1\beta^2(a))(\alpha(S(1_1))\cdot\widehat{1})\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\beta^{-1}(\widehat{1}_1\beta^2(a))(\widehat{1}\cdot S(1_1))\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\beta^{-1}(\widehat{1}_1\beta^2(a))(\alpha^{-1}(1_1)\cdot\widehat{1})\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\beta^{-1}(\widehat{1}_1\beta^2(a))(\sqcap^L(\alpha^{-1}(1_1))\cdot\widehat{1})\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\beta^{-1}(\widehat{1}_1\beta^2(a))\cdot\sqcap^L(\alpha^{-1}(1_1))\sharp\alpha^{-1}(1_{21})\alpha^2(h))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_{A\sharp H}(\widehat{1}_1\beta^2(a)\sharp\alpha^{-1}(\sqcap^L(\alpha^{-1}(1_1))(\alpha^{-1}(1_{21})\alpha^2(h))))\widehat{1}_2\sharp\alpha^{-1}(1_{22})\\
&=\sum\varepsilon_A(\widehat{1}_1\beta^2(a))\widehat{1}_2\sharp\varepsilon_H(\sqcap^L(\alpha^{-1}(1_1))(\alpha^{-1}(1_{21})\alpha^2(h)))\alpha^{-1}(1_{22})\\
&=\sum\sqcap^L(a)\sharp\varepsilon_H(1_{1'}\alpha(1_1))\varepsilon_H(1_{2'}(\alpha^{-1}(1_{21})\alpha^2(h)))\alpha^{-1}(1_{22})\\
&=\sum\sqcap^L(a)\sharp\varepsilon_H(1_{1'}1_{11})\varepsilon_H(\alpha^{-1}(1_{2'}1_{12})\alpha^3(h))1_2\\
&=\sum\sqcap^L(a)\sharp\varepsilon_H(\alpha(1_1)\alpha^3(h))1_2\\
&=\sqcap^L(a)\sharp\sqcap^L(h).
\end{aligned}$$

因此 $(I*S_{A\sharp H})(a\sharp h)=\sqcap^L(a\sharp h)$. 同理可得 $(S_{A\sharp H}*I)(a\sharp h)=\sqcap^R(a\sharp h)$.

对于定义 2.2 中的 (2.9) 式, 利用定义 3.1 中的 (3.1)、(3.2) 式和引理 3.3 及定理中的条件 $H^L\cdot\widehat{1}\subseteq Z(A)$, 有

$$
\begin{aligned}
&\sum(S_{A\sharp H}(a_{11}\sharp h_{11})(a_{12}\sharp h_{12}))S_{A\sharp H}(\gamma^2(a_2\sharp h_2))\\
=\ &\sum((S_H(\alpha^{-2}(h_{112}))\cdot S_A(\beta^{-1}(a_{11}))\sharp S_H(\alpha^{-1}(h_{111})))(a_{12}\sharp h_{12}))\\
&(S_H(h_{22})\cdot S_A(\beta(a_2))\sharp S_H(\alpha(h_{21})))\\
=\ &\sum((S_H(\alpha^{-2}(h_{112}))\cdot S_A(\beta^{-1}(a_{11})))(S_H(\alpha^{-3}(h_{1112}))\cdot\beta^{-1}(a_{12}))\sharp S_H(\alpha^{-2}(h_{1111}))h_{12})\\
&(S_H(h_{22})\cdot S_A(\beta(a_2))\sharp S_H(\alpha(h_{21})))\\
=\ &\sum((S_H(\alpha^{-3}(h_{1122}))\cdot S_A(\beta^{-1}(a_{11})))(S_H(\alpha^{-3}(h_{1121}))\cdot\beta^{-1}(a_{12}))\sharp S_H(\alpha^{-1}(h_{111}))h_{12})\\
&(S_H(h_{22})\cdot S_A(\beta(a_2))\sharp S_H(\alpha(h_{21})))\\
=\ &\sum(S_H(\alpha^{-1}(h_{112}))\cdot S_A(\beta^{-1}(a_{11}))\beta^{-1}(a_{12})\sharp S_H(\alpha^{-1}(h_{111}))h_{12})\\
&(S_H(h_{22})\cdot S_A(\beta(a_2))\sharp S_H(\alpha(h_{21})))\\
=\ &\sum(S_H(\alpha^{-1}(h_{112}))\cdot\sqcap^R(\beta^{-1}(a_1)))((S_H(\alpha^{-3}(h_{1112}))\alpha^{-2}(h_{121}))\cdot\\
&(S_H(\alpha^{-1}(h_{22}))\cdot S_A(a_2)))\sharp(S_H(\alpha^{-2}(h_{1111}))\alpha^{-1}(h_{122}))S_H(\alpha(h_{21}))\\
=\ &\sum(S_H(\alpha^{-2}(h_{1122}))\cdot\sqcap^R(\beta^{-1}(a_1)))(S_H(\alpha^{-2}(h_{1121}))\cdot(\alpha^{-2}(h_{121})\cdot\\
&(S_H(\alpha^{-2}(h_{22}))\cdot S_A(\beta^{-1}(a_2)))))\sharp(S_H(\alpha^{-1}(h_{111}))\alpha^{-1}(h_{122}))S_H(\alpha(h_{21}))\\
=\ &\sum S_H(h_{112})\cdot(\sqcap^R(\beta^{-1}(a_1))(\alpha^{-3}(h_{121})S_H(\alpha^{-2}(h_{22}))\cdot S_A(a_2)))\sharp\\
&S_H(h_{111})(\alpha^{-1}(h_{122})S_H(h_{21}))\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha(h_{11})))(\sqcap^R(\beta^{-1}(a_1))(\alpha^{-2}((\alpha^{-1}(h_{12})S_H(h_2))_1)\cdot S_A(a_2))\sharp(\alpha^{-1}(h_{12})S_H(h_2))_2)\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))(\sqcap^R(\beta^{-1}(a_1))(\alpha^{-3}((\sqcap^L(h_2))_1)\cdot S_A(a_2))\sharp\alpha^{-1}((\sqcap^L(h_2))_2))\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))((\sqcap^R(\beta^{-1}(a_1))\sharp\alpha^{-1}(\sqcap^L(h_2)))(S_A(\beta(a_2))\sharp 1))\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))((\beta^{-1}(\sqcap^R(\beta^{-1}(a_1)))\cdot\alpha^{-1}(\sqcap^L(h_2))\sharp 1)(S_A(\beta(a_2))\sharp 1))\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))((\beta^{-1}(\sqcap^R(\beta^{-1}(a_1)))(\alpha^{-1}(\sqcap^L(h_2))\cdot\widehat{1}))S_A(\beta(a_2))\sharp 1)\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))(\sqcap^R(\beta^{-1}(a_1))((\alpha^{-1}(\sqcap^L(h_2))\cdot\widehat{1})S_A(a_2))\sharp 1)\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))((\beta^{-1}(\sqcap^R(\beta^{-1}(a_1)))S_A(a_2))\cdot\sqcap^L(h_2)\sharp 1)\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))(\sqcap^R(\beta^{-1}(a_1))S_A(\beta(a_2))\sharp\sqcap^L(h_2))\\
=\ &\sum(\widehat{1}\sharp S_H(\alpha^2(h_1)))(S_A(\beta^3(a))\sharp\sqcap^L(h_2))\\
=\ &\sum S_H(\alpha(h_{12}))\cdot S_A(\beta^3(a))\sharp S_H(\alpha(h_{11}))\sqcap^L(h_2)\\
=\ &\sum S_H(\alpha(h_{21}))\cdot S_A(\beta^3(a))\sharp S_H(\alpha^2(h_1))\sqcap^L(\alpha^{-1}(h_{22}))\\
=\ &\sum S_H(\alpha(h_{22}))\cdot S_A(\beta^3(a))\sharp S_H(\alpha^2(h_1))\sqcap^L(\alpha^{-1}(h_{21}))\\
=\ &\sum S_H(\alpha^2(h_2))\cdot S_A(\beta^3(a))\sharp S_H(\alpha(h_{11}))\sqcap^L(\alpha^{-1}(h_{12}))\\
=\ &\sum S_H(\alpha^2(h_2))\cdot S_A(\beta^3(a))\sharp S_H(\alpha^3(h_1))=S_{A\sharp H}(\gamma^4(a\sharp h)),
\end{aligned}
$$

所以 Hom-smash 积 $(A\sharp H,\gamma)$ 是 Hom - 弱 Hopf 代数.

注 3.6 如果线性映射 α 和 β 是恒等映射, 即对任意的 $a\in A$ 和 $h\in H$, 有 $\gamma(a\otimes h)=a\otimes h$, 则 Hom-smash 积 $(A\sharp H,\gamma)$ 是由文献 [8] 定义的弱 Hopf 代数, 并可得文献 [13] 中的例 1.8 或文献 [16] 中的定理 2.2. 如果 (H,α) 和 (A,β) 是 Hom-Hopf 代数, 则 Hom-smash 积 $(A\sharp H,\gamma)$ 是 Hom-Hopf 代数, 并可得文献 [6] 中的例 2.2.

参 考 文 献

[1] Makhlouf A, Silvestrov S. Hom-algebras structures [J]. J. Gen. Lie Theory Appl., 2008, 2: 51–64.

[2] Caenepeel S, Goyvaerts I. Monoidal Hom-Hopf algebras [J]. Comm. Alg., 2011, 39(6): 2216–2240.

[3] Makhlouf A, Silvestrov S. Hom-algebras and Hom-coalgebras[J]. J. Alg. Appl., 2010, 9: 553–589.

[4] Makhlouf A, Silvestrov S. Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras[A]. Silvestrov S, Paal E, Abramov V, Stolin A, eds. Generalized lie theory in mathematics, physics and beyond[C]. Berlin: Springer-Verlag, 2009, 189–206.

[5] Yau D. Hom-bialgebras and comodule Hom-algebras[J]. Inter. Elect. J. Algebra, 2010, 8: 45–64.

[6] 郑乃峰. Hom-ω-smash 积 Hopf 代数的拟三角结构 [J]. 数学年刊, 2013, 34(6): 689–708.

[7] 郑乃峰. Hom-ω-smash 余积 Hopf 代数上的辫化结构 [J]. 数学物理学报, 2013, 33(6): 1068–1088.

[8] Bohm G, Nill F, Szlachanyi K. Weak Hopf algebras I: Integral theory and C^*-structure[J]. J. Alg., 1999, 221: 385–438.

[9] Sweedler M. Hopf algebra[M]. New York: Benjamin, 1969.

[10] 郑乃峰. Hom - 弱 Hopf 代数上的 Hom-Smash 余积 [J]. 数学杂志, 2016, 36(2): 393–402.

[11] Nikshych D, Vainerman L. Finite quantum groupoid and their applications[J]. Math. Sci. Res. Inst. Publ., 2002, 43: 211–262.

[12] Nikshych D. A duality theorem for quantum groupoids[J]. Contemp. Math., 2000, 267: 237–243.

[13] 郑乃峰. 弱 Hopf 代数上的 Smash 双积 [J]. 数学进展, 2009, 38(5): 553–565.

[14] 郑乃峰. 弱 Hopf 代数上的混合积 [J]. 数学年刊, 2010, 31(6): 757–768.

[15] 郑乃峰. 弱 Hopf 代数上的 ω - 交叉积 [J]. 数学年刊, 2012, 33(1): 77–90.

[16] 侯波, 王志玺. 弱 Hopf 代数作用与冲积 [J]. 数学学报, 2007, 50A(1): 89–96.

HOM-SMASH PRODUCTS OVER HOM-WEAK HOPF ALGEBRAS

ZHENG Nai-feng

(*College of Science, Ningbo University, Ningbo 315211, China*)

Abstract: In this paper, we study the concept of weak Hopf algebras over Hom-Hopf algebras. Using the method of establishing weak left H-module Hom-algebras, we construct Hom-smash product and demonstrate that Hom-smash product is a Hom-algebra and Hom-weak Hopf algebra, which generalizes weak Hopf algebra introduced by Bohm etc..

Keywords: Hom-weak Hopf algebra; weak left H-module Hom-algebra; Hom-smash product

2010 MR Subject Classification: 16W30; 16E10