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ON THE BEST CONSTANTS OF HARDY INEQUALITIES ON HALF SPACES IN H-TYPE GROUPS

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Abstract: In this paper, using the corresponding fundamental solution, we obtain some Hardy inequalities on half spaces for Kohn's sublaplacian in H-type groups. Furthermore, the constants we obtain are sharp.

Keywords: Hardy inequality; H-type group; best constant

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1 Introduction

The Hardy inequality in $\mathbb{R}^N$ reads that, for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \tag{1.1}$$

and the constant $\frac{(N-2)^2}{4}$ in (1.1) is sharp. Recently, it was proved by Nazarov (see [12], Proposition 4.1 and [6]) that the following Hardy inequality is valid for $f \in C_0^\infty(\mathbb{R}_+^N)$,

$$\int_{\mathbb{R}_+^N} |\nabla u(x)|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}_+^N} \frac{u(x)^2}{|x|^2} dx, \tag{1.2}$$

where $\mathbb{R}_+^N = \{(x_1, \cdots, x_n) | x_N > 0\}$, and the constant $\frac{N^2}{4}$ is sharp. This shows that the Hardy constant jumps from $\frac{(N-2)^2}{4}$ to $\frac{N^2}{4}$, when the singularity of the potential reaches the boundary. Inequality (1.2) was generalized by Su and Yang [14] to the cone $\mathbb{R}_{k_+}^N := \mathbb{R}^{N-k} \times (\mathbb{R}_+)^k = \{(x_1, \cdots, x_N) | x_{N-k+1} > 0, \cdots, x_N > 0\}$. For more information about this inequality and its applications, we refer to [2, 3] and the references therein.

The aim of this note is to prove similar Hardy type inequality on half spaces for Kohn's sublaplacian in H-type groups G, a remarkable class of stratified groups of step two introduced by Kaplan [11]. Let $G = (\mathbb{R}^m \times \mathbb{R}^n, \circ)$ with group law defined in Section 2. It was

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proved by Han et al. (see [10] and [5, 8, 9, 13] for analogous inequalities on Heisenberg group) that for $u \in C_0^\infty(G)$, there holds

$$\int_{\mathbb{R}^m\times\mathbb{R}^n} |\nabla_G u|^2 dxdt \geq \frac{(Q-2)^2}{4}\int_{\mathbb{R}^m\times\mathbb{R}^n} \frac{u^2}{\rho^2}|\nabla_G \rho|^2 dxdt, \tag{1.3}$$

and the constant $\frac{(Q-2)^2}{4}$ is sharp, where $Q = m+2n$, $\rho(x,y) = (|x|^4 + 16|t|^2)^{\frac{1}{4}}$ and ∇_G is the the horizontal gradient associated with the Kohn's sublaplacian on G (for details, see Section 2). In this note we shall show when the singularity is on the boundary, the Hardy constant also jumps. In fact, we have the following:

Theorem 1.1 Let $\alpha < Q-2$. There holds, for all $u \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n_+)$,

$$\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \frac{|\nabla_G u|^2}{\rho^\alpha} dxdt \geq \left(\frac{(Q+2-\alpha)^2}{4} + 2\alpha\right)\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \frac{u^2}{\rho^{2+\alpha}}|\nabla_G \rho|^2 dxdt, \tag{1.4}$$

and the constant $\frac{(Q+2-\alpha)^2}{4} + 2\alpha$ in (1.4) is sharp.

2 Notation and Preliminaries

We begin by describing the Lie groups and Lie algebras under consideration. For more information about H-type groups, we refer to [1, 11] and references therein. A H-type group G is a Carnot group of step two with the following properties: the Lie algebra $\mathfrak{g}$ of G is endowed with an inner product $\langle , \rangle$ such that, if $\mathfrak{z}$ is the center of $\mathfrak{g}$, then $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$ and moreover, for every fixed $z \in \mathfrak{z}$, the map $J_z : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), \omega\rangle = \langle z, [v,\omega]\rangle, \ \ \forall \omega \in \mathfrak{z}^\perp$$

is an orthogonal map whenever $\langle z, z\rangle = 1$. Set $m = \dim \mathfrak{z}^\perp$ and $n = \dim \mathfrak{z}$. In the sequel we shall fix on G a system of coordinates (x,t) and that the group law has the form

$$(x,t)\circ(x',t') = \begin{pmatrix} x_i + x_i', \ \ i = 1,2,\cdots,m \\ t_j + t_j' + \frac{1}{2} < x, U^{(j)}x' >, \ \ j = 1,2,\cdots,n \end{pmatrix}, \tag{2.1}$$

where the matrices $\{U^{(j)}\}_{j=1}^n$ have the following two properties (see [1])

(1) $U^{(j)}$ is a $m\times m$ Skew symmetric and orthogonal matrix, for every $j = 1,2,\cdots,n$;

(2) $U^{(i)}U^{(j)} + U^{(j)}U^{(i)} = 0$ for every $i, j \in \{1,2,\cdots,n\}$ with $i \neq j$.

A easy computation shows that the vector field in the algebra $\mathfrak{g}$ of $N = (\mathbb{R}^{m+n}, \circ)$ that agrees at the origin with $\frac{\partial}{\partial x_j}(j = 1,\cdots,m)$ is given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^n\left(\sum_{i=1}^m U_{i,j}^{(k)} x_i\right)\frac{\partial}{\partial t_k},$$

and that $\mathfrak{g}$ is spanned by the left-invariant vector fields $X_1,\cdots,X_m,\frac{\partial}{\partial t_1},\cdots,\frac{\partial}{\partial t_n}$. We use the notation $\nabla_G = (X_1,\cdots,X_m)$ and call it the horizontal gradient. The horizontal gradient can be written in the form

$$\nabla_G = \nabla_x - \frac{1}{2}U^{(1)}x\frac{\partial}{\partial t_1} - \cdots - \frac{1}{2}U^{(n)}x\frac{\partial}{\partial t_n} \tag{2.2}$$

with $x=(x_1,\cdots,x_m)$ and $\nabla_x=(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_m})$. The Kohn's sublaplacian on the H-type group G is given by

$$\begin{aligned}\Delta_G=\sum_{j=1}^m X_j^2&=\sum_{j=1}^m\left(\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{k=1}^n\left(\sum_{i=1}^m U_{i,j}^{(k)}x_i\right)\frac{\partial}{\partial t_k}\right)^2\\&=\Delta_x+\frac{1}{4}|x|^2\Delta_t+\sum_{k=1}^n\langle x,U^{(k)}\nabla_x\rangle\frac{\partial}{\partial t_k},\end{aligned}$$

where $\Delta_x=\sum\limits_{j=1}^m\left(\frac{\partial}{\partial x_j}\right)^2$ and $\Delta_t=\sum\limits_{k=1}^n\left(\frac{\partial}{\partial t_k}\right)^2$. Moreover, on functions $f(x,t)=\widetilde{f}(|x|,t)$, we have

$$\langle x,U^{(k)}\nabla_x\rangle\widetilde{f}(|x|,t)=0,\quad k=1,2,\cdots,n.$$

Hence

$$\Delta_G\widetilde{f}(|x|,t)=\Delta_x\widetilde{f}(|x|,t)+\frac{1}{4}|x|^2\Delta_t\widetilde{f}(|x|,t).\tag{2.3}$$

We also have

$$\begin{aligned}|\nabla_G\widetilde{f}(|x|,t)|^2=&\left|\nabla_x\widetilde{f}-\frac{1}{2}\sum_{j=1}^n U^{(j)}x\frac{\partial\widetilde{f}}{\partial t_j}\right|^2\\=&|\nabla_x\widetilde{f}|^2+\frac{1}{4}\left|\sum_{j=1}^n U^{(j)}x\frac{\partial\widetilde{f}}{\partial t_j}\right|^2-\sum_{j=1}^n\langle U^{(j)}x,\nabla_x\widetilde{f}(|x|,t)\rangle\frac{\partial\widetilde{f}}{\partial t_j}\\=&|\nabla_x\widetilde{f}|^2+\frac{1}{4}|x|^2|\nabla_t\widetilde{f}(|x|,t)|^2.\end{aligned}\tag{2.4}$$

To get the last inequality, we use the fact ($r=|x|$ in the equality below)

$$\langle U^{(j)}x,\nabla_x\widetilde{f}(|x|,t)\rangle=\left\langle U^{(j)}x,\frac{x}{|x|}\right\rangle\frac{\partial\widetilde{f}}{\partial r}=0$$

since $U^{(j)}$ $(1\le j\le n)$ is a skew-symmetric matric and

$$\begin{aligned}\left|\sum_{j=1}^n U^{(j)}x\frac{\partial\widetilde{f}}{\partial t_j}\right|^2=&\sum_{j=1}^n\left|U^{(j)}x\frac{\partial\widetilde{f}}{\partial t_j}\right|^2+2\sum_{i<j}\langle U^{(i)}x,U^{(j)}x\rangle\frac{\partial\widetilde{f}}{\partial t_i}\frac{\partial\widetilde{f}}{\partial t_j}\\=&|x|^2|\nabla_t\widetilde{f}(|x|,t)|^2+2\sum_{i<j}\langle(U^{(j)})^TU^{(i)}x,x\rangle\frac{\partial\widetilde{f}}{\partial t_i}\frac{\partial\widetilde{f}}{\partial t_j}\\=&|x|^2|\nabla_t\widetilde{f}(|x|,t)|^2-2\sum_{i<j}\langle U^{(j)}U^{(i)}x,x\rangle\frac{\partial\widetilde{f}}{\partial t_i}\frac{\partial\widetilde{f}}{\partial t_j}\\=&|x|^2|\nabla_t\widetilde{f}(|x|,t)|^2\end{aligned}$$

since $U^{(j)}U^{(i)}$ is also skew-symmetric for every $i\neq j$, for

$$(U^{(j)}U^{(i)})^T=(U^{(i)})^T(U^{(j)})^T=(-U^{(i)})(-U^{(j)})=U^{(i)}U^{(j)}=-U^{(j)}U^{(i)}.$$

For each real number $\lambda > 0$, there is a dilation naturally associated with the group structure which is usually denoted as $\delta_\lambda(\xi) = \delta_\lambda(x,t) = (\lambda x, \lambda^2 t)$, $\xi = (x,t) \in G$. However, for simplicity we will write $\lambda\xi$ to denote $\delta_\lambda(\xi)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = m + 2n$ is the homogeneous dimension of G. The anisotropic dilation structure on G introduces homogeneous norm $\rho(\xi) = \rho(x,t) = (|x|^4 + 16|t|^2)^{\frac{1}{4}}$. With this norm, we can define the Heisenberg ball centered at $\xi = (x,t)$ with radius R,

$$B(\xi, R) = \{v \in G : \rho(\xi^{-1} \circ v) < R\}.$$

For simplicity, we set

$$B_R = B(0,R) = \{v \in G : \rho(v) < R\}. \tag{2.5}$$

Given any $\xi = (x,t) \neq \mathbf{0}$, set $x^* = \frac{x}{\rho(\xi)}$, $t^* = \frac{t}{\rho(\xi)^2}$ and $\xi^* = (x^*, t^*)$. Then $\xi^* \in \Sigma = \{v \in G, \rho(v) = 1\}$, the Heisenberg unit sphere. Furthermore, we have the following polar coordinates on G (see [7]):

$$\int_G f(\xi) dx dt = \int_0^\infty \int_\Sigma f(\lambda\xi^*) \lambda^{Q-1} d\sigma dr$$

for all $f \in L^1(G)$ and for $\beta > -m$ (see [4]),

$$C_\beta := \int_\Sigma |x^*|^\beta d\sigma = \frac{1}{4^{n-\frac{1}{2}}} \frac{\pi^{\frac{n+m}{2}} \Gamma(\frac{m+\beta}{4})}{\Gamma(\frac{m}{2}) \Gamma(\frac{Q+\beta}{4})} > 0. \tag{2.6}$$

A function f on G is said to be radial if $f(x,t) = \widetilde{f}(\rho)$. If f is radial, it is easy to check

$$|\nabla_G f| = |f'(\rho)| \cdot |\nabla_G \rho| = |f'(\rho)| \frac{|x|}{\rho} \tag{2.7}$$

and

$$\Delta_G f = |\nabla_G \rho|^2 \left(f'' + \frac{Q-1}{\rho} f' \right) = \frac{|x|^2}{\rho^2} \left(f'' + \frac{Q-1}{\rho} f' \right). \tag{2.8}$$

3 The Proof

Before the proof of main results, we need the following lemma.

Lemma 3.1 Let $f \in C^\infty(G)$ be a radial function. There holds

(1) $|\nabla_G (t_n f)|^2 = t_n^2 |f'|^2 |\nabla_G \rho|^2 + \frac{|x|^2}{4} f^2 + \frac{t_n |x|^2}{4} \cdot \frac{\partial f^2}{\partial t_n}$;

(2) $\Delta_G (t_n f) = t_n (f'' + \frac{Q+3}{\rho} f') |\nabla_G \rho|^2$.

Proof (1) Since f is radial, we get, by (2.5),

$$\begin{aligned} |\nabla_G (t_n f)|^2 =& |\nabla_x (t_n f)|^2 + \frac{1}{4} |x|^2 |\nabla_t (t_n f)|^2 \\ =& t_n^2 |\nabla_x f|^2 + \frac{|x|^2}{4} \left(t_n^2 |\nabla_t f|^2 + f^2 |\nabla_t t_n|^2 + 2 t_n f \langle \nabla_t t_n, \nabla_t f \rangle \right) \\ =& t_n^2 \left(|\nabla_x f|^2 + \frac{|x|^2}{4} |\nabla_t f|^2 \right) + \frac{|x|^2}{4} f^2 + \frac{t_n |x|^2}{2} \cdot f \frac{\partial f}{\partial t_n} \\ =& t_n^2 |\nabla_G f|^2 + \frac{|x|^2}{4} f^2 + \frac{t_n |x|^2}{4} \cdot \frac{\partial f^2}{\partial t_n}. \end{aligned}$$

It remains to use (2.7) and the desired result follows.

(2) By (2.3) and (2.8), we have

$$\begin{aligned}\Delta_G(t_nf) =&\Delta_x(t_nf)+\frac{|x|^2}{4}\Delta_t(t_nf)=t_n\Delta_xf+\frac{|x|^2}{4}\left(t_n\Delta_tf+2\frac{\partial f}{\partial t_n}\right)\\ =&t_n\left(\Delta_xf+\frac{|x|^2}{4}\Delta_tf\right)+\frac{|x|^2}{4}f'\cdot\frac{\partial\rho}{\partial t_n}=t_n\Delta_Gf+4t_n\frac{|x|^2}{\rho^2}\cdot\frac{f'}{\rho}\\ =&t_n\left(f''+\frac{Q-1}{\rho}f'\right)|\nabla_G\rho|^2+t_n|\nabla_G\rho|^2\cdot\frac{4f'}{\rho}\\ =&t_n\left(f''+\frac{Q+3}{\rho}f'\right)|\nabla_G\rho|^2.\end{aligned}$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 First, we consider the case $\alpha=0$. Using the substitution $u=t_n\rho^{-\frac{Q+2}{2}}f$, we get

$$\begin{aligned}\int_{\mathbb{R}^m\times\mathbb{R}^n_+}|\nabla_Gu|^2=&\int_{\mathbb{R}^m\times\mathbb{R}^n_+}\left[|\nabla_G(t_n\rho^{-\frac{Q+2}{2}})|^2f^2+|\nabla_Gf|^2\frac{t_n^2}{\rho^{Q+2}}+\frac{\langle\nabla_G(t_n^2\rho^{-(Q+2)},\nabla_Gf^2\rangle}{2}\right]\\ \geq&\int_{\mathbb{R}^m\times\mathbb{R}^n_+}\left(|\nabla_G(t_n\rho^{-\frac{Q+2}{2}})|^2f^2+\frac{1}{2}\langle\nabla_G(t_n^2\rho^{-(Q+2)},\nabla_Gf^2\rangle\right)\\ =&\int_{\mathbb{R}^m\times\mathbb{R}^n_+}f^2\left(|\nabla_G(t_n\rho^{-\frac{Q+2}{2}})|^2-\frac{1}{2}\Delta_G(t_n^2\rho^{-(Q+2)})\right).\end{aligned}$$

Notice that, for $g\in C^\infty(G)$,

$$\Delta_Gg^2=\sum_{j=1}^mX_j^2g^2=2g\sum_{j=1}^mX_j^2g+2\sum_{j=1}^m|X_jg|^2=2g\Delta_Gg+2|\nabla_Gg|^2.$$

We have, by Lemma 3.1 (2),

$$\begin{aligned}&|\nabla_G(t_n\rho^{-\frac{Q+2}{2}})|^2-\frac{1}{2}\Delta_G(t_n^2\rho^{-(Q+2)}=-t_n\rho^{-\frac{Q+2}{2}}\Delta_G(t_n\rho^{-\frac{Q+2}{2}})\\ =&-t_n\rho^{-\frac{Q+2}{2}}\cdot t_n\left(\frac{(Q+2)(Q+4)}{4}-\frac{(Q+2)(Q+3)}{2}\right)\rho^{-\frac{Q+6}{2}}|\nabla_G\rho|^2\\ =&\frac{(Q+2)^2}{4}t_n^2\rho^{-(Q+4)}|\nabla_G\rho|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^m\times\mathbb{R}^n_+}|\nabla_Gu|^2\geq&\int_{\mathbb{R}^m\times\mathbb{R}^n_+}f^2\left(|\nabla_G(t_n\rho^{-\frac{Q+2}{2}})|^2-\frac{1}{2}\Delta_G(t_n^2\rho^{-(Q+2)})\right)\\ =&\frac{(Q+2)^2}{4}\int_{\mathbb{R}^m\times\mathbb{R}^n_+}f^2t_n^2\rho^{-(Q+4)}|\nabla_G\rho|^2 \qquad (3.1)\\ =&\frac{(Q+2)^2}{4}\int_{\mathbb{R}^m\times\mathbb{R}^n_+}\frac{u^2}{\rho^2}|\nabla_G\rho|^2.\end{aligned}$$

Now we show the constant $\frac{(Q+2)^2}{4}$ in (3.1) is sharp. Consider the family of function $g_\varepsilon = t_n f_\varepsilon(\rho)$, where

$$f_\varepsilon(\rho) = \begin{cases} \varepsilon^{-(Q+2)/2}, & \rho \le \varepsilon\ ; \\ \rho^{-(Q+2)/2}, & \rho > \varepsilon\ . \end{cases}$$

We take g_ε as the test function. By (2.5) and symmetry,

$$\begin{aligned} 2\int_{\mathbb{R}^m\times\mathbb{R}^n_+} |\nabla_G g_\varepsilon|^2 =& 2\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \left(|\nabla_x g_\varepsilon|^2 + \frac{|x|^2}{4}|\nabla_t g_\varepsilon|^2\right) \\ =& \int_{\mathbb{R}^m\times\mathbb{R}^n} \left(|\nabla_x g_\varepsilon|^2 + \frac{|x|^2}{4}|\nabla_t g_\varepsilon|^2\right) \\ =& \int_{G\setminus B_\varepsilon} \left(|\nabla_x g_\varepsilon|^2 + \frac{|x|^2}{4}|\nabla_t g_\varepsilon|^2\right) dxdt + \varepsilon^{-Q-2}\int_{B_\varepsilon} \frac{|x|^2}{4} dxdt \end{aligned}$$

and

$$2\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2 = \int_{\mathbb{R}^m\times\mathbb{R}^n} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2 \ge \int_{G\setminus B_\varepsilon} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2.$$

By Lemma 3.1 (1), we have, for $\rho > \varepsilon$,

$$\begin{aligned} |\nabla_G g_\varepsilon|^2 =& \frac{(Q+2)^2}{4} t_n^2 \rho^{-(Q+4)}|\nabla_G\rho|^2 + \frac{|x|^2}{4}\rho^{-(Q+2)} + \frac{|x|^2}{4}\cdot t_n \frac{\partial \rho^{-(Q+2)}}{\partial t_n} \\ =& \frac{(Q+2)^2}{4} t_n^2 \rho^{-(Q+4)}|\nabla_G\rho|^2 + \frac{|x|^2}{4}\rho^{-(Q+2)} - 2(Q+2)\frac{|x|^2 t_n^2}{\rho^{Q+6}}. \end{aligned}$$

Since $\displaystyle\int_{G\setminus B_\varepsilon} \left(\frac{|x|^2}{4}\rho^{-(Q+2)} - 2(Q+2)\frac{|x|^2 t_n^2}{\rho^{Q+6}}\right) dxdt = 0$, we have

$$2\int_{\mathbb{R}^m\times\mathbb{R}^n_+} |\nabla_G g_\varepsilon|^2 = \frac{(Q+2)^2}{4}\int_{G\setminus B_\varepsilon} t_n^2\rho^{-(Q+4)}|\nabla_G\rho|^2 + \varepsilon^{-Q-2}\int_{B_\varepsilon}\frac{|x|^2}{4}dxdt.$$

Therefore

$$\begin{aligned} &\inf_{u\in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^n_+)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n_+} |\nabla_G u|^2 dxdt}{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \frac{u^2}{\rho^2}|\nabla_G\rho|^2 dxdt} \le \frac{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n_+} |\nabla_G g_\varepsilon|^2 dxdt}{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n_+} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2 dxdt} \\ =& \frac{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n} |\nabla_G g_\varepsilon|^2 dxdt}{\displaystyle\int_{\mathbb{R}^m\times\mathbb{R}^n} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2 dxdt} \le \frac{\displaystyle\int_{G} |\nabla_G g_\varepsilon|^2 dxdt}{\displaystyle\int_{G\setminus B_\varepsilon} \frac{g_\varepsilon^2}{\rho^2}|\nabla_G\rho|^2 dxdt} \\ =& \frac{(Q+2)^2}{4} + \varepsilon^{-Q-2}\frac{\displaystyle\int_{B_\varepsilon}\frac{|x|^2}{4}dxdt}{\displaystyle\int_{G\setminus B_\varepsilon} t_n^2\rho^{-(Q+4)}|\nabla_G\rho|^2} \to \frac{(Q+2)^2}{4},\ \ \varepsilon\to 0. \end{aligned}$$

This completes the proof.

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H型群内上半空间Hardy不等式的最佳常数问题

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摘要: 本文研究了H型幂零李群上Hardy不等式的问题. 利用基本解的方法, 获得了相关李群上的Hardy不等式, 并且所得到的相关Hardy常数是最佳的.

关键词: Hardy不等式; H型群; 最佳常数

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