Vol. 37 (2017)
No. 4

数 学 杂 志
J. of Math. (PRC)

FINITE GROUPS WHOSE ALL MAXIMAL SUBGROUPS ARE SMSN-GROUPS

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Abstract: A finite group G is called an SMSN-group if its 2-maximal subgroups are subnormal in G. In this paper, the author investigates the structure of finite groups which are not SMSN-groups but all their proper subgroups are SMSN-groups. Using the idea of local analysis, a complete classification of this kind of groups is given, which generalizes some results of the structure of finite groups.

Keywords: power automorphisms; nilpotent groups; minimal non-nilpotent groups; minimal non-SMSN-groups

2010 MR Subject Classification: 20D10; 20E34

Document code: A **Article ID:** 0255-7797(2017)04-0714-09

1 Introduction

All groups in this paper are finite and our notation is standard (see [1]). Let Σ be an abstract group theoretical property, for example, nilpotency, supersolvability, solvability, etc. If all proper subgroups of a group G have the property Σ but G does not have the property Σ, then G is called a minimal non-Σ-group.

One of the hottest topics in group theory is to determinate the structure of minimal non-Σ-groups and many meaningful results about this topic were obtained. The specific papers about this topic can refer to [2–10].

The aim of this paper is to study the structure of a kind of minimal non-Σ-groups. We call the groups whose 2-maximal subgroups are subnormal SMSN-groups. A group G is a minimal non-SMSN-group if every proper subgroup of G is an SMSN-group but G itself is not, and we classify the minimal non-SMSN-groups completely.

* **Received date:** 2015-09-06 **Accepted date:** 2015-12-03

Foundation item: Supported by National Natural Science Foundation of China (11661031); Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-Aged Teachers and Presidents; "333" Project of Jiangsu Province(BRA2015137); "521" Project of Lianyungang City.

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2 Preliminaries

In this section, we give some definitions and some lemmas needed in this paper.

Lemma 2.1 (see [5, Lemma 5]) Every 2-maximal subgroup of a group G is subnormal if and only if either G is nilpotent or G is a Schmidt group with abelian Sylow subgroups.

Lemma 2.2 If G is a solvable minimal non-SMSN-group, then $|\pi(G)| \leq 3$.

Proof If $|\pi(G)| \geq 4$, then every maximal subgroup of G has at least three prime divisors since G is solvable. Applying Lemma 2.1, G is minimal non-nilpotent, a contradiction. Hence $|\pi(G)| \leq 3$.

Lemma 2.3 (see [10]) Any minimal simple group (non-abelian simple group all of whose proper subgroups are solvable) is isomorphic to one of the following simple groups

(1) $\mathrm{PSL}(3,3)$;

(2) $\mathrm{PSL}(2,p)$, where p is a prime with $p>3$ and $5 \nmid p^2-1$;

(3) $\mathrm{PSL}(2,2^q)$, where q is a prime;

(4) $\mathrm{PSL}(2,3^q)$, where q is an odd prime;

(5) The Suzuki group $\mathrm{Sz}(2^q)$, where q is an odd prime.

Lemma 2.4 (see [11]) Suppose that p'-group H acts on a p-group G. Let

$$\Omega(G) = \begin{cases} \Omega_1(G), & p>2, \\ \Omega_2(G), & p=2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

Lemma 2.5 (see [7, Lemma 2.9]) If a p-group G of order p^{n+1} has a unique non-cyclic maximal subgroup, then G is isomorphic to one of the following groups

(I) $C_{p^n} \times C_p = \langle a, b \mid a^{p^n} = b^p = 1, [a,b]=1\rangle$, where $n \geq 2$;

(II) $M_{p^{n+1}} = \langle a, b \mid a^{p^n} = b^p = 1, b^{-1}ab = a^{1+p^{n-1}}\rangle$, where $n \geq 2$ and $n \geq 3$ if $p=2$.

Lemma 2.6 (see [12]) Let G be a group and H a nilpotent subnormal subgroup of G. Then G contains a nilpotent normal subgroup of G containing H.

3 Main Results

In this section, we give the specific classification of the minimal non-SMSN-groups.

Theorem 3.1 A non-solvable group G is a minimal non-SMSN-group if and only if G is isomorphic to A_5, where A_5 is the alternating group of degree 5.

Proof We only prove the necessity part.

Since G is a non-solvable group whose maximal subgroups are all SMSN-groups, then G is a minimal non-solvable group by Lemma 2.1, and so $G/\Phi(G)$ is a minimal simple group.

Case 1 Assume $\Phi(G)=1$. Then G is isomorphic to one of the simple groups mentioned in Lemma 2.3.

Let $G \cong \mathrm{PSL}(3,3)$. Then G has a subgroup which is isomorphic to S_4 by [13, p.13], but S_4 is not an SMSN-group, a contradiction. So $G \ncong \mathrm{PSL}(3,3)$.

Let $G \cong \mathrm{PSL}(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2-1$. If $p \geq 13$, then there exists a maximal subgroup of G which is isomorphic to a dihedral group D_{p-1} or D_{p+1} by [14, Corollary 2.2]. Certainly, 4 divides the order of either D_{p-1} or D_{p+1}, say A, and A is not an SMSN-group applying Lemma 2.1, a contradiction. If $p = 7$, then $p^2 \equiv 1(\mathrm{mod}\ 16)$. By [14, Corollary 2.2], G has a subgroup which is isomorphic to S_4, but S_4 is not an SMSN-group, a contradiction. Hence $p = 5$ and $G \cong A_5$.

Let $G \cong \mathrm{PSL}(2, 2^q)$. By [14, Corollary 2.2], G has maximal subgroups: the dihedral groups of order $2(2^q \pm 1)$; the Frobenius group H of order $2^q(2^q-1)$; the alternating group A_4 of degree 4 when $q = 2$. Clearly, $G \cong A_5$ when $q = 2$ and it is a minimal non-SMSN-group. If $q > 2$, then $3 \mid 2^q+1$. It follows from Lemma 2.1 that G is not a minimal non-SMSN-group.

Let $G \cong \mathrm{PSL}\ (2, 3^q)$. Similar arguments as above, G has a dihedral group B whose Sylow 2-subgroups are neither cyclic nor normal, which contradicts the fact that B is an SMSN-group. So $G \ncong \mathrm{PSL}(2, 3^q)$.

Let $G \cong \mathrm{Sz}(2^q)$. By [15, Theorem 9], G has a Frobenius group K of order $4(2^q \pm 2^{\frac{q+1}{2}}+1)$, but the Sylow 2-subgroups of K are neither cyclic nor normal, a contradiction. So $G \ncong \mathrm{Sz}(2^q)$.

Case 2 Assume $\Phi(G) \neq 1$. It is easy to see that $\Phi(G/\Phi(G)) = 1$ and $G/\Phi(G)$ is a non-solvable minimal non-SMSN-group. Similar arguments as above and by induction, $G/\Phi(G) \cong A_5$. Hence G has two non-nilpotent maximal subgroups M_1 and M_2 such that $M_1/\Phi(G) \cong A_4$ and $M_2/\Phi(G) \cong D_{10}$, where A_4 is the alternating group of degree 4 and D_{10} is the dihedral group of order 10. Since M_1 and M_2 are SMSN-groups, they are minimal non-nilpotent by Lemma 2.1. It makes $|G| = 2^a \cdot 3 \cdot 5$ and $|\Phi(G)| = 2^{a-2}$, where $a \geq 3$. By Lemma 2.1 again, the Sylow 2-subgroups of M_1 are elementary abelian. At the same time, the Sylow 2-subgroups of M_2 are cyclic whose orders are more than 2 by Lemma 2.1, a contradiction.

Theorem 3.2 The minimal non-SMSN-group G whose order has exactly two prime divisors p and q is exactly one of the following types (P and Q are Sylow subgroups)

(1) $G = \langle x, y \mid x^p = y^{q^n} = 1, y^{-1}xy = x^i \rangle$, where $i^q \not\equiv 1(\mathrm{mod}\ p)$, $i^{q^2} \equiv 1(\mathrm{mod}\ p)$, $p > q$, $n \geq 2$ and $0 < i < p$;

(2) $G = \langle x, y \mid x^{pq} = y^q = 1, y^{-1}xy = x^i \rangle$, where $p \equiv 1(\mathrm{mod}\ q)$, $i \equiv 1(\mathrm{mod}\ q)$, $i^q \equiv 1(\mathrm{mod}\ p)$ and $1 < i < p$;

(3) $G = \langle x, y \mid x^{4p} = 1, y^2 = x^{2p}, y^{-1}xy = x^{-1} \rangle$;

(4) $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^i, [x, z] = 1, [y, z] = 1 \rangle$ where $p > q$, $i \not\equiv 1(\mathrm{mod}\ p)$, $i^q \equiv 1(\mathrm{mod}\ p)$ and $n \geq 3$;

(5) $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^i, [x, z] = 1, z^{-1}yz = y^{1+q^{n-2}} \rangle$, where $p > q$, $i \not\equiv 1(\mathrm{mod}\ p)$, $i^q \equiv 1(\mathrm{mod}\ p)$, $n \geq 3$ and $n \geq 4$ if $q = 2$;

(6) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle y \rangle$ with $|y| = q^n$ and $n \geq 2$, $\langle y^q \rangle$ acts irreducibly on P and $\langle y^{q^2} \rangle$ centralizes P;

(7) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group and

$r \geq 2$, $Q = \langle y \rangle$ with $|y| = q^n$ and $n \geq 1$, $[a_1, Q] = 1$, Q acts irreducibly on $\langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ and $\Phi(Q)$ centralizes P;

(8) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle y \rangle$ with $|y| = q^n$ and $n \geq 1$, Q acts irreducibly on $\langle a_1 \rangle \times \cdots \times \langle a_{l-1} \rangle$ and $\langle a_l \rangle \times \cdots \times \langle a_r \rangle$ with $l \geq 2$, and $\Phi(Q)$ centralizes P;

(9) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ $(r \geq 2)$ is a p-group with $|a_1| = |a_2| = \cdots = |a_r| = p^2$, $Q = \langle y \rangle$ with $|y| = q^n$ and $n \geq 1$, Q acts irreducibly on $\Phi(P)$, $\Phi(Q)$ centralizes P, and $G/\Phi(P)$ is a minimal non-abelian group;

(10) $G = P \rtimes Q$, where P is a non-abelian special p-group of rank $2m$, the order of p modulo q being $2m$, $Q = \langle y \rangle$ is cyclic of order $q^r > 1$, y induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q-module, and y centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$;

(11) $G = P \rtimes Q$, where P is a non-abelian special p-group with $\exp(P) \leq p^2$ and $|\Phi(P)| \geq p^2$, $Q = \langle y \rangle$ with $|y| = q^n$ and $n \geq 1$, Q acts irreducibly on $\Phi(P)$, $\Phi(Q)$ centralizes P, and $G/\Phi(P)$ is a minimal non-abelian group;

(12) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle a, b \mid a^q = b^q = 1,\ [a, b] = 1 \rangle, [P, b] = 1$, $\langle a \rangle$ acts irreducibly on P;

(13) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle a, b \mid a^q = b^q = 1, [a, b] = 1 \rangle$, $\langle a \rangle$ and $\langle b \rangle$ act irreducibly on P;

(14) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle, [P, b] = 1$ and $\langle a \rangle$ acts irreducibly on P;

(15) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$, $[P, a^2] = 1$, $\langle a \rangle$ and $\langle b \rangle$ act irreducibly on P;

(16) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, [y, z] = 1 \rangle$ with $n \geq 3$, $z \in Z(G)$, $\langle y \rangle$ acts irreducibly on P and $\langle y^q \rangle$ centralizes P;

(17) $G = P \rtimes Q$, where $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle$ is an elementary abelian p-group with $r \geq 2$, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, z^{-1}yz = y^{1+q^{n-2}} \rangle$ with $n \geq 3$ and $n \geq 4$ if $q = 2$, $\langle z \rangle \leq C_G(P)$, $\langle y \rangle$ acts irreducibly on P and $\langle y^q \rangle$ centralizes P;

(18) $G = PQ$, where $P = \langle x \rangle \ntrianglelefteq G$ with $|P| = p$, $Q = (\langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_{r-1} \rangle) \rtimes \langle a_r \rangle$ is a non-normal q-group with $r \geq 3$, $F(G) = O_q(G) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_{r-1} \rangle$, P acts irreducibly on $O_q(G)$, $a_r^{-1} x a_r = x^i$ and $p > q$, where i is a primitive q-th root of unity modulo p, $F(G)$ is the Fitting subgroup of G.

Proof If G is a solvable minimal non-SMSN-group whose order has exactly two prime divisors, then we assume $G = PQ$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.

Assume that P and Q are neither cyclic nor normal in G. The solvability of G implies that G has a normal subgroup M of prime index, say q. Let M_p be a Sylow p-subgroup of M. Since M is an SMSN-group, we have that M_p is either cyclic or normal in M by Lemma

2.1. Clearly M_p must be normal in M since it is also a Sylow p-group of G. Now it follows from M_p char $M \trianglelefteq G$ that $M_p \trianglelefteq G$, a contradiction. So G has a Sylow subgroup which is either cyclic or normal.

(1) Assume that P and Q are cyclic and let $P = \langle x \rangle$ and $Q = \langle y \rangle$ with $|x| = p^m$, $|y| = q^n$ and $p > q$. In this case, $y^{-1}xy = x^i$ with $i^{q^n} \equiv 1(\bmod\, p^m)$, $0 < i < p^m$ and $(p^m, q^n(i-1)) = 1$. Considering the maximal subgroups $P\langle y^q \rangle$ and $\langle x^p \rangle Q$ of G, if $\langle x^p \rangle Q = \langle x^p \rangle \times Q$, then by Lemma 2.4, G is nilpotent, a contradiction. This implies $\langle x^p \rangle Q = \langle x^p \rangle \rtimes Q$. By Lemma 2.1, $x^p = 1$, $\langle y^q \rangle$ is not normal in G, but $\langle y^{q^2} \rangle$ is normal in G. So $i^q \not\equiv 1(\bmod\, p)$, $i^{q^2} \equiv 1(\bmod\, p)$ and G is of type (1).

(2) Assume that P is a cyclic normal subgroup of G and Q is neither cyclic nor normal in G. If $q > p$, then by Burnside's theorem [1, 10.1.8], $Q \trianglelefteq G$, a contradiction. So $q < p$. If Q has two non-cyclic maximal subgroups Q_1 and Q_2, then by Lemma 2.1, $PQ_1 = P \times Q_1$, $PQ_2 = P \times Q_2$ and so $Q = Q_1Q_2$ is normal in G, a contradiction. Therefore, every maximal subgroup of Q is cyclic or Q has a unique non-cyclic maximal subgroup, and so Q is an elementary abelian q-group of order q^2, the quaternion group Q_8 or one of the types in Lemma 2.5.

Case 1 Assume $P = \langle z \rangle$ and $Q = \langle a, b \mid a^q = b^q = 1, [a, b] = 1 \rangle$. If $\langle a \rangle$ and $\langle b \rangle$ acting on P by conjugation are both trivial, then G is nilpotent, a contradiction. Therefore, we may assume that $\langle a \rangle$ acting on P by conjugation is non-trivial. By Lemma 2.1, $z^p = 1$. If $C_G(P) = P$, then $G/C_G(P)$ is an elementary abelian q-group of order q^2. However, $G/C_G(P) \lesssim \mathrm{Aut}(P)$, and $\mathrm{Aut}(P)$ is cyclic, a contradiction. Hence b is contained in $C_G(P)$. Clearly, $C_G(P) = \langle x \rangle$, $y^{-1}xy = x^i$, $|x| = pq$, $y = a$, $q|p-1$, $i \equiv 1(\bmod\, q)$ and $i^q \equiv 1(\bmod\, p)$, where $x = zb$ is a generator of $C_G(P)$. So G is of type (2).

Case 2 Assume $P = \langle z \rangle$ and $Q = Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Similar arguments as Case 1, we have that $z^p = 1$, b is contained in $C_G(P)$ and $|Z(G)| = 2$. So $C_G(P) = \langle x \rangle$ with $|x| = 4p$, $y = a$, $y^{-1}xy = x^i$ and $i^2 \equiv 1(\bmod\, 4p)$, where $x = zb$ is a generator of $C_G(P)$. By computations, G is of type (3).

Case 3 Assume that $P = \langle x \rangle$ and Q is the type of Lemma 2.5 (I) with $|Q| = q^n$. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, [y, z] = 1 \rangle$, where $n \geq 3$. Then Q has maximal subgroups $H = \langle y \rangle$, $K_0 = \langle y^q, z \rangle$ and $K_s = \langle y^q, zy^s \rangle = \langle zy^s \rangle$ with $s = 1, \cdots, q-1$, where K_0 is the unique non-cyclic maximal subgroup of Q. By hypothesis and Lemma 2.1, $PH \neq P \times H$, $PK_0 = P \times K_0$ and $x^p = 1$. Hence $G = \langle x, y, z \mid x^p = y^{q^{n-1}} = z^q = 1, y^{-1}xy = x^i, [x, z] = 1, [y, z] = 1 \rangle$, where $i \not\equiv 1(\bmod\, p)$, $i^q \equiv 1(\bmod\, p)$. So G is of type (4).

Case 4 Assume that $P = \langle x \rangle$ and Q is the type of Lemma 2.5 (II) with $|Q| = q^n$. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, z^{-1}yz = y^{1+q^{n-2}} \rangle$, where $n \geq 3$ and $n \geq 4$ if $q = 2$. In the similar way as above, we have that $x^p = 1$, $\langle z \rangle \leq C_G(P)$ and $y^{-1}xy = x^i$, where $i \not\equiv 1(\bmod\, p)$ and $i^q \equiv 1(\bmod\, p)$. So G is of type (5).

(3) Assume that P is a non-cyclic normal subgroup of G and $Q = \langle y \rangle$ is non-normal cyclic subgroup of G with $|y| = q^n$. If there exists a subgroup P^* of P with $1 < \Phi(P) < P^* < P$ such that $P^*Q = QP^*$, then $P^* \trianglelefteq G$ since P^* is subnormal in G. By Maschke's theorem

[1, 8.1.2], P has a subgroup K with $1 < K < P$ such that $P/\Phi(P) = P^*/\Phi(P) \times K/\Phi(P)$, $K \trianglelefteq G$, $K \neq P^*$, and at least one of P^*Q and KQ is a non-nilpotent SMSN-group. By Lemma 2.1, it is easy to see that $P^* \cap K = \Phi(P) = 1$, a contradiction. Hence $\Phi(P) = 1$ or $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$ when $\Phi(P) \neq 1$.

Case 1 Assume $\Phi(P) = 1$. If P is a minimal normal subgroup of G, then by hypothesis, the maximal subgroup $P\Phi(Q)$ of G is non-nilpotent. By Lemma 2.1, $\langle y^q \rangle$ acts irreducibly on P and $[P, y^{q^2}] = 1$. So G is of type (6). If P has a non-trivial proper subgroup P_1 which is normal in G, then there exists a subgroup P_2 of P such that $P = P_1 \times P_2$ and $P_2 \trianglelefteq G$ by Maschke's theorem [1, 8.1.2]. Clearly, at least one action that $\langle y \rangle$ acts on P_1 and P_2 by conjugation is non-trivial. If $P_1Q = P_1 \times Q$ and $P_2Q = P_2 \rtimes Q$, then by Maschke's theorem [1, 8.1.2] and Lemma 2.1, it is easy to see that $|P_1| = p$, $[P, y^q] = 1$ and G is of type (7). If $P_1Q = P_1 \rtimes Q$ and $P_2Q = P_2 \rtimes Q$, then by Lemma 2.1, $\langle y \rangle$ acts irreducibly on P_1 and P_2, and $[P, y^q] = 1$. So G is of type (8).

Case 2 Assume $\Phi(P) > 1$ and $Z(P) = P$. By the same arguments as the beginning of (3), it is easy to see that $\Phi(P)$ is the unique normal subgroup of G which is contained in P, and so P is a homocyclic p-group (a product of some cyclic subgroups of the same order). By Lemma 2.1 and Lemma 2.4, we have easily that the exponent of P is p^2, one maximal subgroup $P\Phi(Q)$ of G is nilpotent. Hence another maximal subgroup $\Phi(P)Q$ is non-nilpotent, and $\langle y \rangle$ acts irreducibly on $\Phi(P)$. Clearly the quotient group $G/\Phi(P)$ is a minimal non-abelian group. So G is of type (9).

Case 3 Assume $\Phi(P) > 1$ and $Z(P) < P$. Similarly, $\Phi(P) = Z(P) = P'$ is the unique non-trivial characteristic subgroup of P, that is, P is a special p-group with $\exp(P) \leq p^2$ and $P\Phi(Q)$ is nilpotent. If $\Phi(P)Q$ is nilpotent also, then by a result in [4, Theorem 2], G is of type (10). If $|\Phi(P)| = p$ and $p < q$, then G belongs to type (10). If $\Phi(P)Q$ is non-nilpotent with $|\Phi(P)| = p$ and $p > q$, then G is minimal non-supersolvable. Examining a result in [4, Theorem 10], G is not isomorphic to anyone of them. If $\Phi(P)Q$ is non-nilpotent with $|\Phi(P)| \geq p^2$, then the quotient group $G/\Phi(P)$ is a minimal non-abelian group. So G is of type (11).

(4) Assume that P is a non-cyclic normal subgroup of G and Q is neither cyclic nor normal in G. If $\Phi(P) > 1$, then by Lemma 2.1, PQ_1 and PQ_2 are both nilpotent and so G is nilpotent, a contradiction, where Q_1 and Q_2 are two distinct maximal subgroups of Q. Hence P is an elementary abelian p-group of order p^r with $r \geq 2$. Similar arguments as in (2), Q is an elementary abelian q-group of order q^2, the quaternion group Q_8 or one of the types in Lemma 2.5.

Case 1 Let $Q = \langle a, b \mid a^q = b^q = 1, [a, b] = 1 \rangle$. Clearly, there exists a non-trivial automorphism that $\langle a \rangle$ or $\langle b \rangle$ acts on P by conjugation. We may assume that $\langle a \rangle$ acting on P by conjugation is non-trivial and $\langle b \rangle$ acting on P by conjugation is trivial. So G is of type (12). If $\langle a \rangle$ and $\langle b \rangle$ acting on P by conjugation are both non-trivial, then G is of type (13).

Case 2 Let $Q = Q_8 = \langle a, b \mid a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$. Similar arguments as above, G is of either type (14) or type (15).

Case 3 Let Q be as in Lemma 2.5 (I) with $|Q| = q^n$. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, [y, z] = 1\rangle$, where $n \geq 3$. Similar arguments as Case 3 in (2), $\langle y\rangle$ acts irreducibly on P, $[P, y^q] = 1$ and $z \in Z(G)$. So G is of type (16).

Case 4 Let Q be as in Lemma 2.5 (II) with $|Q| = q^n$. Namely, $Q = \langle y, z \mid y^{q^{n-1}} = z^q = 1, z^{-1}yz = y^{1+p^{n-2}}\rangle$, where $n \geq 3$ and $n \geq 4$ if $p = 2$. Similar arguments as Case 4 in (2), $\langle y\rangle$ acts irreducibly on P, $[P, y^q] = 1$ and $\langle z\rangle \leq C_G(P)$. So G is of type (17).

(5) Assume that $P = \langle x\rangle$ is a non-normal cyclic subgroup of G and Q is neither cyclic nor normal in G. Clearly $p > q$. The solvability of G implies that G has a normal subgroup M of prime index. If $|G : M| = p$, then it is easy to see that G has a normal Sylow q-group since M is an SMSN-group and applying Lemma 2.1, a contradiction. Therefore, $|G : M| = q$. If there exists a cyclic Sylow q-subgroup M_q of M, then M has a normal Sylow p-subgroup M_p, and so M_p is a normal Sylow p-subgroup of G, a contradiction. Hence M_q is non-cyclic and $|Q| \geq q^3$. By Lemma 2.1, M_q is normal in M and M_p has a maximal subgroup P_1 such that P_1 is normal in M, where M_p is a Sylow p-subgroup of M. Hence M_q and P_1 are both subnormal in G. By Lemma 2.6, $F(G) = P_1 \times M_q = O_p(G) \times O_q(G)$. Clearly, $O_p(G) = \langle x^p\rangle$ and $O_q(G) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{r-1}\rangle$ is an elementary abelian q-group with $|O_q(G)| \geq q^2$. If $N_G(P)$ is nilpotent, then $N_G(P) = C_G(P)$ since P is cyclic. By Burnside Theorem [1, 10.1.8], G is p-nilpotent, a contradiction. Hence $N_G(P) = P\langle a_r\rangle$ is a Schmidt subgroup of G, and so $P\langle a_r^q\rangle$ is nilpotent with $|P| = p$ by Lemma 2.1, where a_r is a q-element. Since M is a Schmidt subgroup of G also, $O_q(G)$ is a minimal normal subgroup of G and $G = MN_G(P)$. Hence $O_q(G)\langle a_r\rangle$ is a Sylow q-subgroup of G and $O_q(G) \cap \langle a_r\rangle = \langle a_r^q\rangle$. Furthermore, $|a_r| = q$ since $O_q(G)P$ is a Schmidt subgroup of G. If $\Phi(Q) = 1$, then Q is abelian. Hence $N_G(Q) = C_G(Q) = Q \lessdot G$. So G is q-nilpotent, a contradiction. If $\Phi(Q) \neq 1$, then $Q = (\langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_{r-1}\rangle) \rtimes \langle a_r\rangle$. So G is of type (18).

Conversely, it is clear that the groups of types (1)–(18) are minimal non-SMSN-groups.

Theorem 3.3 The solvable minimal non-SMSN-group G whose order has exactly three prime divisors p, q and r is exactly one of the following types (P, Q and R are Sylow subgroups)

(1) $G = (P \times Q) \rtimes R$, where $[P, R] = 1$, $|P| = p$, Q is an elementary abelian q-group, $R = \langle a\rangle$ is cyclic, R acts irreducibly on Q and $\langle a^r\rangle$ centralizes Q;

(2) $G = (P \times Q) \rtimes R$, where P and Q are both elementary abelian, $R = \langle a\rangle$ is cyclic, R acts irreducibly on P and Q, $\langle a^r\rangle$ centralizes PQ;

(3) $G = P \rtimes (Q \times R)$, where P is an elementary abelian p-group, Q is a group of order q, R is a group of order r, Q and R act irreducibly on P, respectively.

Proof It is easy to see that G has at least one normal Sylow subgroup and we assume that $G = PQR$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, $R \in \mathrm{Syl}_r(G)$, and $P \trianglelefteq G, R \ntrianglelefteq G$. Clearly, we only need consider the following cases.

Case 1 If $Q \trianglelefteq G, PR = P \times R$ and $QR = Q \rtimes R$, then QR is an SMSN-group. By Lemma 2.1, Q is an elementary abelian q-group and $R = \langle a\rangle$ is cyclic. If $|P| \neq p$, then P_1QR is nilpotent by Lemma 2.1 again, a contradiction, where $1 < P_1 < P$. Hence $|P| = p$ and G

is of type (1).

Similarly, if $Q \trianglelefteq G, PR = P \rtimes R$ and $QR = Q \times R$, then G is isomorphic to type (1) also. If $Q \trianglelefteq G, PR = P \rtimes R$ and $QR = Q \rtimes R$, it is easy to see that G is of type (2).

Case 2 If $Q \ntrianglelefteq G$, $PQ = P \rtimes Q$, $PR = P \times R$, $QR = Q \rtimes R$, then by Lemma 2.1, P is an elementary abelian p-group, $Q = \langle a \rangle$ is a cyclic group of order q, $q > r$ and $R = \langle b \rangle$ is cyclic. Since $C_G(P) = P \times R \trianglelefteq N_G(P) = G$, $R \trianglelefteq G$, a contradiction.

If $Q \ntrianglelefteq G$, $PQ = P \rtimes Q$, $PR = P \rtimes R$, $QR = Q \times R$, and $\Phi(R) \neq 1$, then $PQ\Phi(R)$ is nilpotent by Lemma 2.1, a contradiction. Hence $\Phi(R) = 1$, then G is of type (3).

Similarly, if $Q \ntrianglelefteq G$, $PQ = P \rtimes Q$, $PR = P \rtimes R$, $QR = Q \rtimes R$, then P is elementary abelian, $Q = \langle a \rangle$ is a group of order q, $R = \langle b \rangle$ is a group of order r and $r|q-1$. Let $|P| = p^\alpha, \alpha \geq 1$. Then by [16, Theorem 1.5], $p^\alpha \equiv 1(\text{mod } q)$, $p^\alpha \equiv 1(\text{mod } r)$. Hence $p^\alpha - 1 = qm = rn$, where m and n are integers. So $q = rnm^{-1}$, a contradiction.

Conversely, it is clear that the groups of types (1)–(3) are minimal non-SMSN-groups.

By Lemma 2.1, combining Theorem 3.1, Theorem 3.2 and Theorem 3.3, the complete classification of the minimal non-SMSN-groups is as follows.

Corollary 3.4 The minimal non-SMSN-groups are exactly the groups of A_5, types (1) to (18) of Theorem 3.2 and types (1) to (3) of Theorem 3.3, where A_5 is the alternating group of degree 5.

References

[1] Robinson D J S. A course in the theory of groups[M]. New York, Heidelberg, Berlin: Springer-Verlag, 1980.

[2] Schmidt O J. Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind[J]. Mat. Sbornik, 1924, 31: 366–372.

[3] Doerk K. Minimal nicht überauflösbare, endliche Gruppen[J]. Math. Z., 1966, 91: 198–205.

[4] Ballester-Bolinches A, Esteban-Romero R. On minimal non-supersoluble groups[J]. Rev. Mat. Iberoamericana, 2007, 23(1): 127–142.

[5] Lutsenko1 Y V, Skiba A N. Finite groups with subnormal second or third maximal subgroups[J]. Math. Notes., 2012, 91(5): 680–688.

[6] Guo Wenbin, Legchekova E V, Skiba A N. Finite groups in which every 3-maximal subgroup commutes with all maximal subgroups[J]. Math. Notes, 2009, 86(3): 325–332.

[7] Guo Pengfei, Guo Xiuyun. On minimal non-MSN-groups[J]. Front. Math. China, 2011, 6(5): 847–854.

[8] Ma Lijie, Huang Benwen. Structure of a kind of groups of order $p^n m$[J]. J. Math., 2010, 30(4): 671–674 (in Chinese).

[9] Qian Guohua. Finite groups to be able written as the union of four proper subgroups[J]. J. Math., 2011, 31(5): 891–892 (in Chinese).

[10] Thompson J G. Nonsolvable finite groups all of whose local subgroups are solvable[J]. Bull. Amer. Math. Soc., 1968, 74: 383–437.

[11] Laffey T J. A Lemma on finite p-group and some consequences[J]. Math. Proc. Cambridge Phil. Soc., 1974, 75(2): 133–137.

[12] Itô N. Über die Frattini-Gruppe einer endlichen Gruppe[J]. Proc. Japan Acad., 1955, 31(6): 327–328.

[13] Conway J H, Curtis R T, Norton S P, Parker R A, Wilson R A. Atlas of finite groups[M]. Oxford: Oxford University Press, 1985.

[14] King O H. The subgroup structure of finite classical groups in terms of geometric configurations[C]. Surveys in Combinatorics, London Math. Soc., Lecture Note, 2005, 327: 29–56.

[15] Suzuki M. On a class of doubly transitive groups[J]. Ann. Math., 1962, 75(1): 105–145.

[16] Chen Zhongmu. Inner and outer-$\sum$ groups and mininmal non-$\sum$ groups[M]. Chongqing: Southwest Normal University Press, 1988 (in Chinese).

所有极大子群都为SMSN-群的有限群

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摘要: 若有限群G的每个2-极大子群在G中次正规, 则称G为SMSN-群. 本文研究了有限群G的每个真子群是SMSN-群但G本身不是SMSN-群的结构, 利用局部分析的方法, 获得了这类群的完整分类, 推广了有限群结构理论的一些成果.

关键词: 幂自同构; 幂零群; 内幂零群; 极小非SMSN-群

MR(2010)**主题分类号**: 20D10; 20E34 **中图分类号**: O152.1