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THE EXISTENCE OF MILD SOLUTIONS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS OF ORDER $1 < \alpha < 2$

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Abstract: This paper is concerned with the existence of mild solutions for impulsive fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Using the properties of solution operators and Krasnoselskii's fixed point theorem, we obtain the mild solution of the equations which is proved and its existence results.

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1 Introduction

Fractional differential equation had broad applications in resolving real-world problems (see [1–4]), and as such it attracted researchers' attention from different areas. Many authors have studied fractional differential equations from two aspects, one is the theoretical aspects of existence and uniqueness of solutions, the other is the analytic and numerical methods for finding solutions. For more details on this topic one can see the papers [5–14] and references therein.

Because of the applications of differential equations with nonlocal conditions in numerous fields of science, engineering, physics, economy and so on, many authors investigated the existence of solutions of abstract fractional differential equations with nonlocal conditions by using semigroups theorems, solution operator theorems and the relation between solution operators and semigroups constructing by probability density functions as well as fixed point techniques (see [5–8, 10–12, 14]).

In [5], Zhou and Jiao considered the nonlocal Cauchy problem of the following form

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + f(t, x(t)), t \in (0, a], \\ x(0) + g(x) = x_0, \end{cases}$$

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where $0 < q < 1$. The authors established various criteria on existence and uniqueness of mild solutions for nonlocal Cauchy problem by considering a integral equation which is given in terms of probability density and semigroup.

In [6], Wang etc. investigated the following nonlinear integrodifferential evolution equations with nonlocal initial conditions

$$\begin{cases} {}^cD_t^q x(t) = -Ax(t) + t^n f(t, x(t), (Hx)(t)),\ t \in J, h \in Z^+, \\ x(0) = g(x) + x_0 \in X_\alpha, \end{cases}$$

where $0 < q < 1$. By using the fractional calculus, Hölder inequality, p-mean continuity and fixed point theorems, some existence results of mild solutions are obtained.

In [7], Debbouche and Baleanu studied the fractional nonlocal impulsive integro-differential control system of the form

$$\begin{cases} {}^cD_t^\alpha u(t) + A(t, u(t))u(t) = (B\mu)(t) + \Phi(t, f(t, u(\beta(t))), \int_0^t g(t, s, u(\gamma(s)))ds), \\ u(0) + h(u) = u_0, \\ \Delta u(t_i) = I_i(u(t_i)), \end{cases}$$

where $0 < \alpha < 1$. The controllability result of systems was established by using the theory of fractional calculus, fixed point technique and the authors introduced a new concept called (α, u)-resolvent family.

To the best of our knowledge, the existence of mild solutions for impulsive fractional evolution equation with nonlocal conditions of order $1 < \alpha < 2$ is an untreated topics in the literature, motivated by this, we consider the following impulsive fractional evolution equations with nonlocal conditions

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t) + \int_0^t h(s, u(s))ds,\ t \in J = [0, T],\ t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k^-)),\ \Delta u'(t_k) = J_k(u(t_k^-)),\ k = 1, 2, \cdots, m, \\ u(0) + m(u) = u_0 \in X,\ u_0' + n(u) = u_1 \in X, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, D^α is Caputo's fractional derivatives. A is a sectorial operator of type (M, θ, α, μ) defined from domain $D(A) \subset X$ into X, the nonlinear map h defined from $[0, T] \times X$ into X is continuous function. The nonlocal conditions $m : X \longrightarrow X; n : X \longrightarrow X$ are continuous functions.

$$\Delta u(t_k) = u(t_k^+) - u(t_k^-), u(t_k^+) = \lim_{\varepsilon \to 0^+} u(t_k + \varepsilon) \text{ and } u(t_k^-) = \lim_{\varepsilon \to 0^-} u(t_k + \varepsilon)$$

represent the right and left limits of $u(t)$ at $t = t_k$, for $k = 1, 2, \cdots, m$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u'(t_k)$ is similar.

The rest of the paper is organized as follows. In Section 2, some notions and notations that are used throughout the paper and properties of solution operators are presented. In addition, the definitions of the mild solutions are given, and the correctness of the mild solutions is to be proved. The main results of this article are given in Section 3.

2 Preliminary

2.1 Definitions and Theorems

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let X be a complex Banach space with its norm denoted as $\|\cdot\|_X$, $L(X)$ represents the Banach space of all bounded linear operators from X into X, and the corresponding norm is denoted by $\|\cdot\|_{L(X)}$. Let $C(J,X)$ denote the Banach space of functions that are continuous and differentiable from J to X equipped with the norm $\|f\|_C = \sup\limits_{t\in J}\|f(t)\|_X$.

Let $PC(J,X) = \{x : J \to X : x \in C((t_k, t_{k+1}], X),\ k = 0, 1, \cdots, m$ and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k^-) = x(t_k)\}$.

It is easy to check that $PC(J,X)$ is a Banach space with the norm $\|x\|_{PC} = \sup\limits_{t\in J}\|x(t)\|_X$.

In general, the Mittag-Leffler function is defined as [4]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_{H_\alpha} e^{\mu}\frac{\mu^{\alpha-\beta}}{\mu^\alpha - z}d\mu,\ \alpha,\beta>0,\ z\in\mathbb{C},$$

where H_α denotes a Hankel path, a contour starting and ending at $-\infty$, and encircling the disc $|\mu| \le |z|^{\frac{1}{\alpha}}$ counterclockwise.

Definition 2.1 [4] Assume $a, \alpha \in R$, a function $f : [a, +\infty) \to R$ is said to be in the space $C_{a,\alpha}$ if there exist a real number $p > \alpha$ and a function $g \in C([a, +\infty), R)$ satisfying $f(t) = t^p g(t)$. In addition, assuming m is a positive integer, if $f^{(m)} \in C_{a,\alpha}$, then f is said to be in the space $C^m_{a,\alpha}$.

Definition 2.2 For the function $f \in C^m_{a,\alpha}$, and $m \in \mathbb{N}^+$, the fractional derivative of order $\alpha > 0$ of f in the Caputo sense is given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1}f^{(m)}(s)ds, \qquad m-1<\alpha<m.$$

The Laplace transform of the Caputo derivative of order $\alpha > 0$ is given by

$$\mathcal{L}(D_t^\alpha)u(t)(\lambda) = \lambda^\alpha(\mathcal{L}u)(\lambda) - \sum_{j=0}^{m-1}\lambda^{\alpha-j-1}(D^j u)(0), \qquad m-1<\alpha<m.$$

Theorem 2.3 [9] Let A be a densely defined operator in X satisfying the following conditions

(1) For some $0 < \theta < \pi/2$, $\mu + S_\theta = \{\mu + \lambda : \lambda \in C, |Arg(-\lambda)| < \theta\}$.

(2) There exists a constant M such that

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda-\mu|},\ \ \lambda \notin \mu + S_\theta.$$

Then A is the infinitesimal generator of a semigroup $T(t)$ satisfying $\|T(t)\| \le C$. Moreover, $T(t) = \frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda, A)d\lambda$ with c being a suitable path $\lambda \notin \mu + S_\theta$ for $\lambda \in \mathbb{C}$.

Definition 2.4 [9] Let $A: D(A) \subseteq X \to X$ be a closed linear operator. A is said to be a sectorial operator of type (M, θ, α, μ) if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the $\alpha-$resolvent of A exists outside the sector

$$\mu + S_\theta = \{\mu + \lambda^\alpha : \alpha \in C,\ |Arg(-\lambda^\alpha)| < \theta\}$$

and

$$\|(\lambda^\alpha I - A)^{-1}\| \le \frac{M}{|\lambda^\alpha - \mu|},\ \ \lambda \notin \mu + S_\theta.$$

Remark 1 [9] If A is a sectorial operator of type (M, θ, α, μ), then it is not difficult to see that A is the infinitesimal generator of a $\alpha-$resolvent family $\{T_\alpha(t)\}_{t\ge 0}$ in a Banach space, where $T_\alpha(t) = \frac{1}{2\pi i}\int_c e^{\lambda t} R(\lambda^\alpha, A) d\lambda$.

Theorem 2.5 (Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that

(i) $Ax + By \in M$ whenever $x, y \in M$;

(ii) A is compact and continuous;

(iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

2.2 Properties of Solution Operators

In order to study the mild solutions of equation (1.1), we first consider the following initial value problem of impulsive fractional differential equation:

$$\begin{cases} D_t^\alpha u(t) = Au(t) + f(t), \quad t \in J = [0, T],\ \ t \ne t_k, \\ \Delta u(t_k) = y_k,\ \ \Delta u'(t_k) = \overline{y}_k,\ \ k = 1, 2, \cdots, m, \\ u(0) = u_0, u'(0) = u_1. \end{cases} \tag{2.1}$$

Theorem 2.6 Suppose A is a sectorial operator of type (M, θ, α, μ). If f satisfies a uniform Hölder condition with exponent $\beta \in (0, 1]$, then the solution of problem (2.1) is given by

$$u(t) = \begin{cases} S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-\theta) f(\theta) d\theta, \quad t \in [0, t_1], \\ S_\alpha(t)u_0 + K_\alpha(t)u_1 + \int_0^t T_\alpha(t-\theta) f(\theta) d\theta \\ + \sum_{i=1}^k S_\alpha(t-t_i) y_i + \sum_{i=1}^k K_\alpha(t-t_i)\overline{y}_i, \qquad t \in (t_k, t_{k+1}], \end{cases} \tag{2.2}$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda, \tag{2.3}$$

$$K_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-2} R(\lambda^\alpha, A) d\lambda, \tag{2.4}$$

$$T_\alpha(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda. \tag{2.5}$$

In order to prove Theorem 2.6, we give the following lemmas first.

Lemma 2.7 [9] Let A be a sectorial operator of type (M,θ,α,μ). If f satisfies a uniform Hölder condition with exponent $\beta\in(0,1]$, then the unique solution of Cauchy problem

$$\begin{cases} D_t^\alpha u(t) = Au(t)+f(t), \quad t\in J=[0,T], \\ u(0)=u_0,\ u'(0)=u_1 \end{cases}$$

is given by

$$u(t)=S_\alpha(t)u_0+K_\alpha(t)u_1+\int_0^t T_\alpha(t-s)f(s)ds.$$

Lemma 2.8 [12] If A is a sectorial operator of type (M,θ,α,μ), then we have

$$S_\alpha(t)=\frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,A)d\lambda=E_{\alpha,1}(At^\alpha)=\sum_{k=0}^\infty\frac{(At^\alpha)^k}{\Gamma(1+\alpha k)},$$
$$T_\alpha(t)=\frac{1}{2\pi i}\int_c e^{\lambda t}R(\lambda^\alpha,A)d\lambda=t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)=t^{\alpha-1}\sum_{k=0}^\infty\frac{(At^\alpha)^k}{\Gamma(\alpha+\alpha k)}$$

and

$$K_\alpha(t)=\frac{1}{2\pi i}\int_c e^{\lambda t}\lambda^{\alpha-2}R(\lambda^\alpha,A)d\lambda=tE_{\alpha,2}(At^\alpha)=t\sum_{k=0}^\infty\frac{(At^\alpha)^k}{\Gamma(2+\alpha k)}.$$

Lemma 2.9 [12] Let A be a sectorial operator of type $(M,\theta,\alpha,\mu))$, then we have

$$\frac{dK_\alpha(t)}{dt}=S_\alpha(t) \qquad \text{and} \qquad \frac{dS_\alpha(t)}{dt}=AT_\alpha(t).$$

Lemma 2.10 Suppose A is a sectorial operator of type (M,θ,α,μ), then the following equations hold:

$${}^cD_t^\alpha\left[S_\alpha(t)u_0+K_\alpha(t)u_1\right]=A[S_\alpha(t)u_0+K_\alpha(t)u_1] \tag{2.6}$$

and

$${}^cD_t^\alpha\left(\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)=A\int_0^t T_\alpha(t-\theta)f(\theta)d\theta+f(t). \tag{2.7}$$

Proof It follows from (2.3) and (2.4) that

$$\mathcal{L}(S_\alpha(t)u_0)=\lambda^{\alpha-1}R(\lambda^\alpha,A)u_0, \tag{2.8}$$
$$\mathcal{L}(K_\alpha(t)u_1)=\lambda^{\alpha-2}R(\lambda^\alpha,A)u_1. \tag{2.9}$$

Therefore we obtain

$$\begin{aligned}&\mathcal{L}({}^cD_t^\alpha[S_\alpha(t)u_0+K_\alpha(t)u_1])\\ =&\lambda^\alpha\mathcal{L}[S_\alpha(t)u_0+K_\alpha(t)u_1]-\lambda^{\alpha-1}u_0-\lambda^{\alpha-2}u_1\\ =&\lambda^\alpha[\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}u_0+\lambda^{\alpha-2}(\lambda^\alpha-A)^{-1}u_1]-\lambda^{\alpha-1}u_0-\lambda^{\alpha-2}u_1\\ =&\lambda^{\alpha-1}(\lambda^\alpha I-A)^{-1}[\lambda^\alpha-(\lambda^\alpha-A)]u_0+\lambda^{\alpha-2}(\lambda^\alpha I-A)^{-1}[\lambda^\alpha-(\lambda^\alpha-A)]u_1\\ =&A\lambda^{\alpha-1}R(\lambda^\alpha,A)u_0+A\lambda^{\alpha-2}R(\lambda^\alpha,A)u_1.\end{aligned} \tag{2.10}$$

Combing (2.8)–(2.10) yields

$$ {}^cD_t^\alpha\left[S_\alpha(t)u_0+K_\alpha(t)u_1\right]=A[S_\alpha(t)u_0+K_\alpha(t)u_1]. $$

Similarly, we have

$$ \mathcal{L}\left(\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)=\mathcal{L}(T_\alpha(t))\mathcal{L}(f(t))=R(\lambda^\alpha,A)\mathcal{L}(f(t)) \tag{2.11} $$

and

$$ \begin{aligned} &\mathcal{L}\left[{}^cD_t^\alpha\left(\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)\right]=\lambda^\alpha(R(\lambda^\alpha,A)\mathcal{L}(f(t)))-\lambda^{\alpha-1}\cdot 0-\lambda^{\alpha-2}\cdot 0\\ =&(\lambda^\alpha I-A+A)R(\lambda^\alpha,A)\mathcal{L}(f(t))=\mathcal{L}(f(t))+AR(\lambda^\alpha,A)\mathcal{L}(f(t)). \end{aligned} \tag{2.12} $$

Thus it follows from (2.11) and (2.12) that

$$ {}^cD_t^\alpha\left(\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)=A\int_0^t T_\alpha(t-\theta)f(\theta)d\theta+f(t). $$

Proof of Theorem 2.6 For all $t\in(t_k,t_{k+1}]$, $k=0,1,\cdots,m$, by Lemma 2.10, we obtain

$$ \begin{aligned} &{}^cD_t^\alpha\left(S_\alpha(t)u_0+K_\alpha(t)u_1+\sum_{i=1}^k S_\alpha(t-t_i)y_i+\sum_{i=1}^k K_\alpha(t-t_i)\overline{y_i}+\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)\\ =&A\left(S_\alpha(t)u_0+K_\alpha(t)u_1\right)+A\left(\sum_{i=1}^k S_\alpha(t-t_i)y_i+\sum_{i=1}^k K_\alpha(t-t_i)\overline{y}_i\right)\\ &+A\int_0^t T_\alpha(t-\theta)f(\theta)d\theta+f(t)\\ =&A\left(S_\alpha(t)u_0+K_\alpha(t)u_1+\sum_{i=1}^k S_\alpha(t-t_i)y_i+\sum_{i=1}^k K_\alpha(t-t_i)\overline{y}_i+\int_0^t T_\alpha(t-\theta)f(\theta)d\theta\right)+f(t). \end{aligned} $$

That means expression (2.2) satisfies the first formula of problem (1.1).

For $k=1,2,\cdots,m$, it is obvious that

$$ \Delta u(t_k)=u(t_k^+)-u(t_k^-)=S_\alpha(t_k-t_k)y_k+K_\alpha(t_k-t_k)\overline{y}_k=S_\alpha(0)y_k+K_\alpha(0)\overline{y}_k=y_k. $$

According to Lemma 2.9 and equation (2.2), for $t\in(t_k,t_{k+1}]$, $k=1,2,\cdots,m$, we have

$$ \Delta u'(t_k)=u'(t_k^+)-u'(t_k^-)=AT_\alpha(t_k-t_k)y_k+S_\alpha(t_k-t_k)\overline{y}_k=AT_\alpha(0)y_k+S_\alpha(0)\overline{y}_k=\overline{y}_k. $$

And

$$ \begin{aligned} u(0)&=S_\alpha(0)u_0+K_\alpha(0)u_1+\int_0^0 T_\alpha(0-\theta)f(\theta)d\theta=u_0,\\ u'(0)&=AT_\alpha(0)u_0+S_\alpha(0)u_1+[\int_0^0 T_\alpha(0-\theta)f(\theta)d\theta]'=u_1. \end{aligned} $$

Consequently, all the conditions of problem (2.1) are satisfied, thus (2.2) is a solution of problem (2.1).

Hence, we can define the mild solution of equation (1.1) as follow.

Definition 2.11 A function $u \in PC(J, X)$ is said to be a mild solution of system (1.1) if it satisfies the following operator equation

$$u(t) = \begin{cases} S_\alpha(t)[u_0 - m(u)] + K_\alpha(t)[u_1 - n(u)] + \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u(\tau))d\tau)ds, \ \ t \in [0, t_1], \\ S_\alpha(t)[u_0 - m(u)] + K_\alpha(t)[u_1 - n(u)] + \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u(\tau))d\tau)ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_i) I_i(u(t_i^-)) + \sum_{i=1}^{k} K_\alpha(t-t_i) J_i(u(t_i^-)), \ \ t \in (t_k, t_{k+1}]. \end{cases} \tag{2.13}$$

Theorem 2.12 [9] Let A be a sectorial operator of type (M, θ, α, μ). Then the following estimates on $\|S_\alpha(t)\|$ hold.

(i) Suppose $\mu \geq 0$. Given $\phi \in \left(\max\left\{\theta, (1-\alpha)\pi\right\}, \frac{\pi}{2}(2-\alpha)\right)$, we have

$$\begin{aligned} \|S_\alpha(t)\| \leq & \frac{K_1(\theta,\phi) M e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}} \left[(1+\frac{\sin\phi}{\sin(\phi-\theta)})^{\frac{1}{\alpha}} - 1\right]}{\pi \sin^{1+\frac{1}{\alpha}} \theta} (1+\mu t^\alpha) \\ & + \frac{\Gamma(\alpha) M}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta \sin\phi} \end{aligned}$$

for $t > 0$, where

$$K_1(\theta, \phi) = \max\{1, \frac{\sin\theta}{\sin(\theta-\phi)}\}.$$

(ii) Suppose $\mu < 0$. Given $\phi \in \left(\max\left\{\frac{\pi}{2}, (1-\alpha)\pi\right\}, \frac{\pi}{2}(2-\alpha)\right)$, we have

$$\|S_\alpha(t)\| \leq \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha}} - 1]}{\pi|\cos\phi|^{1+\frac{1}{\alpha}}} + \frac{\Gamma(\alpha) M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|^\alpha}\right) \frac{1}{1+|\mu| t^\alpha}$$

for $t > 0$.

Theorem 2.13 [9] Let A be a sectorial operator of type (M, θ, α, μ). Then the following estimates on $\|T_\alpha(t)\|$, $\|K_\alpha(t)\|$ hold.

(i) Suppose $\mu \geq 0$. Given $\phi \in \left(\max\left\{\theta, (1-\alpha)\pi\right\}, \frac{\pi}{2}(2-\alpha)\right)$, we have

$$\begin{aligned} \|T_\alpha(t)\| \leq & \frac{M e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}} \left[(1+\frac{\sin\phi}{\sin(\theta-\phi)})^{\frac{1}{\alpha}} - 1\right]}{\pi \sin\theta} (1+\mu t^\alpha)^{\frac{1}{\alpha}} t^{\alpha-1} \\ & + \frac{M t^{\alpha-1}}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta\sin\phi}, \\ \|K_\alpha(t)\| \leq & \frac{M K_1(\theta,\phi) e^{[K_1(\theta,\phi)(1+\mu t^\alpha)]^{\frac{1}{\alpha}}} \left[(1+\frac{\sin\phi}{\sin(\theta-\phi)})^{\frac{1}{\alpha}} - 1\right]}{\pi \sin^{\frac{\alpha+2}{\alpha}} \theta} \\ & \times (1+\mu t^\alpha)^{\frac{\alpha-1}{\alpha}} t^{\alpha-1} + \frac{M\alpha\Gamma(\alpha)}{\pi(1+\mu t^\alpha)|\cos\frac{\pi-\phi}{\alpha}|^\alpha \sin\theta\sin\phi} \end{aligned}$$

for $t > 0$, where

$$K_1(\theta, \phi) = \max\{1, \frac{\sin\theta}{\sin(\theta - \phi)}\}.$$

(ii) Suppose $\mu < 0$. Given $\phi \in \left(\max\left\{\frac{\pi}{2}, (1-\alpha)\pi\right\}, \frac{\pi}{2}(2-\alpha)\right)$, we have

$$\|T_\alpha(t)\| \leq \left(\frac{eM[(1+\sin\phi)^{\frac{1}{\alpha}} - 1]}{\pi|\cos\phi|} + \frac{M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{t^{\alpha-1}}{1+|\mu|t^\alpha},$$

$$\|K_\alpha(t)\| \leq \left(\frac{eMt[(1+\sin\phi)^{\frac{1}{\alpha}} - 1]}{\pi|\cos\phi|^{1+\frac{2}{\alpha}}} + \frac{\alpha\Gamma(\alpha)M}{\pi|\cos\phi||\cos\frac{\pi-\phi}{\alpha}|}\right)\frac{1}{1+|\mu|t^\alpha}$$

for $t > 0$.

3 Main Results

To prove our main results, we list the following basic assumptions of this paper.

Because of the estimation on $\|S_\alpha(t)\|$, $\|K_\alpha(t)\|$ and $\|T_\alpha(t)\|$ in Section 2, it is easy to know they are bounded. So we make the following assumptions:

(H1) The operators $S_\alpha(t), K_\alpha(t), T_\alpha(t)$ generated by A are compact in $\overline{D(A)}$ when $t \geq 0$ and

$$\sup_{t\in J}\|S_\alpha(t)\| \leq \widetilde{M}, \ \sup_{t\in J}\|K_\alpha(t)\| \leq \widetilde{M}, \ \sup_{t\in J}\|T_\alpha(t)\| \leq \widetilde{M}.$$

(H2) $h : [0, T] \times X \to X$ is continuous and for any $k > 0$ there exists positive function $v_k \in L^\infty([0, T], \mathbb{R}^+)$ such that

$$\sup_{\|u\|\leq k}\|h(t, u)\| \leq v_k(t).$$

(H3) $m, n : X \to \overline{D(A)}$ are continuous, and there exist positive constants b, d such that

$$\|m(u) - m(v)\| \leq b\|u - v\|, \ \|n(u) - n(v)\| \leq d\|u - v\| \text{ for any } u, v \in X.$$

(H4) $I_k, J_k : X \to X$ are continuous, and there exist positive numbers d_k, f_k such that

$$\|I_k(x)\|_X \leq d_k\|x\|_X, \ \|J_k(x)\|_X \leq f_k\|x\|_X \text{ for all } x \in X, \ k = 1, 2, \cdots, m.$$

(H5)

$$\widetilde{M}\sum_{i=1}^m (d_i + f_i) < 1.$$

Theorem 3.1 Suppose that conditions (H1)–(H5) are satisfied. If $\widetilde{M}(b + d) < \frac{1}{2}$, then system (1.1) has at least one mild solution on J.

Proof We define operator $\Gamma : PC(J, X) \to PC(J, X)$ by

$$(\Gamma u)(t) = \begin{cases} S_\alpha(t)[u_0 - m(u)] + K_\alpha(t)[u_1 - n(u)] + \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u(\tau))d\tau)ds, \ t \in [0, t_1], \\ S_\alpha(t)[u_0 - m(u)] + K_\alpha(t)[u_1 - n(u)] + \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u(\tau))d\tau)ds \\ + \sum_{i=1}^k S_\alpha(t - t_i)I_i(u(t_i^-)) + \sum_{i=1}^k K_\alpha(t - t_i)J_i(u(t_i^-)), \ \ t \in (t_k, t_{k+1}]. \end{cases}$$

Next, we will prove that Γ has a fixed point.

Set
$$B_r = \{u \in PC(J,X) : \|u\|_X \leq r\}$$
for $r > 0$. Then for each r, B_r is a bounded, close and convex subset in X.

Step 1 We prove that there exists a positive integer $r \in \mathbb{R}^+$ such that $\Gamma(B_r) \subset B_r$.

If this is not true, then, for each positive integer r, there exists $u^r \in B_r$ and $t \in [0,T]$ such that $\|(\Gamma u^r)(t)\| > r$, however, on the other hand, we have

$$\begin{aligned}
r <&\|(\Gamma u^r)(t)\|_X \\
\leq&\|S_\alpha(t)\|\|u_0^r - m(u^r)\|_X + \|K_\alpha(t)\|\|u_1^r - n(u^r)\|_X + \int_0^t \|T_\alpha(t-s)\|\|(\int_0^s h(\tau, u^r(\tau))d\tau)\|ds \\
&+ \sum_{i=1}^{k} \|S_\alpha(t-t_i)\|\|I_i(u^r(t_i^-))\|_X + \sum_{i=1}^{k} \|K_\alpha(t-t_i)\|\|J_i(u^r(t_i^-))\|_X \\
\leq&\widetilde{M}(\|u_0^r\| + \|m(u^r)\| + \|u_1^r\| + \|n(u^r)\|) + T^2\widetilde{M}\|v_r\|_{L^\infty(J,\mathbb{R}^+)} \\
&+ \sum_{i=1}^{m}(d_i + f_i)\widetilde{M}\|u^r(t_i^-)\|_X \leq \widetilde{M}\left[\Omega + r\Big(\sum_{i=1}^{m}(d_i+f_i)\Big)\right],
\end{aligned}$$

where
$$\Omega = \|u_0^r\| + \|m(u^r)\| + \|u_1^r\| + \|n(u^r)\| + T^2\|v_r\|_{L^\infty(J,\mathbb{R}^+)}.$$

Since Ω is a positive constant, so dividing the above formula on both sides by r and taking the lower limit as $r \to +\infty$, we get
$$1 < \widetilde{M}\sum_{i=1}^{m}(d_i + f_i).$$

This is a contraction to (H5). Hence, for some positive integer r, $\Gamma(B_r) \subset B_r$.

We decompose $\Gamma = \Gamma_1 + \Gamma_2$, respectively,

$$(\Gamma_1 u)(t) = S_\alpha(t)[u_0 - m(u)] + K_\alpha(t)[u_1 - n(u)] \ \ t \in J.$$
$$(\Gamma_2 u)(t) = \begin{cases} \int_0^t T_\alpha(t-s)(\int_0^s h(\tau,u(\tau))d\tau)ds, \ \ t \in [0,t_1], \\ \int_0^t T_\alpha(t-s)(\int_0^s h(\tau,u(\tau))d\tau)ds \\ +\sum_{i=1}^{k} S_\alpha(t-t_i)I_i(u(t_i^-)) + \sum_{i=1}^{k} K_\alpha(t-t_i)J_i(u(t_i^-)), \quad t \in (t_k, t_{k+1}]. \end{cases}$$

Step 2 We prove that Γ_1 is a contraction mapping.

Take $u, v \in B_r$ arbitrarily, then for each $t \in [0,T]$, we obtain

$$\begin{aligned}
\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\|_X &\leq \|S_\alpha(t)\|\|m(u) - m(v)\| + \|K_\alpha(t)\|\|n(u) - n(v)\| \\
&\leq \widetilde{M}(b+d)\|u - v\|.
\end{aligned}$$

And since $\widetilde{M}(b+d) < 1$, so Γ_1 is a contraction mapping.

Step 3 We prove that Γ_2 is continuous on B_r.

Let $\{u_n\}_{n=1}^{+\infty} \subset B_r$, with $u_n \to u$ in B_r. Noting that the function h, I_k, J_k are continuous, we have

$$h(s, u_n(s)) \to h(s, u(s)), I_k(u_n(t_k^-)) \to I_k(u(t_k^-)), J_k(u_n(t_k^-)) \to J_k(u(t_k^-))$$

as $n \to \infty$. Now, for all $t \in J_k, k = 1, 2, \cdots, m$, we have

$$\begin{aligned}
&\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\| \\
\leq &\| \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u_n(\tau))d\tau)ds - \int_0^t T_\alpha(t-s)(\int_0^s h(\tau, u(\tau))d\tau)ds\| \\
&+ \sum_{i=1}^{k} \|S_\alpha(t-t_i)\| \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\| \\
&+ \sum_{i=1}^{k} \|K_\alpha(t-t_i)\| \|J_i(u_n(t_i^-)) - J_i(u(t_i^-))\| \\
\leq &\widetilde{M}T^2 \|h(\tau, u_n(\tau)) - h(\tau, u(\tau))\| + \sum_{i=1}^{m} \widetilde{M} \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\| \\
&+ \sum_{i=1}^{m} \widetilde{M} \|J_i(u_n(t_i^-)) - J_i(u(t_i^-))\| \to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

Thus Γ_2 is continuous.

Next, we prove Γ_2 is compact. To this end, we use the Ascoli-Arzela theorem. We prove that $(\Gamma_2 u)(t) : u \in B_r$ is relatively compact in X for all $t \in J$.

Step 4 We prove the uniform boundedness of the map Γ_2.

For any $u \in B_r, t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
\|(\Gamma_2 u)(t)\|_X \leq &\int_0^t \|T_\alpha(t-s)\| \| \int_0^s h(\tau, u(\tau))d\tau\| ds + \sum_{i=1}^{k} \|S_\alpha(t-t_i)\| \|I_i(u(t_i^-))\| \\
&+ \sum_{i=1}^{k} \|K_\alpha(t-t_i)\| \|J_i(u(t_i^-))\| \\
\leq &\widetilde{M}[T^2 \|v_r\|_{L^\infty(J, \mathbb{R}^+)} + r \sum_{i=1}^{m} (d_i + f_i)] < \infty.
\end{aligned}$$

So it is proved.

Step 5 Let us prove that the map $\Gamma_2(B_r)$ is equicontinuous.

The function $\{\Gamma_2 u : u \in B_r\}$ are equicontinuous at $t = 0$. For $0 < t_2 < t_1 \leq T$, $t_1, t_2 \in (t_k, t_{k+1}]$, $k = 1, 2, \cdots, m$ and $u \in B_r$, we have

$$\begin{aligned}
&\|(\Gamma_2 u)(t_1) - (\Gamma_2 u)(t_2)\| \\
\leq &\int_0^{t_2} \|T_\alpha(t_1 - s) - T_\alpha(t_2 - s)\| [\int_0^s \|h(\tau, u(\tau))\| d\tau] ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_2}^{t_1}\|T_\alpha(t_2-s)\|[\int_0^s\|h(\tau,u(\tau))\|d\tau]ds+\sum_{i=1}^m\|S_\alpha(t_1-t_i)-S_\alpha(t_2-t_i)\|\|I_i(u(t_i^-))\|\\
&+\sum_{i=1}^m\|K_\alpha(t_1-t_i)-K_\alpha(t_2-t_i)\|\|J_i(u(t_i^-))\|\\
\leq& T^2\|v_r\|_{L^\infty(J,\mathbb{R}^+)}\|T_\alpha(t_1-s)-T_\alpha(t_2-s)\|+\widetilde{M}T\|v_r\|_{L^\infty(J,\mathbb{R}^+)}(t_1-t_2)\\
&+\sum_{i=1}^m d_i\|S_\alpha(t_1-t_i)-S_\alpha(t_2-t_i)\|r+\sum_{i=1}^m f_i\|K_\alpha(t_1-t_i)-K_\alpha(t_2-t_i)\|r.
\end{aligned}
$$

Actually, the right side is independent of $u \in B_r$ and tends to zero as $t_2 \to t_1$ since the continuity of function $t \to \|S_\alpha(t)\|$, $t \to \|T_\alpha(t)\|$ and $t \to \|K_\alpha(t)\|$.

In short, we have proved that $\Gamma_2(B_r)$ is relatively compact, for $\{\Gamma_2 u : u \in B_r\}$ is a family of equicontinuous function. Hence by Arzela-Ascoli theorem, Γ_2 is a compact operator. All the conditions of Krasnoselskii's fixed point theorem are satisfied, thus, system (1.1) has at least one mild solution.

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一类$1 < \alpha < 2$非局部条件下的脉冲分数阶微分方程 mild解的存在性

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摘要: 本文研究了一类$1 < \alpha < 2$非局部条件下的脉冲分数阶偏微分方程mild解的存在性问题. 利用解算子的相关性质及Krasnoselskii不动点理论的方法, 获得了这类方程的mild解并予以证明, 且得到了解的存在性结果.

关键词: mild解; 分数阶微分方程; 非局部条件; 不动点理论

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