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THE GROWTH ON ENTIRE SOLUTIONS OF FERMAT TYPE Q-DIFFERENCE DIFFERENTIAL EQUATIONS

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Abstract: This paper is devoted to consider the entire solutions on Fermat type q-difference differential equations. Using the classical and difference Nevanlinna theory and functional equations theory, we obtain some results on the growth of the Fermat type q-difference differential equations.

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1 Introduction

Let $f(z)$ be a meromorphic function in the complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [5, 16]. As we all know that Nevanlinna theory was extensively applied to considering the growth, value distribution, and solvability of meromorphic solutions of differential equations [6]. Recently, difference analogues of Nevanlinna theory were established, which also be used to consider the corresponding properties of meromorphic solutions on difference equations or q-difference equations, such as [2, 4, 7–12, 14, 17].

Let us recall the classical Fermat type equation

$$f(z)^2 + g(z)^2 = 1. \tag{1.1}$$

Equation (1.1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exist. However, the above result fails to give more precise informations when $g(z)$ has a special relationship with $f(z)$. Yang and Li [15] first considered the entire solutions of the Fermat type differential equation

$$f(z)^2 + f'(z)^2 = 1, \tag{1.2}$$

and they proved the following result.

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Theorem A [15, Theorem 1] The transcendental meromorphic solutions of (1.2) must satisfy $f(z)=\frac{1}{2}\left(Pe^{-iz}+\frac{1}{P}e^{iz}\right)=\sin(z+B)$, where P is non-zero constant and $e^{iB}=\frac{i}{P}$.

Tang and Liao [13] further investigated the entire solutions of a generalization of (1.2) as follows

$$f(z)^2+P(z)^2f^{(k)}(z)^2=Q(z), \tag{1.3}$$

where $P(z),Q(z)$ are non-zero polynomials and obtained the next result.

Theorem B [13, Theorem 1] If the differential equation (1.3) has a transcendental meromorphic solution f, then $P(z)\equiv A$, $Q(z)\equiv B$, k is an odd and $f(z)=b\sin(az+d)$, where a,b,d are constants such that $Aa^k=\pm1$, $b^2=B$.

Recently, the difference analogues of Nevanlinna theory were used to consider the solutions properties of Fermat type difference equations. Liu, Cao and Cao [8] investigated the finite order entire solutions of the difference equation

$$f(z)^2+f(z+c)^2=1, \tag{1.4}$$

here and in the following, c is a non-zero constant and $P(z),Q(z)$ are non-zero polynomial, unless otherwise specified. The result can be stated as follows.

Theorem C [8, Theorem 1.1] The transcendental entire solutions with finite order of (1.4) must satisfy $f(z)=\sin(Az+B)$, where B is a constant and $A=\frac{(4k+1)\pi}{2c}$, k is an integer.

Furthermore, Liu and Yang [10] considered a generalization of (1.4) as follows

$$f(z)^2+P(z)^2f(z+c)^2=Q(z), \tag{1.5}$$

and obtained the following result.

Theorem D Let $P(z)$, $Q(z)$ be non-zero polynomials. If the difference equation (1.5) admits a transcendental entire solution of finite order, then $P(z)\equiv\pm1$ and $Q(z)$ reduces to a constant q.

If an equation includes the q-difference $f(qz)$ and the derivatives of $f(z)$ or $f(z+c)$, then this equation can be called q-difference differential equation. Liu and Cao [11] considered the entire solutions on Fermat type q-difference differential equation

$$f'(z)^2+f(qz)^2=1, \tag{1.6}$$

and obtained the following result.

Theorem E [11, Theorem 3.1] The transcendental entire solutions with finite order of (1.6) must satisfy $f(z)=\sin(z+B)$ when $q=1$, and $f(z)=\sin(z+k\pi)$ or $f(z)=-\sin(z+k\pi+\frac{\pi}{2})$ when $q=-1$. There are no transcendental entire solutions with finite order when $q\neq\pm1$.

By comparing with the above five theorems, we state the following questions which will be considered in this paper.

Question 1 From Theorem A to Theorem E, we remark that the order of all transcendental entire solutions with finite order of different equations are equal to one. Hence, considering a generalization of equation (1.6), such as

$$f'(z)^2 + P(z)^2 f(qz)^2 = Q(z), \tag{1.7}$$

it is natural to ask if the finite order of the entire solutions of (1.7) is equal to one or not?

Question 2 From Theorem B to Theorem E, the existence of finite order entire solutions of (1.3) and (1.5) forces the polynomial $P(z)$ reduce to a constant. Is it also remain valid for equation (1.7)?

However, Examples 1 and 2 below show that Questions 1 and 2 are false in generally.

Example 1 Entire function $f(z) = \sin z^n$ solves

$$f'(z)^2 + n^2 z^{2(n-1)} f(qz)^2 = n^2 z^{2(n-1)},$$

where q satisfies $q^n = 1$. It implies that the solutions order of (1.7) may take arbitrary numbers and $P(z)^2 = n^2 z^{2(n-1)}$ is not a constant.

Example 2 We can construct a general solution from Example 1. Entire function $f(z) = \sin(h(z))$ solves

$$f'(z)^2 + [h'(z)]^2 f(qz)^2 = [h'(z)]^2,$$

where q satisfies $q^n = 1$ and $h(z)$ is a non-constant polynomial.

Example 3 Function $f(z) = \sinh z$ is also an entire solution of $f'(z)^2 - f(qz)^2 = 1$ and $f(z) = \cosh z$ is an entire solution of $f'(z)^2 - f(qz)^2 = -1$, where $q = -1$.

From Example 1 to Example 3, we also remark that if $P(z)^2 = \pm 1$, the transcendental entire solutions $f(z)$ are of order one, if $P(z) = nz^{(n-1)}$, the transcendental entire solutions $f(z)$ are of order n. Hence, it is reasonable to conjecture that the order of entire solutions of (1.7) is equal to $\rho(f) = 1 + \deg P(z)$. In this paper, we will answer the above conjecture and obtain the following result.

Theorem 1.1 If $|q| > 1$, then the entire solution of (1.7) should be a polynomial. If there exists a finite order transcendental entire solution f of (1.7), then $\rho(f) = 1 + \deg P(z)$ and $|q| = 1$.

In the following, we will consider another q-difference differential equation

$$f'(z+c)^2 + P(z)^2 f(qz)^2 = Q(z), \tag{1.8}$$

and obtain the following result.

Theorem 1.2 If $|q| > 1$, then the entire solution of (1.8) should be a polynomial. If there exist a finite order transcendental entire solution f of (1.8), then $\rho(f) = 1 + \deg P(z)$ and $|q| = 1$.

Example 4 Function $f(z) = \sin z$ is an entire solution of $f'(z+c)^2 + f(qz)^2 = 1$, where $c = \pi$ and $q = -1$.

Finally, we consider other q-difference equation

$$f(z+c)^2 + P(z)^2 f(qz)^2 = Q(z). \tag{1.9}$$

Theorem 1.3 If $|q| > 1$, then the entire solution $f(z)$ of (1.9) should be a polynomial.

If $P(z)^2 = 1$ in (1.9) , the following example shows that we can not give the precise expression of finite order entire solution and the order of $f(z)$ does not satisfy $\rho(f) = 1 + \deg P(z)$ and $|q| = 1$.

Example 5 [11] If $q = -1$, $c = \frac{\pi}{2}$, thus $f(z) = \sin z$ satisfies $f(z + \frac{\pi}{2})^2 + f(-z)^2 = 1$. If $q = \frac{1+i\sqrt{3}}{2}$, $c = \frac{1-i\sqrt{3}}{2}$, and $p(z) = \frac{1}{3}z^3 + z^2 + z + \frac{3i}{4}\pi + \frac{1}{3} + ki\pi$, thus

$$p(z + \frac{c}{q}) + p(qz) = \frac{3i\pi}{2} + 2ki\pi$$

and k is an integer. Thus

$$f(z) = \frac{e^{p(z - \frac{1-i\sqrt{3}}{2})} - e^{-p(z - \frac{1-i\sqrt{3}}{2})}}{2}$$

satisfies

$$f(z + \frac{1-i\sqrt{3}}{2})^2 + f(\frac{1+i\sqrt{3}}{2}z)^2 = 1.$$

Remark 1 The proofs of Theorem 1.2 and Theorem 1.3 are similar as the proof of Theorem 1.1. Hence we will not give the details here.

2 Some Lemmas

For the proofs of Theorems 1.1, 1.2 and 1.3, we need the following results.

Lemma 2.1 [3, Lemma 3.1] Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let f be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \geq 1$ such that

$$T(r, f) \leq K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f),$$

then

$$\rho(f) \leq \frac{\log K_2}{-\log \alpha} + \limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 2.2 [11, Lemma 2.15] Let $p(z)$ be a non-zero polynomial with degree n. If $p(qz) - p(z)$ is a constant, then $q^n = 1$ and $p(qz) \equiv p(z)$. If $p(qz) + p(z)$ is a constant, then $q^n = -1$ and $p(qz) + p(z) \equiv 2a_0$, where a_0 is the constant term of $p(z)$.

Lemma 2.3 [2, Theorem 2.1] Let $f(z)$ be transcendental meromorphic function of finite order ρ. Then for any $\varepsilon > 0$, we have

$$T(r, f(z+c)) = T(r, f) + O(r^{\rho-1+\varepsilon}) + O(\log r) = T(r, f) + S(r, f). \tag{2.1}$$

Lemma 2.4 [16, Theorem 1.62] Let $f_j(z)$ be meromorphic functions, $f_k(z)$ ($k = 1, 2, \cdots, n-1$) be not constants, satisfying $\sum\limits_{j=1}^{n} f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$ and

$$\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $\lambda < 1$ and $k = 1, 2, \cdots, n-1$, then $f_n(z) \equiv 1$.

3 Proof of Theorem 1.1

If $|q| > 1$ and $f(z)$ is an entire solution of (1.7), we use the observation (see [1]) that

$$T(r, f(qz)) = T(|q|r, f(z)) + O(1)$$

holds for any meromorphic function f and any constant q. If $f(z)$ is a transcendental entire function, then from (1.7) and Valiron-Mohon'ko theorem, we have

$$T(|q|r, f(z)) = T(r, f(qz)) + O(1) \leq T(r, f'(z)) + S(r, f) \leq T(r, f(z)) + S(r, f).$$

Let $\alpha = \frac{1}{|q|}$ and $|q| > 1$. Then we have

$$T(|q|\alpha r, f(z)) \leq T(\alpha r, f(z)) + S(\alpha r, f(z)).$$

Hence, we have $T(r, f(z)) \leq T(\alpha r, f(z)) + S(\alpha r, f(z))$. From Lemma 2.1, we have $\rho(f) = 0$. Combining Hadamard factorization theorem, we have $f'(z) + iP(z)f(qz) = Q_1(z)$ and $f'(z) - iP(z)f(qz) = Q_2(z)$, thus $f'(z) = \frac{Q_1(z)+Q_2(z)}{2}$ is a polynomial, which is a contradiction with $f(z)$ is a transcendental entire function. Thus $f(z)$ should be a polynomial.

Assume that $f(z)$ is a transcendental entire solution of (1.7) with finite order, then

$$[f'(z) + iP(z)f(qz)][f'(z) - iP(z)f(qz)] = Q(z). \tag{3.1}$$

Thus both $f'(z) + iP(z)f(qz)$ and $f'(z) - iP(z)f(qz)$ have finitely many zeros. Combining (3.1) with the Hadamard factorization theorem, we assume that

$$f'(z) + iP(z)f(qz) = Q_1(z)e^{h(z)}$$

and

$$f'(z) - iP(z)f(qz) = Q_2(z)e^{-h(z)},$$

where $h(z)$ is a non-constant polynomial provided that $f(z)$ is of finite order transcendental and $Q_1(z)Q_2(z) = Q(z)$, where $Q_1(z)$, $Q_2(z)$ are non-zero polynomials. Thus we have

$$f'(z) = \frac{Q_1(z)e^{h(z)} + Q_2(z)e^{-h(z)}}{2} \tag{3.2}$$

and

$$f(qz) = \frac{Q_1(z)e^{h(z)} - Q_2(z)e^{-h(z)}}{2iP(z)}. \tag{3.3}$$

From (3.2), we have

$$f'(qz) = \frac{Q_1(qz)e^{h(qz)} + Q_2(qz)e^{-h(qz)}}{2}. \tag{3.4}$$

Taking first derivative of (3.3), we have

$$f'(qz) = \frac{A(z)e^{h(z)} - B(z)e^{-h(z)}}{2iqP(z)^2}, \tag{3.5}$$

where

$$A(z) = P(z)Q_1'(z) + Q_1(z)[P(z)h'(z) - P'(z)] \tag{3.6}$$

and

$$B(z) = P(z)Q_2'(z) - Q_2(z)[P(z)h'(z) + P'(z)]. \tag{3.7}$$

From (3.4) and (3.5), we have

$$\frac{A(z)e^{h(qz)+h(z)}}{iqP(z)^2Q_2(qz)} - \frac{B(z)e^{h(qz)-h(z)}}{iqP(z)^2Q_2(qz)} - \frac{Q_1(qz)}{Q_2(qz)}e^{2h(qz)} \equiv 1. \tag{3.8}$$

Obviously, if $h(qz)$ is a constant, then $h(z)$ is a constant, thus $f(z)$ should be a polynomial. If $h(qz)$ is a non-constant entire function, then $h(qz) - h(z)$ and $h(qz) + h(z)$ are not constants simultaneously. The following, we will discuss two cases.

Case 1 If $h(qz) - h(z)$ is not a constant, from Lemma 2.4, we know that

$$\frac{A(z)e^{h(qz)+h(z)}}{iqP(z)^2Q_2(qz)} \equiv 1. \tag{3.9}$$

Since $f(z)$ is a finite order entire solution, then $h(z)$ should satisfies $h(z) = a_nz^n + \cdots + a_0$ is a non-constant polynomial, thus $|q| = 1$ follows for avoiding a contradiction. From Lemma 2.2, we have $h(qz) + h(z) = 2a_0$. Hence, we have

$$A(z) = iqP(z)^2Q_2(qz)e^{-2a_0}. \tag{3.10}$$

In addition, from (3.8), we also get

$$\frac{B(z)e^{h(qz)-h(z)}}{iqP(z)^2Q_2(qz)} + \frac{Q_1(qz)}{Q_2(qz)}e^{2h(qz)} \equiv 0, \tag{3.11}$$

which implies that

$$B(z) = -iqQ_1(qz)P(z)^2e^{2a_0}. \tag{3.12}$$

Thus

$$A(z)B(z) = q^2P(z)^4Q(qz). \tag{3.13}$$

Substitute (3.6) and (3.7) into (3.13), we have

$$\begin{aligned} & \{P(z)Q_1'(z) + Q_1(z)[P(z)h'(z) - P'(z)]\}\{P(z)Q_2'(z) - Q_2(z)[P(z)h'(z) + P'(z)]\} \\ = \ & q^2P(z)^4Q(qz). \end{aligned} \tag{3.14}$$

Since $f(z)$ is a finite order entire solution, by comparing with the degree of both hand side of (3.14), we have

$$\deg(h(z)) = 1 + \deg P(z).$$

It implies that $\rho(f) = 1 + \deg P(z)$.

Case 2 If $h(qz) + h(z)$ is not a constant, from Lemma 2.4, we know that

$$-\frac{B(z)e^{h(qz)-h(z)}}{iqP(z)^2Q_2(qz)} \equiv 1.$$

Hence $|q| = 1$ follows for avoiding a contradiction. Assume that $h(z) = a_nz^n + \cdots + a_0$, thus $h(qz) = h(z)$. Hence we have

$$-B(z) = iqP(z)^2Q_2(qz). \tag{3.15}$$

In addition, from (3.8), we also get

$$A(z) = iqQ_1(qz)P(z)^2. \tag{3.16}$$

Thus, similar as the above, we also get $\rho(f) = 1 + \deg P(z)$.

References

[1] Bergweiler W, Ishizaki K, Yanagihara N. Meromorphic solutions of some functional equations [J]. Meth. Appl. Anal, 1998, 5: 248–258.

[2] Chiang Y M, Feng S J. On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane [J]. Ramanujan. J., 2008, 16: 105–129.

[3] Gundensen G, Heittokangas J, Laine I, Rieppo J, Yang D G. Meromorphic solutions of generalized Schröder equations [J]. Aequations Math., 2002, 63: 110–135.

[4] Halburd R G, Korhonen R J. Difference analogue of the lemma on the logarithmic derivative with applications to difference equations [J]. J. Math. Anal. Appl., 2006, 314: 477–487.

[5] Hayman W K. Meromorphic functions [M]. Oxford: Clarendon Press, 1964.

[6] Laine I. Nevanlinna theory and complex differential equations [M]. Berlin, New York: Walter de Gruyter, 1993.

[7] Liu K. Meromorphic functions sharing a set with applications to difference equations [J]. J. Math. Anal. Appl., 2009, 359: 384–393.

[8] Liu K, Cao T B, Cao H Z. Entire solutions of Fermat type differential-difference equations [J]. Arch. Math., 2012, 99: 147–155.

[9] Liu K, Yang L Z, Liu X L. Existence of entire solutions of nonlinear difference equations [J]. Czech. Math. J., 2011, 61(2): 565–576.

[10] Liu K, Yang L Z. On entire solutions of some differential-difference equations [J]. Comput. Meth. Funct. Theory, 2013, 13: 433–447.

[11] Liu K, Cao T B. Entire solutions of Fermat type difference differential equations [J]. Electron. J. Diff. Equ., 2013, 2013: 1–10.

[12] Liu K, Yang L Z. Some results on complex differential-difference theory [J]. J. Math., 2013, 33(5): 830–836.

[13] Tang J F, Liao L W. The transcendental meromorphic solutions of a certain type of nonlinear differential equations [J]. J. Math. Anal. Appl., 2007, 334: 517–527.

[14] Yang C C, Laine I. On analogies between nonlinear difference and differential equations [J]. Proc. Japan Acad., Ser. A, 2010, 86: 10–14.

[15] Yang C C, Li P. On the transcendental solutions of a certain type of nonlinear differential equations [J]. Arch. Math., 2004, 82: 442–448.

[16] Yang C C, Yi H X. Uniqueness theory of meromorphic functions [M]. Nederland: Kluwer Academic Publishers, 2003.

[17] Zhang J L, Korhonen R J. On the Nevanlinna characteristic of $f(qz)$ and its applications [J]. J. Math. Anal. Appl., 2010, 369: 537–544.

费马 q-差分微分方程整函数解的增长性研究

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摘要: 本文研究了费马 q-差分微分方程的整函数解的相关问题. 利用经典和差分的Nevanlinna理论和函数方程理论的研究方法, 获得了 q-差分微分方程整函数解增长性的几个结果.

关键词: q-差分微分方程; 整函数解; 有穷级

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