

## SOME OPERATOR INEQUALITIES OF MONOTONE FUNCTIONS CONTAINING FURUTA INEQUALITY

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**Abstract:** In this paper, we study the relations between the operator inequalities and the operator monotone functions. By using the fundamental conclusions based on majorization, namely, product lemma and product theorem for operator monotone functions, we can give some operator inequalities. This result contains the Furuta inequality, which has a huge impact on positive operator theory.

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### 1 Introduction

Let  $J$  be an interval such that  $J \neq (-\infty, \infty)$ .  $P(J)$  denotes the set of all operator monotone functions on  $J$ . We set  $P_+(J) = \{f \in P(J) \mid f(t) \geq 0, t \in J\}$ . If  $f \in P_+(a, b)$  and  $-\infty < a$ , then  $f$  has the natural extension to  $[a, b]$ , which belongs to  $P_+[a, b]$ . We therefore identify  $P_+(a, b)$  with  $P_+[a, b]$ .

It is well-known that if  $f(t) \in P_+(0, \infty)$ , then  $\frac{t}{f(t)}$  (if  $f \neq 0$ ) and  $f(t^\alpha)^{\frac{1}{\alpha}}$  are both in  $P_+(0, \infty)$ , and that if  $f(t), \phi(t), \varphi(t)$  are all in  $P_+(0, \infty)$ , then so are

$$\phi(f(t))\varphi\left(\frac{t}{f(t)}\right) \quad (\text{if } f \neq 0), f(t^\alpha)\phi(t^{1-\alpha})$$

and  $f(t)^\alpha\phi(t)^{1-\alpha}$  for  $0 < \alpha < 1$  (see [1–5]). Throughout this work, we assume that a function is continuous and increasing means “strictly increasing”. Further more, for convenience, let  $B(\mathcal{H})$  denote the  $C^*$ -algebra of all bounded linear operators acting on a Hilbert space  $\mathcal{H}$ . A capital letter  $A$  means an element belongs to  $B(\mathcal{H})$ ,  $\Phi$  means a positive linear map from  $B(\mathcal{H})$  to  $B(\mathcal{H})$  and we assume  $\Phi(I) = I$  always stand (see [7, 8]). In this paper, we also assume that  $J = [a, b)$  or  $J = (a, b]$  with  $-\infty \leq a < b \leq +\infty$ .

**Definition 1.1** [9, 10] Let  $P_+^{-1}(J), LP_+(J)$  denote the following sets, respectively,

$$P_+^{-1}(J) = \left\{ h \mid h \text{ is increasing on } J, h((a, b)) = (0, \infty), h^{-1} \in P(h(J)) \right\},$$

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where  $h^{-1}$  stands for the inverse function of  $h$ .

$$LP_+(J) = \{h \mid h \text{ is defined on } J, h(t) > 0 \text{ on } (a, b), \log h \in P(a, b)\}.$$

**Definition 1.2** Let  $h(t)$  and  $g(t)$  be functions defined on  $J$ , and  $g(t)$  is increasing, then  $h$  is said to be majorized by  $g$ , in symbol  $h \preceq g$  if the composite  $h \circ g^{-1}$  is operator monotone on  $g(J)$ , which is equivalent to

$$\sigma(A), \sigma(B) \subset J, \quad g(A) \leq g(B) \implies h(A) \leq h(B).$$

**Lemma 1.1** (Product lemma) (see [9, 10]) Let  $h, g$  be non-negative functions defined on  $J$ . Suppose the product  $hg$  is increasing,  $(hg)(a + 0) = 0$  and  $(hg)(b - 0) = \infty$ . Then

$$g \preceq hg \text{ on } J \iff h \preceq hg \text{ on } J.$$

Moreover, for every  $\psi_1, \psi_2$  in  $P_+[0, \infty)$ ,

$$g \preceq hg \text{ on } J \implies \psi_1(h)\psi_2(g) \preceq hg \text{ on } J.$$

**Theorem 1.1** (Product theorem) (see [9,10])

$$P_+^{-1}(J) \cdot P_+^{-1}(J) \subset P_+^{-1}(J), \quad LP_+^{-1}(J) \cdot P_+^{-1}(J) \subset P_+^{-1}(J).$$

Further, let  $g_i(t) \in LP_+(J)$  for  $1 \leq i \leq m$  and  $h_j(t) \in P_+^{-1}(J)$  for  $1 \leq j \leq n$ . Then for every  $\psi_i, \phi_j \in P_+[0, \infty)$ , we have

$$\prod_{i=1}^m \psi_i(g_i) \prod_{j=1}^n \phi_j(h_j) \preceq \prod_{i=1}^m g_i \prod_{j=1}^n h_j \in P_+^{-1}(J).$$

## 2 Main Results

Before to prove our main results, we give the following lemmas.

**Lemma 2.1** (L-H inequality) (see [2, 12]) If  $0 \leq \alpha \leq 1, A \geq B \geq 0$ , then  $A^\alpha \geq B^\alpha$ .

**Lemma 2.2** (Furuta inequality) (see [6, 9]) Let  $A \geq B \geq 0$ , then

- (1)  $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^\alpha \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^\alpha$ ;
- (2)  $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^\alpha \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^\alpha$ ,

where  $r \geq 0, p \geq 1$  with  $0 < \alpha \leq \frac{1+r}{p+r}$ .

**Lemma 2.3** (Hansen inequality) (see [13]) Let  $X$  and  $A$  be bounded linear operators on  $\mathcal{H}$ , and such that  $X \geq 0, \|A\| \leq 1$ . If  $f$  is an operator monotone function on  $[0, \infty)$ , then

$$A^* f(X) A \leq f(A^* X A).$$

**Theorem 2.1** Put  $J \neq (-\infty, \infty), \eta \in P_+(J) \cap P_+^{-1}(J), f_i \in P_+(J), i = 1, 2, \dots, n, g(t) \in P_+^{-1}(J) \cup \{f_1(t)\}$ , and  $k_n(t) = f_1(t)f_2(t) \cdots f_n(t)$ . If  $h(t)$  is defined on  $J$  such that  $f_1(t)h(t) \in P_+^{-1}(J)$ , then

(i) the function  $\phi_n$  on  $(0, \infty)$  defined by

$$\phi_n(k_n(t)h(t)g(t)) = k_n(t)\eta(t) \quad (t \in J) \quad (2.1)$$

belongs to  $P_+(0, \infty)$ ;

(ii) if  $A \leq C \leq B$ , then

$$\begin{aligned} \phi_n(k_n(C)^{\frac{1}{2}}h(A)g(A)k_n(C)^{\frac{1}{2}}) &\leq \phi_n(k_n(C)^{\frac{1}{2}}h(C)g(C)k_n(C)^{\frac{1}{2}}) \\ &\leq \phi_n(k_n(C)^{\frac{1}{2}}h(B)g(B)k_n(C)^{\frac{1}{2}}). \end{aligned} \quad (2.2)$$

**Proof** (i) Since  $f_1(t) \preceq t \preceq f_1(t)h(t)$ , by product lemma  $h(t) \preceq f_1(t)h(t)$ , therefore  $h(t)$  is nondecreasing. When  $g \in P_+^{-1}(J)$ , since  $\eta \in P_+(J) \cap P_+^{-1}(J)$ , we have  $\eta(t) \preceq t \preceq g(t)$ . Now putting  $\psi_0(s) = s$ ,  $\psi_1(g) = \eta$ ,  $\psi_2(f_1h) = f_1$ , obviously, we have  $\psi_0, \psi_1, \psi_2 \in P_+(0, \infty)$ . By taking  $s$  in  $\psi_0(s)$  as  $f_2 \cdots f_n$ , and from product theorem, we obtain

$$k_n\eta = \psi_0(f_2 \cdots f_n)\psi_1(g)\psi_2(f_1h) \preceq k_n(t)h(t)g(t).$$

Therefore we have  $\phi_n$  belongs to  $P_+(0, \infty)$  for  $\phi_n$  given in (i).

When  $g(t) = f_1(t)$ , by taking  $\psi_0(s) = s$ ,  $\psi_1(g(t)h(t)) = \eta(t)$ , we have  $\psi_0, \psi_1 \in P_+(0, \infty)$ , and then  $\phi_n \in P_+(0, \infty)$  by product theorem.

(ii) First we prove that

$$C \leq B \implies \phi_n(k_n(C)^{\frac{1}{2}}h(C)g(C)k_n(C)^{\frac{1}{2}}) \leq \phi_n(k_n(C)^{\frac{1}{2}}h(B)g(B)k_n(C)^{\frac{1}{2}}).$$

Since  $\phi_n, k_n, h, g$  are all nonnegative, nondecreasing functions and  $J$  is a right open interval, by considering  $C + \epsilon, B + \epsilon$ , we may assume that  $k_n(C)^{\frac{1}{2}}, h(C), h(B), g(C), g(B)$  are positive semi-definite and invertible. Through (i),

$$\phi_1(f_1(t)h(t)g(t)) = f_1(t)\eta(t). \quad (2.3)$$

Since  $0 \leq f_1(C) \leq f_1(B) \implies f_1(C)^{\frac{1}{2}}f_1(B)^{-1}f_1(C)^{\frac{1}{2}} \leq 1$ , by Lemma 2.3, we have

$$\begin{aligned} \phi_1(k_1(C)^{\frac{1}{2}}h(B)g(B)k_1(C)^{\frac{1}{2}}) &= \phi_1(f_1(C)^{\frac{1}{2}}f_1(B)^{-\frac{1}{2}}f_1(B)h(B)g(B)f_1(B)^{-\frac{1}{2}}f_1(C)^{\frac{1}{2}}) \\ &\geq f_1(C)^{\frac{1}{2}}f_1(B)^{-\frac{1}{2}}\phi_1(f_1(B)h(B)g(B))f_1(B)^{-\frac{1}{2}}f_1(C)^{\frac{1}{2}} \\ &= f_1(C)^{\frac{1}{2}}f_1(B)^{-\frac{1}{2}}f_1(B)\eta(B)f_1(B)^{-\frac{1}{2}}f_1(C)^{\frac{1}{2}} \\ &= f_1(C)^{\frac{1}{2}}\eta(B)f_1(C)^{\frac{1}{2}} \\ &\geq f_1(C)\eta(C) = \phi_1(k_1(C)^{\frac{1}{2}}h(C)g(C)k_1(C)^{\frac{1}{2}}). \end{aligned}$$

This implies the right part of (2.2) holds for  $n = 1$ . Next we assume the right part of (2.2) holds for  $n - 1$ . Since  $k_{n-1}(t)\eta(t) \in P_+^{-1}(J)$  and  $f_n \in P_+(J)$ , so  $f_n \preceq t \preceq k_{n-1}(t)\eta(t)$ , and this means that there exists  $\Psi_n \in P_+(0, \infty)$  such that  $f_n(t) = \Psi_n(k_{n-1}(t)\eta(t))$ . Put  $s = k_{n-1}(t)\eta(t)$ , we can obtain  $\phi_n(\phi_{n-1}^{-1}(s)\Psi_n(s)) = s\Psi_n(s)$ . Since the following inequality holds

$$\phi_{n-1}(k_{n-1}(C)^{\frac{1}{2}}h(C)g(C)k_{n-1}(C)^{\frac{1}{2}}) \leq \phi_{n-1}(k_{n-1}(C)^{\frac{1}{2}}h(B)g(B)k_{n-1}(C)^{\frac{1}{2}}).$$

Denote the left side of the upper inequalities as  $H$ , the right one as  $K$ , we have

$$\Psi_n(H)^{\frac{1}{2}}\Psi_n(K)^{-1}\Psi_n(H)^{\frac{1}{2}} \leq I.$$

By  $H = \phi_{n-1}(k_{n-1}(C)h(C)g(C)) = k_{n-1}(C)\eta(C)$ , we obtain

$$\Psi_n(H) = f_n(C), \quad \phi_{n-1}^{-1}(K) = k_{n-1}(C)^{\frac{1}{2}}h(B)g(B)k_{n-1}(C)^{\frac{1}{2}}. \tag{2.4}$$

By Lemma 2.3 again, we obtain

$$\begin{aligned} & \phi_n(\Psi_n(H)^{\frac{1}{2}}\phi_{n-1}^{-1}(K)\Psi_n(H)^{\frac{1}{2}}) \\ &= \phi_n(\Psi_n(H)^{\frac{1}{2}}\Psi_n(K)^{-\frac{1}{2}}\Psi_n(K)\phi_{n-1}^{-1}(K)\Psi_n(K)^{-\frac{1}{2}}\Psi_n(H)^{\frac{1}{2}}) \\ &\geq \Psi_n(H)^{\frac{1}{2}}\Psi_n(K)^{-\frac{1}{2}}\phi_n(\Psi_n(K)\phi_{n-1}^{-1}(K))\Psi_n(K)^{-\frac{1}{2}}\Psi_n(H)^{\frac{1}{2}} \\ &= \Psi_n(H)^{\frac{1}{2}}\Psi_n(K)^{-\frac{1}{2}}K\Psi_n(K)\Psi_n(K)^{-\frac{1}{2}}\Psi_n(H)^{\frac{1}{2}} \\ &= \Psi_n(H)^{\frac{1}{2}}K\Psi_n(H)^{\frac{1}{2}} \geq H\Psi_n(H). \end{aligned}$$

From the above inequalities and (2.4), we get

$$\begin{aligned} & \phi_n(f_n(C)^{\frac{1}{2}}k_{n-1}(C)^{\frac{1}{2}}h(B)g(B)k_{n-1}(C)^{\frac{1}{2}}f_n(C)^{\frac{1}{2}}) \\ &\geq f_n(C)k_{n-1}(C)\eta(C) = \phi_n(k_n(C)h(C)g(C)). \end{aligned}$$

Therefore the right part of (2.2) holds for  $n$ , one can proof the left part of (2.2) similarly.

**Remark** In Theorem 2.1, let  $n = 2$ ,  $f_1(t) = g(t) = 1$ ,  $f_2(t) = t^r (r \geq 0)$ ,  $h(t) = t^p (p \geq 1)$ , and  $\eta(t) = t$ , then we have  $\phi_2(t^{p+r}) = t^{1+r}$ . So Furuta inequality can be obtained by (2.2) and L-H inequality.

**Lemma 2.4** (see [10, 11]) Put  $J \neq (-\infty, \infty)$ , then  $g \in LP_+(J)$  if and only if there exists a sequence  $\{g_n\}$  of a finite product of functions in  $P_+(J)$  which converges pointwise to  $g$  on  $J$ , further more,  $\{g_n\}$  converges uniformly to  $g$  on every bounded closed subinterval of  $J$ .

**Theorem 2.2** Put  $J \neq (-\infty, \infty)$ ,  $f(t) > 0$  for  $t \in J$  and  $\eta(t), h(t), k(t), g(t)$  are nonnegative functions on  $J$  such that  $\eta \in P_+(J) \cap P_+^{-1}(J), f \in P_+(J), fh \in P_+^{-1}(J), \frac{k}{f} \in LP_+(J), g \in P_+^{-1}(J) \cup \{f\}$ , then

(i) the function  $\phi$  on  $(0, \infty)$  defined by

$$\phi(k(t)h(t)g(t)) = k(t)\eta(t) \quad (t \in J) \tag{2.5}$$

belongs to  $P_+(0, \infty)$ ;

(ii) If  $A \leq C \leq B$ , then for  $\varphi \in P(0, \infty)$  such that  $\varphi \preceq \phi$  on  $(0, \infty)$ ,

$$\varphi(k(C)^{\frac{1}{2}}h(A)g(A)k(C)^{\frac{1}{2}}) \leq \varphi(k(C)^{\frac{1}{2}}h(C)g(C)k(C)^{\frac{1}{2}}) \leq \varphi(k(C)^{\frac{1}{2}}h(B)g(B)k(C)^{\frac{1}{2}}). \tag{2.6}$$

**Proof** (i) First consider  $g \in P_+^{-1}(J)$ . Put  $l = \frac{k}{f}$ , then  $k = lf$  and

$$k(t)h(t)g(t) = l(t)f(t)h(t)g(t), \quad k(t)\eta(t) = l(t)f(t)\eta(t).$$

Let  $\psi_0(s) = s, \psi_1(f(t)h(t)) = f(t), \psi_2(g(t)) = \eta(t)$ , then  $\psi_0, \psi_1, \psi_2 \in P_+(0, \infty)$ . By taking  $s = l(t)$  and applying product theorem, we get

$$\psi_0(l(t))\psi_1(f(t)h(t))\psi_2(g(t)) \preceq l(t)f(t)h(t)g(t),$$

which equals to  $k(t)\eta(t) \preceq k(t)h(t)g(t)$ . So we have  $\phi \in P_+(0, \infty)$  for  $\phi$  such that

$$\phi(k(t)h(t)g(t)) = k(t)\eta(t) \quad (t \in I).$$

If  $g = f$ , taking  $\psi_0(s) = s, \psi_1(h(t)g(t)) = \eta(t)$ , obviously, we have  $\psi_0, \psi_1 \in P_+(0, \infty)$ , and then  $\psi_0(k)\psi_1(hg) \preceq khg$ . Hence we also have  $\phi \in P_+(0, \infty)$  from product theorem.

(ii) From Lemma 2.4, we obtain there exists a sequence  $\{l_n\}$ , where  $l_n(t)$  is a finite product of functions in  $P_+(J)$ , such that  $l_n(t)$  converges pointwise to  $l(t)$ . Put  $k_n(t) = f(t)l_n(t)$  then we easily get  $k_n(t)$  converges to  $k(t) = f(t)l(t)$ . Define  $\phi_n(k_n(t)h(t)g(t)) = k_n(t)\eta(t) \quad (t \in J), \phi_n \in P_+(0, \infty)$ . By Theorem 2.1, we have

$$\begin{aligned} \phi_n(k_n(C)^{\frac{1}{2}}h(A)g(A)k_n(C)^{\frac{1}{2}}) &\leq \phi_n(k_n(C)^{\frac{1}{2}}h(C)g(C)k_n(C)^{\frac{1}{2}}) \\ &\leq \phi_n(k_n(C)^{\frac{1}{2}}h(B)g(B)k_n(C)^{\frac{1}{2}}). \end{aligned}$$

Since  $k_n(t)h(t)g(t) \in P_+^{-1}(J)$  therefore  $k_n(t)h(t)g(t)$  is increasing on  $J$  and converges uniformly to  $k(t)h(t)g(t)$  on every compact interval of  $J$ . Since  $k(t)h(t)g(t) \in P_+^{-1}(J)$  is increasing, the inverse of  $k_n(t)h(t)g(t)$  converges uniformly to one of  $k(t)h(t)g(t)$  on every compact interval of  $J$ . It is also clear that  $k_n(t)\eta(t)$  converges uniformly to  $k(t)\eta(t)$  on every compact interval of  $J$ . Therefore  $\phi_n$  converges uniformly to  $\phi$  on every compact interval of  $J$ , since  $k_n(C)$  converges to  $k(C)$  in the operator norms, (2.6) holds for  $\phi$  hence for any  $\varphi$  given by  $\varphi \preceq \phi$ .

- Lemma 2.5** (Choi inequality ) (see [6, 7]) Let  $\Phi$  be a positive unital linear map, then
- (C1) when  $A > 0$  and  $-1 \leq p \leq 0$ , then  $\Phi(A)^p \leq \Phi(A^p)$ ;
  - (C2) when  $A \geq 0$  and  $0 \leq p \leq 1$ , then  $\Phi(A)^p \geq \Phi(A^p)$ ;
  - (C3) when  $A \geq 0$  and  $1 \leq p \leq 2$ , then  $\Phi(A)^p \leq \Phi(A^p)$ .

**Corollary 2.1** Put  $J \neq (-\infty, \infty), f(t) > 0$  for  $t \in J$  and  $\eta(t), h(t), k(t), g(t)$  are nonnegative functions on  $J$  such that  $\eta \in P_+(J) \cap P_+^{-1}(J), f \in P_+(J), fh \in P_+^{-1}(J), \frac{k}{f} \in LP_+(J), g \in P_+^{-1}(J) \cup \{f\}$ , the function  $\phi$  on  $(0, \infty)$  defined as (2.5),  $\Phi$  is a positive unital linear map. If

$$A_0^{p_0} \leq A_1^{p_0}, A_1^{p_1} \leq A_2^{p_1}, \quad 0 \leq p_0 \leq p_1 \leq p_2,$$

then for  $\varphi \in P(0, \infty)$  such that  $\varphi \preceq \phi$ ,

$$\begin{aligned} &\varphi(k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}h(\Phi(A_0^{p_0}))g(\Phi(A_0^{p_0}))k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}) \\ &\leq \varphi(k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}h(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})g(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}) \\ &\leq \varphi(k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}h(\Phi(A_2^{p_2})^{\frac{p_0}{p_2}})g(\Phi(A_2^{p_2})^{\frac{p_0}{p_2}})k(\Phi(A_1^{p_1})^{\frac{p_0}{p_1}})^{\frac{1}{2}}). \end{aligned} \tag{2.7}$$

**Proof** By Choi inequality and L-H inequality, we obtain

$$\Phi(A_2^{p_2})^{\frac{p_0}{p_2}} = \Phi(A_2^{p_2})^{\frac{p_1}{p_2} \cdot \frac{p_0}{p_1}} \geq \Phi(A_2^{p_1})^{\frac{p_0}{p_1}} \geq \Phi(A_1^{p_1})^{\frac{p_0}{p_1}} \geq \Phi(A_1^{p_0}) \geq \Phi(A_0^{p_0}),$$

which contains  $\Phi(A_2^{p_2})^{\frac{p_0}{p_2}} \geq \Phi(A_1^{p_1})^{\frac{p_0}{p_1}} \geq \Phi(A_0^{p_0})$ , from Theorem 2.2, we thus get (2.7).

**Corollary 2.2** Put

$$\eta(t) \in P_+(J) \cap P_+^{-1}(J), \quad \frac{h(t)}{t^p}, \quad \frac{k(t)}{t^r} \in LP_+(0, \infty), \quad p, r \geq 0$$

and  $p + r \geq 1, g(t) \in P_+^{-1}(0, \infty)$ . The function  $\phi$  on  $(0, \infty)$  is defined by

$$\phi(k(t)h(t)g(t)) = k(t)\eta(t) \quad (t \in J), \varphi \in P(0, \infty)$$

such that  $\varphi \preceq \phi$ . Then (2.5) and (2.6) in Theorem 2.2 hold.

**Proof** Put  $c = \min\{1, p\}$ , then  $f(t) = t^{1-c} \in P_+(0, \infty)$ . Thus we get

$$f(t)h(t) = \frac{h(t)}{t^p} t^{1+p-c} \in P_+^{-1}(0, \infty), \quad \frac{k(t)}{f(t)} = \frac{k(t)}{t^r} t^{r+c-1} \in LP_+(0, \infty),$$

which means the conditions of Theorem 2.2 is satisfied. Therefore (2.5) and (2.6) in Theorem 2.2 hold.

**Corollary 2.3** Put  $\eta(t) \in P_+(J) \cap P_+^{-1}(J), \frac{h(t)}{t^p}, \frac{k(t)}{t^r} \in LP_+(0, \infty), p, r \geq 0$  and  $p + r \geq 1, s \geq 1$ , we obtain

$$\log(k(C)^{\frac{1}{2}}h(A)A^s k(C)^{\frac{1}{2}}) \leq \log(k(C)^{\frac{1}{2}}h(C)C^s k(C)^{\frac{1}{2}}) \leq \log(k(C)^{\frac{1}{2}}h(B)B^s k(C)^{\frac{1}{2}}). \quad (2.8)$$

**Proof** Put  $g(t) = t^s (s \geq 1), \eta(t) = t$  in Corollary 2.2. Then we only need to show  $\log s \preceq \phi(s), s \in (0, \infty)$ . The definition of  $\phi$  is given in (2.5). The upper majorization relationship is equivalent to

$$\log(k(t)h(t)t^s) \preceq \phi(k(t)h(t)t^s) = k(t)t.$$

It is obviously that  $\log k(t), \log h(t), \log t^s$  are operator monotone on  $(0, \infty)$  and  $k(t)t = \frac{k(t)}{t^r} t^{r+1} \in P_+^{-1}(0, \infty)$ , then

$$\log(k(t)h(t)t^s) = \log k(t) + \log h(t) + \log t^s \preceq t \preceq k(t)t, \quad t \in (0, \infty).$$

Therefore (2.8) holds.

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## 一些蕴含Furuta不等式的算子单调函数的算子不等式

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**摘要:** 本文研究了算子不等式与算子单调函数之间的联系. 利用关于算子单调函数的乘积引理, 乘积定理等基本控制原理, 给出许多算子不等式, 这些不等式可包含正算子理论中应有十分广泛的Furuta不等式.

**关键词:** 算子单调函数; 积引理; 积定理; 控制

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