Vol. 37 ( 2017 )
No. 4

数 学 杂 志
J. of Math. (PRC)

# ROSSBY WAVES WITH THE CHANGE OF $\beta$ AND THE INFLUENCE OF TOPOGRAPHY IN A TWO-LAYER FLUID

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**Abstract:** In this paper, we study the problem of the change of Rossby parameter and the topography in a two-layer fluid. Based on the traveling wave method and the perturbation method, the Rossby wave amplitude is obtained to satisfy the homogeneous KdV equation and the homogeneous mKdV equation, which describe the evolution of the amplitude of solitary Rossby waves. The effects of Rossby parameters and topography on Rossby wave are generalized.

**Keywords:** change of $\beta$; homogeneous KdV equation; homogeneous mKdV equation; topography

**2010 MR Subject Classification:** 76B65; 86A10

**Document code:** A **Article ID:** 0255-7797(2017)04-0751-10

## 1 Introduction

The solitary Rossby waves was applied to the planetary-scale wave phenomenon. Since Long [1] derived the Korteweg-de Vries (KdV)equation on a baratropic atmosphere, theories of solitary Rossby waves were developed by Larse [2], Benney [3–7], Clarke [8] and Redekopp [9, 10] primarily in the context of the atmospheric models. The basic theory demonstrates that the amplitude of long Rossby waves propagating in a zonal shear flow is governed by either the KdV or modified Korteweg-de Vries (mKdV) equation depending on the vertical density distribution in the atmospheric model. Hukuda [11] studied the effect of vertical shear in an analysis restricted to neutral modes propagating in a weak horizontal shear flow without critical layers. Pedlosky [12, 13] presented theory that is the finite-amplitude behavior of unstable baroclinic waves in a quasi-geostrophic two-layer model, it was shown that in the absence of dissipation the equilibrated finite-amplitude state exhibits an oscillation, both of the mean flow and the baroclinic waves. Mitsudera and Grimshaw [14] also presented a weakly nonlinear, long-waves theory to describe a complicated system, they described the

* **Received date:** 2015-12-18 **Accepted date:** 2016-04-20

**Foundation item:** Supported by Natural Science Foundation of China (11362012; 11562014); the scientific research project of the Inner Mongolia Autonomous Region high school (NJZY16096); the Scientific research project of Inner Mongolia University of Technology (ZD201411).

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generation and evolution of mesoscale phenomena in a baroclinic current when it is interacting with a localized longshore topographic feature. Gottwald and Grimshaw [15] gave that the influence of topography on the interaction of long, weakly nonlinear, quasigeostrophic baroclinic waves can be described by a pair of linearly coupled KdV equations, with a forcing term in one of the equations. Patoine and Warn [16] showed the interaction of long, quasi-stationary baroclinc waves with topography can be described by an inhomogeneous KdV equation. Liu and Tan [17] discussed the change of the Rossby parameter beta with latitude and extended the beta-plane approximation. Liu and Tan [18] used a barotropic semi-geostrophic model with topogrphic forcing the stability and solution of the nonlinear Rossby waves were discussed. They found that the effect of the W-E oriented topography and the N-S oriented topography on the stability and phase speed of the waves are quite different, the Rossby waves forced by the topography can been described by the KdV equation. Luo [19] studied a kind of the solitary Rossby waves excited by the change of $\beta$ excluding effects of shear basic flow and topography, showed the $\beta$ parameter with the change of latitude may be one reason of producing dipole blocking in the mid-high latitudes. Luo [20] investigated the planetary-scale baroclinic envelope Rossby solitons for zonal wavenumber 2 in a two-layer, it is found that when the shear of basic state westerly winds between the upper and lower layers is weak, both the upper-and lower-layer envelope Rossby solitons are almost in phase and exhibit vortex pair block structure which have a weak baroclinicity. But he did not discussed the topography effect on the Rossby waves. Charney and Straus [21] showed that the forced flow of a barotropic fluid over wary topography in a periodic beta-plane channel may possess a multiplicity of stationary equilibrium states of which more than one may be stable. Lv [22] showed the solitary Rossby waves caused by the shearing basic flow and orography with small variable slope were mainly of meridional wave number one and two, different shears of flow can excite different stream line patterns of solitary waves, and the orography with variable slope is also important factors of formation of solitary Rossby waves. Solutions of solitary waves play such an important role in soliton theory that many mathematicians and physicists were interested in this topic, such as Hirota's bilinear method, the Jacobi elliptic function expansion method et al. [23, 24] were proposed and used widely.

## 2 The Governing Equation

The non-dimensional quasi-geostrophic with topography potential vorticity equation in each fluid layer on a $\beta$-plane can be written in the form [11]

$$\frac{\partial}{\partial t}[\nabla^2\psi_1 - F(\psi_1-\psi_2)] + J[\psi_1, \nabla^2\psi_1 - F(\psi_1-\psi_2) + \beta y] = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}[\nabla^2\psi_2 + F(\psi_1-\psi_2)] + J[\psi_2, \nabla^2\psi_2 - F(\psi_1-\psi_2) + \beta y + Mh(x,y)] = 0. \tag{2.2}$$

In equations (2.1), (2.2), the variables with subscripts 1, 2 refer to the quantities defined in the upper and lower-layers, respectively. The $\psi_n(x, y, t)$ represents the geostrophic stream function in the nth layer, where $n = 1, 2$. $\nabla^2$ is the horizontal Laplace operator, $h(x, y)$ is the function of topography. The internal rotational Froude number and the dimensionless gradient of planetary vorticity $\beta$, defined as

$$F = \frac{f_0^2 L^2}{\frac{\triangle \rho g D}{2\rho}}, \quad \beta = \beta' \frac{L^2}{U},$$

$M = \dfrac{g f_0}{c_0^2} \dfrac{DL}{U}$ is the effect parameter of topography, $c_0^2 = gH$, where $f_0$ is a constant Coriolis parameter in moderate position, $L$ is the north-south extent of the zonal field, $\dfrac{\triangle \rho}{\rho}$ the density difference anomaly between the upper and lower layers, $g$ the gravitational acceleration, $D$ is the total fluid depth and $U$ is a characteristic zonal velocity, $H$ is the scale height. The boundary conditions is wall at the northern and southern [12]

$$\frac{\partial \psi_n}{\partial x} = 0, \quad y = 0, 1. \tag{2.3}$$

Introducing a scaled coordinate in the form [9, 10]

$$\xi = \epsilon(x - ct), \tag{2.4}$$

To consider effects of topography, assuming [22],

$$h(x, y) = h(y), \tag{2.5}$$

here $c$ is the phase speed of a long Rossby wave and $\epsilon$ the dimensionless Rossby number assumed to be smaller than unity, it is a measure of the Rossby wave amplitude. Inserting (2.4) and (2.5) into (2.1), (2.2) and (2.3) gives an set of equations

$$[\frac{\partial \psi_1}{\partial \xi}\frac{\partial}{\partial y} - (c + \frac{\partial \psi_1}{\partial y})\frac{\partial}{\partial \xi}][\epsilon^2 \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial y^2} - F(\psi_1 - \psi_2)] + \beta \frac{\partial \psi_1}{\partial \xi} = 0, \tag{2.6}$$

$$[\frac{\partial \psi_2}{\partial \xi}\frac{\partial}{\partial y} - (c + \frac{\partial \psi_2}{\partial y})\frac{\partial}{\partial \xi}][\epsilon^2 \frac{\partial^2 \psi_2}{\partial \xi^2} + \frac{\partial^2 \psi_2}{\partial y^2} + F(\psi_1 - \psi_2)] + \beta \frac{\partial \psi_2}{\partial \xi} + M \frac{\partial \psi_2}{\partial \xi}\frac{dh}{dy} = 0, \tag{2.7}$$

$$\frac{\partial \psi_n}{\partial \xi} = 0, \quad y = 0, 1. \tag{2.8}$$

The total flow $\psi_(\xi, y)$ is composed of the basic shear flow plus the long wave disturbance. The disturbance turn out to be solitary waves described by nonlinear equation depending on the balance between the effects of nonlinear and dispersion.

## 3 Expansion of Solution

### 3.1 The Even Order of $\epsilon$ Expansion

First, we seek a solution in the form [25]

$$\psi_n(\xi, y) = -\int_0^y U_n dy + \epsilon^2 \psi_{n1}(\xi, y) + \epsilon^4 \psi_{n1}(\xi, y) + \cdots, \tag{3.1}$$

$$c = c_0^* + \epsilon^2 c_1 + \cdots, \tag{3.2}$$

where $U_n(y)$ represents a zonal in the nth layer, $c_0^*$ is the phase speed of an infinitely long Rossby wave. Inserting (3.1), (3.2) into (2.6), (2.7) gives

$$\begin{aligned}&(U_n-c_0^*)\frac{\partial}{\partial\xi}[\frac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21})]+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}\\&+(-1)^{n-1}F(U_1-U_2)]\tfrac{\partial\psi_{n1}}{\partial\xi}+\epsilon^2\{(U_n-c_0^*)\frac{\partial}{\partial\xi}[\frac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^nF(\psi_{12}-\psi_{22}]\\&+[\beta+M\tfrac{(-1)^n+1}{2}\tfrac{dh}{dy}-\tfrac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)]\frac{\partial\psi_{n2}}{\partial\xi}+(U_n-c_0^*)\frac{\partial}{\partial\xi}(\frac{\partial^2\psi_{n1}}{\partial\xi^2})\\&+(\tfrac{\partial\psi_{n1}}{\partial\xi}\tfrac{\partial}{\partial y}-\tfrac{\partial\psi_{n1}}{\partial y}\tfrac{\partial}{\partial\xi}-c_1\tfrac{\partial}{\partial y})(\tfrac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21})\}+O(\epsilon^4)=0,\end{aligned}\tag{3.3}$$

here $n=1,2$, from equation (3.3) the problem of the lowest order are written as $O(\epsilon^0)$:

$$\begin{cases}(U_n-c_0^*)\dfrac{\partial}{\partial\xi}[\dfrac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21})]\\+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)]\dfrac{\partial\psi_{n1}}{\partial\xi}=0,\\\dfrac{\partial\psi_{n1}}{\partial\xi}=0,\ \ y=0,1.\end{cases}\tag{3.4}$$

Separating variables in the form [9]

$$\psi_{n1}(\xi,y)=A(\xi)\phi_n(y),\tag{3.5}$$

where $\phi_n(y)$ satisfies

$$\begin{cases}(U_n-c_0^*)[\dfrac{d^2\phi_n}{dy^2}+(-1)^nF(\phi_1-\phi_2)]+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}\\+(-1)^{n-1}F(U_1-U_2)]\phi_n=0,\\\phi_n(y)=0,\ \ y=0,1.\end{cases}\tag{3.6}$$

Equation (3.6) determines the model structure of a long Rossby wave while the wave amplitude $A(\xi)$ is as yet undetermined to this order. Proceeding to the next order $O(\epsilon^2)$, obtaining $O(\epsilon^2)$:

$$\begin{cases}(U_n-c_0^*)\dfrac{\partial}{\partial\xi}[\dfrac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^nF(\psi_{12}-\psi_{22})]+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}\\+(-1)^{n-1}F(U_1-U_2)]\tfrac{\partial\psi_{n2}}{\partial\xi}\\=-(U_n-c_0^*)\phi_n\dfrac{d^3A}{d\xi^3}-[\phi_n\dfrac{d^3\phi_n}{dy^3}+(-1)^nF\phi_n\dfrac{d}{dy}(\phi_1-\phi_2)+(-1)^{n+1}F(\phi_1-\phi_2)\dfrac{d\phi_n}{dy}\\-\dfrac{d\phi_n}{dy}\dfrac{d^2\phi_n}{dy^2}]A\dfrac{dA}{d\xi}+[c_1\dfrac{d^2\phi_n}{dy^2}+(-1)^nF(\phi_1-\phi_2)]\dfrac{dA}{d\xi},\\\dfrac{\partial\psi_{n2}}{\partial\xi}=0,\ \ y=0,1,\end{cases}\tag{3.7}$$

Using equation (3.6) gives

$$\begin{aligned}
&(U_n-c_0^*)\frac{\partial}{\partial\xi}[\frac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^nF(\psi_{12}-\psi_{22})]+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}\\
&+(-1)^{n+1}F(U_1-U_2)]\tfrac{\partial\psi_{n2}}{\partial\xi}\\
=\ &(U_n-c_0^*)\phi_n\frac{d^3A}{d\xi^3}+\phi_n^2\frac{d}{dy}\{\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n+1}F(U_1-U_2)}{U_n-c_0}\}A\frac{dA}{d\xi}\\
&-\phi_n\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n+1}F(U_1-U_2)}{U_n-c_0}c_1\frac{dA}{d\xi}.
\end{aligned} \tag{3.8}$$

By application of the solvability condition having

$$e_1\frac{d^3A}{d\xi^3}+e_2A\frac{dA}{d\xi}+e_3c_1\frac{dA}{d\xi}=0, \tag{3.9}$$

where

$$\begin{aligned}
e_1&=\int_0^1\sum_{n=1}^2\phi_n^2dy,\\
e_2&=-\int_0^1\sum_{n=1}^2[\frac{\phi_n^3}{U_n-c_0^*}\frac{d}{dy}(\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*})]dy,\\
e_3&=\int_0^1\sum_{n=1}^2[\frac{\phi_n^2}{(U_n-c_0^*)^2}(\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*})dy.
\end{aligned} \tag{3.10}$$

In the above derivation, assuming that any critical level does not exist, i.e., $U_n-c_0^*\neq 0$. Equation (3.9) is an KdV equation including topography forcing term $M\frac{dh}{dy}$. Pursuing another possible expansion.

### 3.2 The Order of $\epsilon$ Expansion

Another solution to equations (2.6)–(2.8) may be sought in the form

$$\begin{aligned}
&\psi_n(\xi,y)=-\int_0^y U_n(y)dy+\epsilon\psi_{n1}(\xi,y)+\epsilon^2\psi_{n2}(\xi,y)+\epsilon^3\psi_{n3}(\xi,y)+\cdots,\\
&c=c_0^*+\epsilon^2c_1^*+\cdots.
\end{aligned} \tag{3.11}$$

Substitution of (3.11) into equations (2.6), (2.7) yields

$$\begin{aligned}
&(U_n-c_0^*)\frac{\partial}{\partial\xi}[\frac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21})]+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}\\
&+(-1)^{n-1}F(U_1-U_2)]\tfrac{\partial\psi_{n1}}{\partial\xi}+\epsilon\{(U_n-c_0^*)[\frac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^nF(\psi_{12}-\psi_{22}]\\
&+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)]\frac{\partial\psi_{n2}}{\partial\xi}\\
&+(\frac{\partial\psi_{n1}}{\partial\xi}\frac{\partial}{\partial y}-\frac{\partial\psi_{n1}}{\partial y})(\frac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^{n+1}F(\psi_{11}-\psi_{21}))\}
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^2\{(U_n-c_0^*)\tfrac{\partial}{\partial\xi}[\tfrac{\partial^2\psi_{n3}}{\partial y^2}+(-1)^nF(\psi_{13}-\psi_{23})]\\
&+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)]\frac{\partial\psi_{n3}}{\partial\xi}+(U_n-c_0^*)\frac{\partial^3\psi_{n1}}{\partial\xi^3}\\
&+(\frac{\partial\psi_{n1}}{\partial\xi}\frac{\partial}{\partial y}-\frac{\partial\psi_{n1}}{\partial y}\frac{\partial}{\partial\xi})(\frac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^{n+1}F(\psi_{12}-\psi_{22}))\\
&+(\frac{\partial\psi_{n2}}{\partial\xi}\frac{\partial}{\partial y}-\frac{\partial\psi_{n2}}{\partial y}\frac{\partial}{\partial\xi}-c_1^*\frac{\partial}{\partial\xi})(\frac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^{n+1}F(\psi_{11}-\psi_{21}))\}+O(\epsilon^3)=0.
\end{aligned}
\tag{3.12}
$$

From equation (3.12), the problem of the lowest order are written as $O(\epsilon^0)$:

$$
\begin{cases}
(U_n-c_0^*)\dfrac{\partial}{\partial\xi}[\dfrac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21})]\\
+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)]\dfrac{\partial\psi_{n1}}{\partial\xi}=0,\\
\dfrac{\partial\psi_{n1}}{\partial\xi}=0, y=0,1.
\end{cases}
\tag{3.13}
$$

Separating variables in the form

$$
\psi_{n1}(\xi,y)=A(\xi)\phi_n(y)\ (n=1,2),
\tag{3.14}
$$

here $\phi_n(y)$ satisfies

$$
\begin{cases}
(U_n-c_0^*)[\dfrac{d^2\phi_n}{dy^2}+(-1)^nF(\phi_1-\phi_2)]+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}\\
+(-1)^{n-1}F(U_1-U_2)]\phi_n=0,\\
\phi_n(y)=0,\ \ y=0,1.
\end{cases}
\tag{3.15}
$$

That the lowest order problem agrees with that of the KdV equation case. From the next order problem, we obtain $O(\epsilon^1)$:

$$
\begin{cases}
(U_n-c_0^*)\dfrac{\partial}{\partial\xi}[\dfrac{\partial^2\psi_{n2}}{\partial y^2}+(-1)^nF(\psi_{12}-\psi_{22})]+[\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}\\
+(-1)^{n-1}F(U_1-U_2)]\frac{\partial\psi_{n2}}{\partial\xi}\\
-(\dfrac{\partial\psi_{n1}}{\partial\xi}\dfrac{\partial}{\partial y}-\dfrac{\partial\psi_{n1}}{\partial y}\dfrac{\partial}{\partial\xi})(\dfrac{\partial^2\psi_{n1}}{\partial y^2}+(-1)^nF(\psi_{11}-\psi_{21}))=0,\\
\dfrac{\partial\psi_{n2}}{\partial\xi}=0,\ \ y=0,1.
\end{cases}
\tag{3.16}
$$

The solution of equation (3.16) is [11]

$$
\psi_{n2}(\xi,y)=\frac{1}{2}A^2(\xi)\chi_n(y)\ (n=1,2),
\tag{3.17}
$$

where $\chi_n(y)$ satisfies

$$
\begin{cases}
(U_n-c_0^*)(\dfrac{d^2\chi_n}{dy^2}+(-1)^nF(\chi_1-\chi_2))+(\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}\\
+(-1)^{n-1}F(U_1-U_2))\chi_n\\
=\dfrac{d}{dy}(\dfrac{\beta+M\dfrac{(-1)^n+1}{2}\dfrac{dh}{dy}-\dfrac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*})\phi_n^2,\\
\chi_n(0)=\chi_n(1)=0.
\end{cases}
\tag{3.18}
$$

The last problem yields $O(\epsilon^2)$:

$$\begin{aligned}
&(U_n-c_0^*)\frac{\partial}{\partial\xi}[\frac{\partial^2\psi_{n3}}{\partial y^2}+(-1)^nF(\psi_{13}-\psi_{23})]+[\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}\\
&+(-1)^{n-1}F(U_1-U_2)]\frac{\partial\psi_{n3}}{\partial\xi}\\
=\ &-(U_n-c_0^*)\phi_n\frac{d^3A}{d\xi^3}+[\frac{3}{2}\phi_n\chi_n\frac{d}{dy}(\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*})\\
&-\frac{1}{2}\phi_n^3\frac{d}{dy}(\frac{1}{U_n-c_0^*}\frac{d}{dy}(\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*}))]A^2\frac{dA}{d\xi}\\
&-[\phi_n\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*}]c_1^*\frac{dA}{d\xi}.
\end{aligned}\tag{3.19}$$

As in equation (3.8), application of the solvability condition yields, the mKdV equation

$$e_1^*\frac{d^3A}{d\xi^3}+e_2^*A^2\frac{dA}{d\xi}+e_3^*c_1^*\frac{dA}{d\xi}=0,\tag{3.20}$$

where

$$\begin{aligned}
e_1^*\ &=\ \int_0^1\sum_{n=1}^2\phi_n^2dy=e_1,\\
e_2^*\ &=\ -\int_0^1\sum_{n=1}^2[\frac{3}{2}\frac{\phi_n^2\chi_n}{U_n-c_0^*}\frac{d}{dy}\{\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*}\}\\
&\quad-\frac{1}{2}\frac{\phi_n^4}{U_n-c_0^*}\frac{d}{dy}(\frac{1}{U_n-c_0^*}\frac{d}{dy}(\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0^*}))]dy,\\
e_3^*\ &=\ \int_0^1\sum_{n=1}^2[\frac{\phi_n^2}{(U_n-c_0^*)^2}\frac{\beta+M\frac{(-1)^n+1}{2}\frac{dh}{dy}-\frac{d^2U_n}{dy^2}+(-1)^{n-1}F(U_1-U_2)}{U_n-c_0}]dy=e_3.
\end{aligned}\tag{3.21}$$

## 4 The Coefficient of KdV and mKdV Equation under the Change of $\beta$

In the section, discuss the change of $\beta$ and the topography. The $\beta$ and the topographic basic field chosen here is [17, 22]

$$\beta=\beta_0-\delta_0y,\ \ \frac{dh}{dy}=h_{y0}+by,\tag{4.1}$$

where $\beta_0=\dfrac{2\Omega\cos\varphi_0}{a}$, $\delta_0=\dfrac{2\Omega\sin\varphi_0}{a^2}$, $h_{y0}$ =constant, it is a slow change of the orographic slope, $b\ll 1$, it is a measure of the shear basic flow and topography that are weakly change.

The KdV equation has the solitary waves solution [11]

$$A(\xi) = \mathrm{sgn}(e_1, e_2)\mathrm{sech}^2(\lambda\xi), \ \ c_1 = -\frac{|e_2|}{3e_3}\mathrm{sgn}(e_1). \tag{4.2}$$

The wave steepness $\lambda$ is

$$\lambda = |\frac{e_2}{12e_1}|^{\frac{1}{2}}.$$

The solitary wave solution of the mKdV equation [11]

$$A(\xi) = \pm\mathrm{sech}^2(\lambda^*\xi), \ \ c_1^* = -\frac{|e_2^*|}{6e_3^*}, \tag{4.3}$$

where

$$\lambda^* = |\frac{e_2}{6e_1}|^{\frac{1}{2}}.$$

In order to determine these coefficients of the inhomogeneous KdV and mKdV equation, substitution of (4.1) into (3.10), (3.21), are approximate evaluated to yield

$$\begin{aligned}
&e_1 = e_1^* = \int_0^1 \sum_{n=1}^2 \phi_n^2 dy,\\
&e_2 = -\int_0^1 \sum_{n=1}^2 [\frac{\phi_n^3}{U_n - c_0^*}\frac{d}{dy}(\frac{\beta_0 - \delta_0 y + M\frac{(-1)^n+1}{2}(h_{yo}+by) - \frac{d^2U_n}{dy^2} + (-1)^{n-1}F(U_1-U_2)}{U_n - c_0^*})]dy,\\
&e_2^* == -\int_0^1 \sum_{n=1}^2 [\frac{3}{2}\frac{\phi_n^2\chi_n}{U_n - c_0^*}\frac{d}{dy}\{\frac{\beta_0 - \delta_0 y + M\frac{(-1)^n+1}{2}(h_{y0}+by) - \frac{d^2U_n}{dy^2} + (-1)^{n-1}F(U_1-U_2)}{U_n - c_0^*}\}\\
&-\frac{1}{2}\frac{\phi_n^4}{U_n - c_0^*}\frac{d}{dy}(\frac{1}{U_n - c_0^*}\frac{d}{dy}(\frac{\beta_0 - \delta_0 y + M\frac{(-1)^n+1}{2}(h_{yo}+by) - \frac{d^2U_n}{dy^2} + (-1)^{n-1}F(U_1-U_2)}{U_n - c_0^*}))]dy,\\
&e_3 = e_3^* = \int_0^1 \sum_{n=1}^2 [\frac{\phi_n^2}{(U_n - c_0^*)^2}\frac{\beta_0 - \delta_0 y + M\frac{(-1)^n+1}{2}(h_{y0}+by) - \frac{d^2U_n}{dy^2} + (-1)^{n-1}F(U_1-U_2)}{U_n - c_0^*}]dy.
\end{aligned} \tag{4.4}$$

For the baroclinic mode $e_2, e_2^*$ tell us that $\delta_0 \neq 0, b \neq 0$ in spite of the absence of the basic flow $U_n$ =constant, the KdV (mKdV) solition can exist. The change of $\beta$ and topography is the important factors that induce Rossby solitary waves.

## 5 Concluding Remarks

These solitary Rossby waves of the two layer fluid are described by the homogeneous KdV or mKdV equation depending on the baroclinicity fluid, when the change of $\beta$ and the fluid with the topography. In the general case where the basic flow has the shear, Rossby solitary waves are described by the KdV (mKdV) equation, but, if considering the change of $\beta$, the Rossby solitary can exit in the absence of horizontal shear in the basic flow. The inhomogeneous terms of the KdV and mKdV equation are induce by the bottom topography effect. The internal rotational Froude number $F$ has a certain effect on the Rossby solitary waves steepness, with the decrease of $F$, the solitary waves steepness is increase. There is no effect on the basic flow pattern of solitary waves. The horizontal shear and the vertical shear of the basic flow are also the factor causing solitary waves steepness increase. Finally, the further modifications the Rossby waves will be considered; the instability of the Rossby solitary with the change $\beta$; the Rossby waves in the $n$-level model.

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# 两层流体中具有$\beta$变化和地形影响的Rossby波

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**摘 要**: 本文研究了两层流体中具有变化的Rossby参数和地形 Rossby 波的问题. 利用行波法和摄动的方法, 获得了 Rossby 波振幅满足齐次 KdV 方程和齐次 mKdV 方程, 推广了 Rossby 参数和地形对 Rossby 孤立波的影响.

**关 键 词**: 变化的$\beta$; 齐次 KdV 方程; 齐次 mKdV 方程; 地形

MR(2010)**主 题 分 类 号**: 76B65; 86A10 **中 图 分 类 号**: O351; P433