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SOME RESULTS ON THE p-DIVISIBLE kG-MODULE

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Abstract: In this paper, we study the p-divisible kG-module, which is essentially controlled by the prime p. With the Heller operator, we prove that the nth-Heller operator permutates the isomorphism classes of the indecomposable non-projective p-divisible kG-modules; and with the methods of induction and restriction, we prove that Green correspondence induces a bijection between the isomorphism classes of the indecomposable p-divisible kG-modules and that of the indecomposable p-divisible kH-modules whenever H is strongly p-embedded in G, which generalizes Green correspondence for the indecomposable relative projective modules.

Keywords: p-divisible module; endo-p-permutation module; Heller operator; Green correspondence

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1 Introduction

The terminology of the p-divisible kG-module was introduced in [2] to be an absolutely p-divisible kG-module, as a tool to study the nilpotent elements in the Green ring, confined to the algebraic closed field k of characteristic p, it is essentially controlled by the prime p. This terminology was also studied in [1] from the sight of splitting trace module, where with the so-called splitting trace module the authors obtain some results on the problem of decomposing a tensor product into a direct sum of indecomposable modules, and then some results on the almost-split sequence related to the finite group representation is obtained. Landrock also contributed to this kind of modules. The class of p-divisible kG-modules is a big one, any (relative) projective kG-module is p-divisible.

Here we prove the properties on the p-divisible kG-module. We show that how can the induction and the restriction of a p-divisible kG-module remain to be p-divisible. We confirm that the nth-Heller operator preserves the p-divisible kG-modules, that is, besides the class of indecomposable non-projective kG-modules, the class of the p-divisible kG-modules is another one which is closed under Heller operators; and confirm that the vertex of the indecomposable end-p-permutation kG-module being p-divisible at the same time must be the proper p-subgroup of G. Finally, we prove that, Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules induces a bijection between the

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isomorphism classes of indecomposable p-divisible kG-modules and that of indecomposable p-divisible kH-modules whenever H is strongly p-embedded in G.

Throughout the paper, we fix a prime p, a finite group G such that $p||G|$, and an algebraic closed field k of characteristic p. All modules involved are finitely generated, the order of any finite group involve in the p-divisible kG-module is divided by p. Our main results are as follows in Section 2.

Theorem A Any indecomposable endo-p-permutation kG-module V with vertex P is p-divisible if and only if P is the proper p-subgroup of G, moreover, in the case of P being the Sylow p-subgroup of G, p does not divide $\dim_k(V)$, and $\mathrm{Res}_P^G(V)$ is a capped endo-permutation kP-module (Theorem 2.9).

The following result shows that, $\Omega^n(-)$ permutates the isomorphism classes of indecomposable non-projective p-divisible kG-modules.

Theorem B Let V be a kG-module, then for any $n \in Z$, V is p-divisible if and only if $\Omega^n(V)$ is p-divisible (Theorem 2.11).

The following result shows that the class of the indecomposable p-divisible modules is closed under the bijection of Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules whenever H is strongly p-embedded in G.

Theorem C Let $G \geq H \geq N_G(P)$, where P is a Sylow p-subgroup of G, if H is strongly p-embedded, then Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules induces a bijection between the isomorphism classes of indecomposable p-divisible kG-modules and that of indecomposable p-divisible kH-modules (Theorem 2.14).

2 The p-Divisible kG-Module

Definition 2.1 For a kG-module V, if the dimension of any indecomposable direct summand of V is divisible by P, where $p = \mathrm{char}k$, we say V is a p-divisible kG-module.

Notation The terminology of the p-divisible kG-module here is based on "absolutely" (see [2]), while for the algebraic closed field k, any indecomposable kG-module is already absolutely indecomposable therein, so that, in this paper, a p-divisible kG-module is essentially controlled by the prime p. We often denote a p-divisible kG-module with (p-divisible) for short. Obviously, the trivial kG-module k is not p-divisible.

Lemma 2.2 Let U, V be two p-divisible kG-modules, W be a kG-module, and P be a proper p-subgroup of G, then

(1) any P-projective kG-module is p-divisible, particularly, any projective kG-module is p-divisible;

(2) V^* is p-divisible;

(3) $U \oplus_k V$ is p-divisible, and vice versa;

(4) $U \otimes_k W$ is p-divisible;

(5) $\mathrm{Hom}_k(U, V)$ is p-divisible, in particular, $\mathrm{End}_k(U)$ is p-divisible;

(6) any direct summand of U is p-divisible, especially, k is not the direct summand of U and not the direct summand of $\mathrm{End}_k(U)$.

Proof (1) It comes from the fact that any direct summand of the P-projective kG-module (respectively, the projective kG-module) remains to be P-projective (respectively, projective), and from the fact that the dimension of any indecomposable P-projective kG-module (respectively, any projective kG-module) can be divided by $|G:P|_p$ (respectively, by $|G|_p$) (see [5, Exercise 23.1, Exercise 21.2(a)]).

(2) First, we see that $\dim_k(V^*) = \dim_k(V)$ from the dual basis, secondly, we have $(V^*)^* \cong V$ as kG-module, then if V is indecomposable, so is V^*, hence we have the result from the fact that all the dimensions of the indecomposable direct summands of V^* are just that of V.

(3) We confirm this result because of Krull-Schmidt theorem.

(4) In the case that U is an indecomposable kG-module, so is $U \otimes_k W$ by [1, Corollary 4.3, Corollary 4.7] since k here is algebraically closed. For the general case it is also true because of (3).

(5) It is true because of (4) and the kG-module isomorphism $\mathrm{Hom}_k(U,V) \cong U^* \otimes_k V$.

(6) It follows from (3).

Proposition 2.3 Let $G \geq H$ and V be a kG-module, if $\mathrm{Res}^G_H(V)$ is a p-divisible kH-module, then V is a p-divisible kG-module.

Proof It follows from Krull-Schmidt theorem.

Proposition 2.4 Let $G \geq H$ and V be a p-divisible kG-module, if H contains a Sylow p-subgroup of G, then $\mathrm{Res}^G_H(V)$ is a p-divisible kH-module.

Proof Proof by contradiction. If $\mathrm{Res}^G_H(V)$ is not p-divisible, then p does not divide $\dim_k(V_1)$, and then $k|\mathrm{End}_k(V_1)$ for some direct summand V_1 of $\mathrm{Res}^G_H(V)$ by [1, Corollary 4.7], we see that $k|\mathrm{End}_k(\mathrm{Res}^G_H(V))$, so that

$$\mathrm{Ind}^G_H(k)|\mathrm{Ind}^G_H(\mathrm{Res}^G_H(\mathrm{End}_k(V))).$$

On the other hand,

$$\mathrm{Ind}^G_H(\mathrm{Res}^G_H(\mathrm{End}_k(V))) \cong \mathrm{Ind}^G_H(k) \otimes_k \mathrm{End}_k(V)$$

by Frobenius Reciprocity, so that $\mathrm{Ind}^G_H(k) \otimes_k \mathrm{End}_k(V)$ is also p-divisible by Lemma 2.2(5)–(4), that is, $\mathrm{Ind}^G_H(\mathrm{Res}^G_H(\mathrm{End}_k(V)))$ is p-divisible, it means that $\mathrm{Ind}^G_H(k)$ is p-divisible by Lemma 2.2(3), it contradicts with the order of $\mathrm{Ind}^G_H(k)$ if H contains some Sylow p-subgroup of G.

For any $k(G/N)$-module V, where $G \trianglerighteq N$, the inflation kG-module $\inf(V)$ has the same k-vector space as V, while its G-action is from the composition of the canonical group homomorphisms: $G \to G/N \to \mathrm{End}_k(V)$.

Proposition 2.5 Let N be a normal subgroup of G with $p||G:N|$, then V is a p-divisible $k(G/N)$-module if and only if $\inf(V)$ is a p-divisible kG-module.

Proof We see that V is an indecomposable $k(G/N)$-module if and only if $\inf(V)$ is an indecomposable kG-module, moreover, the dimensions of V and $\inf(V)$ are the same since as k-module they are the same, then V is p-divisible if and only if $\inf(V)$ is p-divisible.

Let $G \geq H$, a kG-module M is said to be H-projective, if there exists a kH-module N such that $M|\mathrm{Ind}_H^G(N)$; moreover, if M is indecomposable, then the minimal subgroup H such that M is H-projective is unique under the G-conjugation, we call H the vertex of M, it must be a p-group, in the case, the kH-module N is called the corresponding source module (see [5, 6]).

Corollary 2.6 Let U be not a p-divisible kG-module, if P is a p-subgroup of G, then $U \otimes_k V$ is P-projective if and only if the kG-module V is P-projective.

Proof The proof of sufficiency is obvious; for the necessity, we see $k|(U^* \otimes_k U)$ as the proof of Proposition 2.4, then

$$k \otimes_k V|((U^* \otimes_k U) \otimes_k V),$$

that is,

$$V|U^* \otimes_k (P\text{-projective}),$$

hence, V is a direct summand of the P-projective kG-module, so that V is P-projective.

By Green indecomposability theorem (see [5, Corollary 23.6]), $\mathrm{Ind}_Q^P(k)$ is p-divisible if Q is the proper subgroup of the p-group P. In general, we have the following result.

Proposition 2.7 Let $G \geq H$, V be a kG-module, and U be a kH-module such that $V = \mathrm{Ind}_H^G(U)$;

(1) if the Sylow p-subgroup of H is a proper p-subgroup of G, then V is p-divisible;

(2) if U is p-divisible and H contains some Sylow p-subgroup of G such that $p||G : H \cap {}^gH|$ for any $g \in G - H$, then V is p-divisible;

(3) if U is not p-divisible while V is p-divisible, then the Sylow p-subgroup of H is a proper p-subgroup of G.

Proof (1) By Lemma 2.2(1), $\mathrm{Ind}_H^G(U)$ is p-divisible since $\mathrm{Ind}_H^G(U)$ is P-projective, where P is a Sylow p-subgroup of H.

(2) We see that

$$\begin{aligned}
\mathrm{Res}_H^G(V) &= \mathrm{Res}_H^G(\mathrm{Ind}_H^G(U)) \\
&\cong \bigoplus_{g\in[H\backslash G/H]} \mathrm{Ind}_{H\cap\, {}^gH}^H(\mathrm{Res}_{H\cap\, {}^gH}^H\ ({}^gU)) \\
&= \bigoplus_{g\in[H\backslash G/H]} \mathrm{Ind}_{H\cap\, {}^gH}^H\ ({}^gU) \\
&= U \oplus (\bigoplus_{1\neq g\in[H\backslash G/H]} \mathrm{Ind}_{H\cap\, {}^gH}^H\ ({}^gU)),
\end{aligned}$$

where each $\mathrm{Ind}_{H\cap\, {}^gH}^H\ ({}^gU)$ must be p-divisible by (1). So that $\mathrm{Res}_H^G(V)$ is p-divisible by Lemma 2.2 (3), and then V is also p-divisible by Proposition 2.3.

(3) On the contrary, if H contains a Sylow p-subgroup of G, then $U|\mathrm{Res}_H^G(\mathrm{Ind}_H^G(U))$, it means that $\mathrm{Ind}_H^G(U)$ cannot be p-divisible. Indeed, if $\mathrm{Ind}_H^G(U)$ is p-divisible, so is

$$\mathrm{Res}_H^G(\mathrm{Ind}_H^G(U))$$

by Proposition 2.4, and so is U by Lemma 2.2 (3). Contradiction!

Recall that a subgroup H of G is strongly p-embedded if p divides the order of H but does not divide $|H \cap {}^xH|$, for all $x \in G - H$. Note here that the strongly p-embedded subgroup H of G contains the normalizer in G of any p-subgroup of G, and such H exists whenever the Sylow p-subgroup of G is trivial intersection (that is, T.I set) (see [9]).

Corollary 2.8 (1) Let $G \geq H$, then $\mathrm{Ind}_H^G(k)$ is p-divisible if and only if the Sylow p-subgroup of H is a proper p-subgroup of G;

(2) let $G \geq H$ and U be a p-divisible kH-module such that $V = \mathrm{Ind}_H^G(U)$, if H is strongly p-embedded, then V is p-divisible.

Proof It follows from Proposition 2.7.

A kG-module M is called a p-permutation kG-module if $\mathrm{Res}_Q^G(M)$ is a permutation module for every p-subgroup Q of G (see [3]); an endo-p-permutation kG-module N is a kG-module N with $\mathrm{End}_k(N)$ being a p-permutation kG-module under the conjugation action of G, in the case, if G is a p-group, we call it an endo-permutation module (see [5]); any p-permutation kG-module is an endo-p-permutation kG-module, and the p-permutation kG-module plays a crucial role for the equivalence of the categories of blocks of kG (see [7, 8]).

Theorem 2.9 Any indecomposable endo-p-permutation kG-module V with the vertex P is p-divisible if and only if P is the proper p-subgroup of G, moreover, in the case of P being the Sylow p-subgroup of G, P does not divide $\dim_k(V)$, and $\mathrm{Res}_P^G(V)$ is a capped endo-permutation kP-module.

Proof If P is the Sylow p-subgroup of G, $\mathrm{Res}_P^G(V)$ is an endo-permutation kP-module, and the source module of V has the vertex P and is the direct summand of $\mathrm{Res}_P^G(V)$, then $\mathrm{Res}_P^G(V)$ is a capped endo-permutation kP-module, so that, $\dim_k(V) \equiv \pm 1 \pmod{p}$ by [5, Corollary 28.11], that is, p does not divide $\dim_k(V)$, V is not p-divisible.

On the contrary, if P is the proper p-subgroup of G, V is p-divisible by Lemma 2.2 (1).

Given a kG-module M, the Heller translate $\Omega(M)$ is the kernel of the projective cover $P_M \to M$, so that there is a short exact sequence of kG-modules $0 \to \Omega(M) \to P_M \to M \to 0$; since projective covers of M are unique up to isomorphism, $\Omega(M)$ is well-defined up to isomorphism; similarly, one may define $\Omega^{-1}(M)$ to be the cokernel of the injective hull $M \to I_M$. The Heller operator Ω provides a way to construct new indecomposable kG-modules from the old ones. Let $\Omega^0(M)$ be the non-projective kG-module such that $M = \Omega^0(M) \oplus_k$ (projective), $\Omega^1(M) = \Omega(M)$; one can obtain the nth-Heller translates

$$\Omega^n(M) := \Omega(\Omega^{n-1}(M)) \ \ (n \geq 2)$$

and

$$\Omega^n(M) := \Omega^{-1}(\Omega^{n+1}(M)) \ \ (n \leq -2)$$

(see [6]).

Lemma 2.10 Let V and W be kG-modules, then for any $m, n \in Z$, $\Omega^m(V) \otimes_k \Omega^n(W) \cong \Omega^{m+n}(V \otimes_k W) \oplus_k$ (projective), particularly, there exists a projective kG-module U such that $V \otimes_k \Omega^n(k) \cong \Omega^n(V) \oplus_k U$.

Proof Set m=0, $W = k$ in the former, we can obtain the latter. While the latter is also Proposition 11.7.2 of [6], now we prove the former by the latter.

Set $V \otimes_k \Omega^m(k) \cong \Omega^m(V) \oplus_k X$ and $W \otimes_k \Omega^n(k) \cong \Omega^n(W) \oplus_k Y$, where X and Y are projective kG-modules. Then

$$\begin{aligned} & (\Omega^m(V) \oplus_k X) \otimes_k (\Omega^n(W) \oplus_k Y) \cong \Omega^m(V) \otimes_k \Omega^n(W) \oplus_k \text{(projective)} \\ \cong \;& (V \otimes_k \Omega^m(k)) \otimes_k (W \otimes_k \Omega^n(k)) \cong V \otimes_k W \otimes_k (\Omega^m(k) \otimes_k \Omega^n(k)) \\ \cong \;& (V \otimes_k W) \otimes_k (\Omega^{m+n}(k) \oplus_k \text{(projective)}) \\ \cong \;& \Omega^{m+n}(V \otimes_k W) \oplus_k \text{(projective)}, \end{aligned}$$

it means that $\Omega^m(V) \otimes_k \Omega^n(W) \cong \Omega^{m+n}(V \otimes_k W) \oplus_k$ (projective) since $\Omega^{m+n}(V \otimes_k W)$ has no projective direct summands.

The following result shows that, $\Omega^n(-)$ permutates the isomorphism classes of indecomposable non-projective p-divisible kG-modules.

Theorem 2.11 Let V be a kG-module, then for any $n \in Z$, V is p-divisible if and only if $\Omega^n(V)$ is p-divisible.

Proof If V is p-divisible, so is $V \otimes_k \Omega^n(k)$, and then $\Omega^n(V) \oplus_k U$ is also p-divisible by Lemma 2.10, that is, $\Omega^n(V)$ is p-divisible for each $n \in Z$.

On the contrary, if $\Omega^n(V)$ is p-divisible, so is $\Omega^n(V) \oplus_k U$ in Lemma 2.10, and then $V \otimes_k \Omega^n(k)$ is p-divisible, so that $(V \otimes_k \Omega^n(k)) \otimes_k \Omega^{-n}(k)$ is also p-divisible. While

$$\begin{aligned} & (V \otimes_k \Omega^n(k)) \otimes_k \Omega^{-n}(k) \cong V \otimes_k (\Omega^n(k) \otimes_k \Omega^{-n}(k)) \\ \cong \;& V \otimes_k (\Omega^0(k \otimes_k k) \oplus_k U_1) \cong V \otimes_k (k \oplus_k U_1) \\ \cong \;& V \oplus_k (V \otimes_k U_1), \end{aligned}$$

where U_1 is some projective kG-module. We have $V|(V \otimes_k \Omega^n(k)) \otimes_k \Omega^{-n}(k)$, then V is also p-divisible.

Corollary 2.12 Let $0 \to U \to W \to V \to 0$ be a short exact sequence of the kG-modules, if W is projective, then U is a p-divisible kG-module if and only if V is p-divisible.

Proof One can check the following kG-module isomorphisms by Schanuel′s lemma

$$\begin{aligned} U &\cong \Omega(V) \oplus_k \text{(projective)}, \\ V &\cong \Omega^{-1}(U) \oplus_k \text{(projective)}, \end{aligned}$$

then the result follows from Theorem 2.11 and Lemma 2.2(1)(3).

A V-projective kG-module W is the one such that $W|(V \otimes_k X)$ for some kG-module X; the V-projective kG-module generalizes the notion of the (relative) projective kG-modules (see [4]).

Proposition 2.13 Let V be a p-divisible kG-module and W be a V-projective kG-module, then W is p-divisible; if moreover, V is Q-projective, where Q is a p-subgroup of G, then W is also Q-projective.

Proof If W is indecomposable and p does not divide $\dim_k(W)$, then $k|(W^*\otimes_k W)$, and then $k|(W^*\otimes_k V\otimes_k X)$ since W is V-projective, where X is a kG-module; while $W^*\otimes_k V\otimes_k X$ is p-divisible by Lemma 2.2 (4), so that, k is p-divisible by Lemma 2.2 (3), too; it contradicts with Lemma 2.2 (6). In general, W is always p-divisible.

If V is Q-projective, so is $V\otimes_k X$ (see [5], Lemma 14.3), and then the direct summand W is also Q-projective.

Let $G\geq H\geq N_G(P)$, where P is a Sylow p-subgroup of G. The following result shows that the class of the indecomposable p-divisible modules is closed under the bijection of Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules whenever H is strongly p-embedded in G.

Theorem 2.14 Let $G\geq H\geq N_G(P)$, where P is a Sylow p-subgroup of G, if H is strongly p-embedded, then Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules induces a bijection between the isomorphism classes of indecomposable p-divisible kG-modules and that of indecomposable p-divisible kH-modules.

Proof In the case of H being strongly p-embedded, for the indecomposable p-divisible kG-module V with the vertex P, $\mathrm{Res}_H^G(V)$ is a p-divisible kH-module by Proposition 2.4, it means that the Green correspondent of V remains to be p-divisible by Lemma 2.2 (3); similarly, if U is a p-divisible kH-module with the vertex P, then $\mathrm{Ind}_H^G(U)$ is p-divisible by Corollary 2.8, and then the Green correspondent of U remains to be p-divisible by Lemma 2.2 (3) again.

Since H contains the normalizer of any proper p-subgroup Q of G (see [9]), Green correspondence sets up a bijection between the isomorphism classes of the indecomposable kG-modules with the vertex Q and that of the indecomposable kH-modules with the same vertex Q, moreover, these indecomposable modules with the vertex Q are p-divisible (Lemma 2.2 (1)).

Sum up the above, for any p-subgroup Q of G, whether or not it is a proper p-subgroup, the indecomposable p-divisible modules with the vertex Q are closed under the bijection of Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules, so that, Green correspondence between the indecomposable kG-modules and the indecomposable kH-modules induces a bijection between the isomorphism classes of indecomposable p-divisible kG-modules and that of indecomposable p-divisible kH-modules.

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关于p-可除kG-模的一些结论

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摘要: 本文研究了p-可除kG-模, 这是一类由群阶的素数因子来控制的模类. 利用Heller算子, 证明了n次Heller算子置换非投射不可分解p-可除kG-模的同类; 利用模的诱导和限制方法, 证明了若H是G的强p-嵌入子群, 则Green对应建立了不可分解p-可除kG-模的同构类与不可分解p-可除kH-模的同构类之间的一一对应. 推广了不可分解相对投射kG-模上的Green对应.

关键词: p-可除kG-模; 置换模; Heller 算子; Green 对应

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