Vol. 37 (2017)
No. 4

数 学 杂 志
J. of Math. (PRC)

A HYPERGEOMETRIC EQUATION ON THE LINE BUNDLE OVER $\mathrm{SL}(n+1,\mathbb{R})/S(\mathrm{GL}(1,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}))$

YANG Xiang-hui[1], HE Min-hua[1], ZHU Li[1,2]

(*1. School of Science, Wuhan Institute of Technology, Wuhan 430205, China*)

(*2. School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*)

Abstract: In this paper, we study the differential equation on the line bundle over the pseudo-Riemannian symmetric space $\mathrm{SL}(n+1,\mathbb{R})/S(\mathrm{GL}(1,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}))$. We use Lie algebraic method, i.e., Casimir operator to obtain the desired differential operator. The differential equation turns out to be a hypergeometric differential equation, which generalizes the differential equations in [1, 3, 5].

Keywords: Casimir operator; pseudo-Riemannian symmetric space; line bundle; hypergeometric equation

2010 MR Subject Classification: 22E46; 33C05

Document code: A **Article ID:** 0255-7797(2017)04-0667-05

1 Introduction

Hypergeometric functions play important roles in harmonic analysis over pseudo-Riemannian symmetric spaces. Hyperbolic spaces are examples of pseudo-Riemannian symmetric spaces. There are a lot of work on hyperbolic spaces such as [3, 4]. Using a geometric method, Faraut obtained a second order differential equation in the explicit case of hyperbolic spaces $\mathrm{U}(p,q;\mathbb{F})/\mathrm{U}(1;\mathbb{F})\times\mathrm{U}(p-1,q;\mathbb{F})$ with $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ in [3]. Later in an algebraic way, i.e., through Casimir operator of $\mathfrak{sl}(n+1,\mathbb{R})$, van Dijk and Kosters obtained a hypergeometric equation on the pseudo-Riemannian smmetric space $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$ in [5].

A natural extension of [3, 5] is harmonic analysis on the sections of vector bundles over pseudo-Riemannian symmetric spaces. Charchov obtained a hypergeometric equation on the sections of line bundles over complex hyperbolic spaces $\mathrm{U}(p,q;\mathbb{C})/\mathrm{U}(1;\mathbb{C})\times\mathrm{U}(p-1,q;\mathbb{C})$ in his doctor thesis [1]. The differential equation in [1] is the same as the one in [3]. In this paper we will follow the method in [6] to obtain the hypergeometric equation on the sections of line bundles over $\mathrm{SL}(n+1,\mathbb{R})/\mathrm{GL}(n,\mathbb{R})$. When the parameter λ is zero, our result degenerates to the differential equation in [5]. Our hypergeometric equation will be used to obtaining the Plancherel formula on the sections of the line bundle over $\mathrm{SL}(n+1,\mathbb{R})/S(\mathrm{GL}(1,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}))$ in a future paper.

* **Received date:** 2016-05-27 **Accepted date:** 2016-07-11

Foundation item: Supported by the Research Foundation of Education Department of Hubei Province (Q20121512).

Biography: Yang Xianghui (1980–), female, born at Xiantao, Hubei, lecturer, major in functional differential equations.

2 Preliminaries and Main Result

Let $G = \mathrm{SL}(n+1,\mathbb{R})$ and $H_1 = \mathrm{SL}(n,\mathbb{R})$. We imbed H_1 in G as usual, i.e., for any $h \in H_1$, $h \mapsto \begin{pmatrix} 1 & \\ & h \end{pmatrix} \in G$. Let H be the subgroup of G:

$$H = S(\mathrm{GL}(1,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})) = \left\{ \begin{pmatrix} \det h^{-1} & \\ & h \end{pmatrix} : h \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$

In what follows ${}^tA$ denotes the transpose of a matrix A. Let X_1 be the algebraic manifold of

$$\mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1} \ (\mathbb{R}_*^{n+1} = \mathbb{R}_*^{n+1} \setminus \{\mathbf{0}\})$$

defined by

$$X_1 = \{(x,y) \in \mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1} : \langle x, y\rangle = x_0y_0 + x_1y_1 + \cdots + x_ny_n = 1\},$$

where $x = {}^t(x_0, x_1, \cdots, x_n)$, $y = {}^t(y_0, y_1, \cdots, y_n)$, G acts on $\mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1}$ by

$$g \cdot (x,y) = (gx, {}^tg^{-1}y) \tag{2.1}$$

for any $g \in G$ and any $(x,y) \in \mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1}$. With this action, X_1 is transitive under G. Let $x^0 = (e_0, e_0) \in X_1$ where e_0 is the first standard unit vector in $\mathbb{R}^{n+1}$, i.e., $e_0 = {}^t(1, 0, \cdots, 0)$. Then the stabilizer of x^0 in G is H_1. An elementary proof shows that $X_1 \simeq G/H_1$. We also have $X \simeq G/H$ where $X = \{x \in \mathrm{M}_{n+1}(\mathbb{R}) : \mathrm{rank} x = \mathrm{tr} x = 1\}$, here $\mathrm{M}_{n+1}(\mathbb{R})$ is the space of all real $(n+1) \times (n+1)$ matrices. G acts on $\mathrm{M}_{n+1}(\mathbb{R})$ by conjugation (see [5])

$$g \cdot x = gxg^{-1} \quad (g \in G, x \in \mathrm{M}_{n+1}(\mathbb{R})). \tag{2.2}$$

Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ be the Lie algebra of G. The Killing form of $\mathfrak{g}$ is $B(X,Y) = 2(n+1)\mathrm{tr}XY$ for X, $Y \in \mathfrak{g}$. The Killing form induces a measure on X_1. With this measure, the Casimir operator Ω of $\mathfrak{g}$ induces a second order differential operator on X_1. We call it the Laplace operator and denote it as $\Box_1$.

For $\lambda \in \mathbb{R}$, set $\chi_0(t) = t^{\sqrt{-1}\lambda}$, $t \in \mathbb{R}_*$ be a continuous unitary character of $\mathbb{R}_*$. Define a character χ_λ of H as $\chi_\lambda(h) = \chi_0(h_0) = h_0^{\sqrt{-1}\lambda}$ for

$$h = \begin{pmatrix} h_0 & \\ & h_1 \end{pmatrix} \in H.$$

Let $\mathcal{D}(X_1)$ be the space of complex-valued C^∞-functions on X_1 with compact support. The action of G on X_1 induces a representation U of G in $\mathcal{D}(X_1)$:

$$U(g)f(x) = f(g^{-1}x), \quad g \in G, \ \ x \in X_1, \ \ f \in \mathcal{D}(X_1)$$

and by inverse transposition a representation U of G in $\mathcal{D}'(X_1)$.

We define

$$\mathcal{D}'(X_1,\chi_\lambda)=\{T\in\mathcal{D}'(X_1):U(h)T=\chi_\lambda(h)^{-1}T,\quad h\in H\}.$$

Because $\chi_\lambda=1$ on H_1, the above distributions T can be viewed as the bi-H_1-invariant distributions on G satisfying $U(h)T=\chi_\lambda(h)^{-1}T,\ h\in H$.

If $\mu\in\mathbb{C}$, define

$$\mathcal{D}'(X_1,\chi_\lambda,\mu)=\{T\in\mathcal{D}'(X_1,\chi_\lambda):\Box_1'T=\mu T\},$$

where $\Box_1'$ is the transpose of the Laplace operator $\Box_1$.

Definition 2.1 The χ_λ-spherical distributions T on X_1 are the distributions on G satisfying the following properties

- T is H_1-invariant,
- $T(hx)=\chi_\lambda(h)T(x)$, $h\in H$, $x\in X_1$,
- $\Box_1'T=\mu T$ for some $\mu\in\mathbb{C}$.

As in [2], we define a mapping $Q_1:X_1\to\mathbb{R}$ by $Q_1(x,y)=x_0y_0$. We take the open subsets $X_1^0=\{(x,y)\in X_1:Q_1(x,y)<1\}$ and $X_1^1=\{(x,y)\in X_1:Q_1(x,y)>0\}$ of X_1. There is an averaging mapping $\mathcal{M}_1:f\mapsto\mathcal{M}_1f$ defined by

$$\mathcal{M}_1f(t)=\int_{X_1}f(x,y)\delta(Q_1(x,y)-t)d(x,y),$$

where δ is the Dirac measure and $d(x,y)$ is a G-invariant measure on X_1. Define $\xi:X_1\to\mathbb{R}^2$ by $\xi(x,y)=(\xi_1(x,y),\xi_2(x,y))=(x_0,y_0)$. Then $\chi_0\circ\xi_1(x,y)=x_0^{\sqrt{-1}\lambda}$. Let $\mathcal{M}'_{1,\sqrt{-1}\lambda}=(\chi_0\circ\xi_1)\cdot\mathcal{M}_1'$ where $\mathcal{M}_1'$ is the adjoint of $\mathcal{M}_1$. Then we have the main theorem of this paper.

Theorem 2.1 There is a second order differential operator L_λ on $\mathbb{R}$ such that the following formula holds

$$\Box_1'\circ\mathcal{M}'_{1,\sqrt{-1}\lambda}=\mathcal{M}'_{1,\sqrt{-1}\lambda}\circ L_\lambda, \tag{2.3}$$

where

$$L_\lambda=4t(t-1)\frac{d^2}{dt^2}+[4((n+1)t-1)+4\sqrt{-1}\lambda(t-1)]\frac{d}{dt}+\frac{2n}{n+1}\sqrt{-1}\lambda(\sqrt{-1}\lambda+n+1). \tag{2.4}$$

3 Proof of Main Result

We take a basis of $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{R})$ as

$$\left\{E_{11}-\frac{1}{n}\sum_{i=2}^{n+1}E_{ii},E_{jj}-E_{j+1,j+1}(2\le j\le n),E_{1s},E_{s1}(2\le s\le n+1),E_{kl}(2\le k\ne l\le n+1)\right\},$$

where $E_{\alpha\beta}=(\delta_{\alpha\mu}\delta_{\beta\nu})_{\mu\nu}$ is as usual.

On X_1 we take the coordinates $\{x_0,y_0,x_1,y_1,\cdots,x_{n-1},y_{n-1},x_n\}$. Using (2.1), we follow the way in [6] to express $E_{\alpha\beta}$ as differential operators on X_1 in terms of the coordinates $\{x_0, y_0, x_1, y_1, \cdots, x_{n-1}, y_{n-1}, x_n\}$. The results are

$$E_{11} - \frac{1}{n}\sum_{i=2}^{n+1} E_{ii} = x_0\frac{\partial}{\partial x_0} - y_0\frac{\partial}{\partial y_0} + \frac{1}{n}\left(-\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} + \sum_{i=1}^{n-1} y_i\frac{\partial}{\partial y_i}\right), \tag{3.1}$$

$$E_{1j} = x_{j-1}\frac{\partial}{\partial x_0} - y_0\frac{\partial}{\partial y_{j-1}}, \quad 2 \le j \le n, \tag{3.2}$$

$$E_{1,n+1} = x_n\frac{\partial}{\partial x_0}, \tag{3.3}$$

$$E_{j1} = x_0\frac{\partial}{\partial x_{j-1}} - y_{j-1}\frac{\partial}{\partial y_0}, \quad 2 \le j \le n, \tag{3.4}$$

$$E_{n+1,1} = x_0\frac{\partial}{\partial x_n} - \frac{1 - x_0y_0 - x_1y_1 - \cdots - x_{n-1}y_{n-1}}{x_n}\frac{\partial}{\partial y_0}. \tag{3.5}$$

Following [1], let the function F on X_1 be the form $F(x, y) = F(x_0, y_0)$. We calculate the action of the Laplace operator $\Box_1$ or the Casimir operator Ω on such functions. Because F depends on x_0, y_0 only, we take Ω as

$$2(n+1)\Omega = \frac{n}{n+1}\left(E_{11} - \frac{1}{n}\sum_{i=2}^{n+1} E_{ii}\right)^2 + \sum_{k=2}^{n+1}(E_{1k}E_{k1} + E_{k1}E_{1k}) + \text{other terms}, \tag{3.6}$$

where the 'other terms' are the combinations of $E_{kl}(2 \le k \ne l \le n+1)$. With the coordinates $\{x_0, y_0, x_1, y_1, \cdots, x_{n-1}, y_{n-1}, x_n\}$, using (3.1)–(3.5), we have

$$\begin{aligned} 2(n+1)\Omega F(x_0, y_0) = &\left\{\frac{n}{n+1}\left(x_0^2\frac{\partial^2}{\partial x_0^2} + y_0^2\frac{\partial^2}{\partial y_0^2}\right) + \left(\frac{2}{n+1}x_0y_0 - 2\right)\frac{\partial^2}{\partial x_0\partial y_0}\right. \\ &\left. + \frac{n(n+2)}{n+1}x_0\frac{\partial}{\partial x_0} + \frac{n(n+2)}{n+1}y_0\frac{\partial}{\partial y_0}\right\} F(x_0, y_0). \end{aligned} \tag{3.7}$$

Now taking function $F(x_0, y_0)$ with the form $F(x_0, y_0) = x_0^{\sqrt{-1}\lambda}F_0(x_0y_0)$ and $t = x_0y_0$, we obtain

$$4(n+1)\Omega(x_0^{\sqrt{-1}\lambda}F_0(x_0y_0)) = x_0^{\sqrt{-1}\lambda}L_\lambda F_0(x_0y_0) \tag{3.8}$$

with

$$L_\lambda = 4t(t-1)\frac{d^2}{dt^2} + [4((n+1)t-1) + 4\sqrt{-1}\lambda(t-1)]\frac{d}{dt} + \frac{2n}{n+1}\sqrt{-1}\lambda(\sqrt{-1}\lambda + n + 1). \tag{3.9}$$

For $f \in \mathcal{D}(X_1)$, $T \in \mathcal{D}'(\mathbb{R})$,

$$\begin{aligned} &\int_{X_1} \Box_1[(T\circ Q_1)\cdot \xi_1^{\sqrt{-1}\lambda}](x, y)f(x, y)d(x, y) \\ = &\int_{X_1}(L_\lambda T)(Q_1(x, y))\cdot \xi_1^{\sqrt{-1}\lambda}(x, y)f(x, y)d(x, y) = \int_{\mathbb{R}}(L_\lambda T)(t)\int_{Q_1(x)=t}\xi_1^{\sqrt{-1}\lambda}(x)f(x)dxdt \\ = &\langle L_\lambda T, \mathcal{M}_1\xi_1^{\sqrt{-1}\lambda}f\rangle_{\mathbb{R}} = \langle \mathcal{M}_1' L_\lambda T, \xi_1^{\sqrt{-1}\lambda}f\rangle_{X_1} = \langle \xi_1^{\sqrt{-1}\lambda}\mathcal{M}_1' L_\lambda T, f\rangle_{X_1}, \end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\int_{X_1}\square_1[(T\circ Q_1)\cdot\xi_1^{\sqrt{-1}\lambda}](x,y)f(x,y)d(x,y)\\
=&\int_{X_1}(T\circ Q_1)(x,y)\cdot\xi_1^{\sqrt{-1}\lambda}(x,y)(\square_1 f)(x,y)d(x,y)\\
=&\langle T,\mathcal{M}_1(\xi_1^{\sqrt{-1}\lambda}\square_1 f)\rangle_{\mathbb{R}}=\langle\mathcal{M}_1'T,\xi_1^{\sqrt{-1}\lambda}\square_1 f\rangle_{X_1}=\langle\square_1\mathcal{M}'_{1,\sqrt{-1}\lambda}T,f\rangle_{X_1}. \qquad (3.11)
\end{aligned}$$

Comparing (3.10) and (3.11), we have $\square_1\circ\mathcal{M}'_{1,\sqrt{-1}\lambda}=\mathcal{M}'_{1,\sqrt{-1}\lambda}\circ L_\lambda$. This completes the proof of Theorem 2.1.

References

[1] Charchov I. Harmonic analysis on line bundles over complex hyperbolic spaces[D]. Leiden: Univ. Leiden, 1999.

[2] van Dijk G. $(\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$ is a generalized gelfand pair[J]. Russian J. Math. Phys., 2008, 15(4): 548–551.

[3] Faraut J. Distributions sphériques sur les espaces hyperboliques[J]. J. Math. Pures Appl., 1979, 58: 369–444.

[4] Han Yingbo, Feng Shuxiang. On complete hypersurfaces in hyperbolic space form $H^{n+1}(-1)$[J]. J. Math., 2013, 33(5): 767–772.

[5] Kosters M T, van Dijk G. Spherical distributions on the Pseudo-Riemannian space $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$[J]. J. Funct. Anal., 1986, 68: 168–213.

[6] Lang S. $\mathrm{SL}_2(\mathbb{R})$[A]. Volume 105 of Graduate Texts in Mathematics[C]. Reprint of 1975 ed., New York: Springer-Verlag, 1985.

$\mathrm{SL}(n+1,\mathbb{R})/S(\mathrm{GL}(1,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}))$ 上线丛的一个超几何方程

杨向辉[1], 何敏华[1], 朱 理[1,2]

(1. 武汉工程大学理学院, 湖北 武汉 430205)

(2. 武汉大学数学与统计学院, 湖北 武汉 430072)

摘要: 本文研究了伪黎曼对称空间$\mathrm{SL}(n+1,\mathbb{R})/S(\mathrm{GL}(1,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R}))$线丛上的微分方程. 利用李代数方法, 即Casimir 算子得到这个微分算子. 这个微分算子是一个超几何方程, 这个结论推广了文献[1, 3, 5]中的微分方程.

关键词: Casimir算子; 伪黎曼对称空间; 线丛; 超几何方程

MR(2010)**主题分类号:** 22E46; 33C05 **中图分类号:** O152.5