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Bazilevič 函数的对数系数

牛潇萌, 李书海
(赤峰学院数学与统计学院, 内蒙古 赤峰 024000)

摘 要: 本文研究了 Bazilevič 函数类 $B_\alpha(C,D)$ 的对数系数. 利用构造一个非负函数和对复变函数模的积分进行估计的方法, 获得了 $B_\alpha(C,D)$ 的对数系数, 推广了一些已有的相关结果.

关键词: 单叶函数; 对数系数; Bazilevič 函数
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1 引言

设 $f(z)$ 与 $g(z)$ 在单位圆盘 $U=\{z:|z|<1\}$ 内解析, 如果存在 U 内满足 $|\omega(z)|\le|z|$ 的解析函数 $\omega(z)$, 使得 $g(z)=f(\omega(z))$, 则称 $g(z)$ 从属于 $f(z)$, 记作 $g(z)\prec f(z)$.

用 $P(C,D)$ $(-1\le D<C\le 1)$ 表示在单位圆盘 U 内解析并且满足条件 $p(z)\prec\frac{1+Cz}{1+Dz}$ 的所有函数 $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$ 的全体. 显然 $P(1,-1)=P$ 为熟知的正实部函数类 [1].

设 S 表示在单位圆盘 U 内单叶解析函数 $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ 构成的函数类. 若函数 $f(z)\in S$ 满足条件 $\frac{zf'(z)}{f(z)}\in P$, 则称 $f(z)$ 属于星象函数类 S^*; 设函数 $f(z)\in S$, 如果存在函数 $g(z)\in S^*$, 使得 $\frac{zf'(z)}{g(z)}\in P$, 则称 $f(z)$ 属于近于凸函数类 C. 设 b 为复数且 $b\ne 0$, $f(z)\in S$, 如果存在 $g(z)\in S^*$ 使得 $\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{g(z)}-1\right)\right\}\in P$, 则称 $f(z)$ 属于复阶近于凸函数类 $C(b)$[2]. 设函数 $f(z)\in S$, $\alpha>0$, 如果存在函数 $g(z)\in S^*$, 使得 $\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\in P$, 则称 $f(z)$ 属于 Bazilevič 函数类 B_α. 一些学者从不同的角度出发, 研究了一些有趣的 Bazilevič 函数子族 [3–5].

设 $\alpha>0$, $\beta\in R$, $f(z)\in S$, 如果存在 $g(z)\in S^*$ 使得 $\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{\mathrm{i}\beta}\in P$, 则称 $f(z)\in B(\alpha,\beta)$[3].

文献 [5] 给出了如下 α 型 β 级 Bazilevič 函数类 $B_\alpha(\beta)$.

定义 1.1 [5] 设 $f(z)\in S$, $\alpha\ge 0$, $0\le\beta<1$, 若存在 $g(z)\in S^*$, 使得

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\right\}>\beta, \tag{1.1}$$

则称 $f(z)\in B_\alpha(\beta)$, 其中的幂函数取主值. 显然 $B_\alpha(0)=B_\alpha$.

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作者简介: 牛潇萌 (1982–), 女, 蒙古族, 内蒙古赤峰, 讲师, 主要研究方向: 复分析及其应用.

本文引入如下 Bazilevič 函数.

定义 1.2 设 $f(z)\in S$, $\alpha>0$, $-1\leq D<C\leq 1$, 如果存在 $g(z)\in S^*$, 使得

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\in P(C,D),$$

则称 $f(z)\in B_\alpha(C,D)$. 显然 $B_\alpha(1,-1)=B_\alpha$, $B_\alpha(1-2\beta,-1)=B_\alpha(\beta)$.

用 Y 表示圆对称函数类 [6]. 设 $f(z)\in S$, 若

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}b_nz^n, \tag{1.2}$$

则称 b_n 为 $f(z)$ 的对数系数. 对数系数的估计在单叶函数的系数估计中有重要作用. Keobe 函数 $k(z)=z(1-z)^{-2}$ 的对数系数为 $b_n=1/n$. 对 b_n $(n\geq 2)$ 的估计, 现在已经证明

(1) 当 $f(z)\in S^*$ 时, $b_n\leq\frac{1}{n}$ [7];

(2) 当 $f(z)\in C$ 时, $b_n\leq A\frac{\log n}{n}$, 其中 A 表示一个绝对常数 [8];

(3) 当 $f(z)\in B_\alpha$ 时, $b_n\leq A(1+\alpha)\frac{\log n}{n}$, 其中 A 表示一个绝对常数 [9];

(4) 当 $f(z)\in Y$ 时, $b_n\leq A\frac{\log n}{n}$, 其中 A 表示一个绝对常数 [10];

(5) 当 $f(z)\in B(\alpha,\beta)$ 时, $b_n\leq A\frac{\log n}{n}$, 其中 A 表示一个绝对常数 [11];

(6) 当 $f(z)\in C(b)$ 时, $b_n\leq A\frac{\log n}{n}$, 其中 A 表示一个绝对常数 [12].

不同的地方, A 表示不同常数.

本文研究 $B_\alpha(C,D)$ 的对数系数, 推广了文献 [9] 的结果.

2 引理

为了得到 $B_\alpha(C,D)$ 的对数系数, 需要如下引理

引理 2.1 [13] 如果 $p(z)\in P(C,D)$, 其中 $-1\leq D<C\leq 1$, 则 $\frac{1-C}{1-D}<\mathrm{Re}(p(z))<\frac{1+C}{1+D}$.

引理 2.2 [11] 设 $f(z)\in S$, 则 $\mathrm{Re}\frac{zf'(z)}{f(z)}=\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right)+1$.

引理 2.3 [11] 设 $f(z)\in S$, $\alpha\in\mathbb{C}$. 则对 $z=re^{\mathrm{i}\theta}$, $0<r<1$, 有

$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{f(z)}{z}\right)^{\alpha}\right)=\alpha\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right).$$

引理 2.4 [8] 设 $f(z)\in S$, 则对 $z=re^{\mathrm{i}\theta}$, $\frac{1}{2}\leq r<1$, 有

(1) $\frac{1}{2\pi}\displaystyle\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2d\theta\leq 1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}$;

(2) $\frac{1}{2\pi}\displaystyle\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2d\theta dr\leq 1+2\log\frac{1}{1-r}$.

引理 2.5 [14] 设 $g(z)\in S^*$, 则 $\frac{\partial}{\partial\theta}\arg g(z)>0$ 且 $\displaystyle\int_0^{2\pi}\frac{\partial}{\partial\theta}\arg g(z)=2\pi$.

引理 2.6 [15] 设 $f(z)\in S$, 则对 $0<r<1$, 有 $\sum\limits_{k=1}^{\infty}k|b_k|^2r^{2k}\leq\log\frac{1}{1-r}$.

引理 2.7 设 $f(z)\in B_\alpha(C,D)$, $g(z)\in S^*$ 使得 $\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\in P(C,D)$. 则对 $z=re^{\mathrm{i}\theta}$, $0\leq r<1$, 有

(1) $J_1=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Re}\frac{zf'(z)}{f(z)}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\leq 3$;

(2) $J_2=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Im}\frac{zf'(z)}{f(z)}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\leq 3+4\alpha+8\alpha\log\frac{1}{1-r}$;

(3) $J_3=\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{2\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}e^{\mathrm{i}n\theta}d\theta\right|\leq 4\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}$.

证 (1) 由引理 2.2 和 引理 2.3 可知

$$\begin{aligned}J_1&\leq\frac{1}{2\pi}\left|\int_0^{2\pi}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right)e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\\&=\frac{1}{2\pi}\left|\int_0^{2\pi}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{1}{\alpha\mathrm{i}}\frac{\partial}{\partial\theta}e^{\mathrm{i}\arg\left(\frac{f(z)}{z}\right)^{\alpha}}e^{-\mathrm{i}\arg\left(\frac{g(z)}{z}\right)^{\alpha}}d\theta\right|=J_{11}+J_{12}.\end{aligned}$$

易知 $J_{11}\leq 1$, 利用分部积分公式和引理 2.5 可得

$$\begin{aligned}J_{12}&=\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{\partial}{\partial\theta}\arg\left(\frac{g(z)}{z}\right)e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\\&\leq\frac{1}{2\pi}\int_0^{2\pi}\left\{\left|\frac{\partial}{\partial\theta}\arg g(z)\right|+\left|\frac{\partial}{\partial\theta}\arg z\right|\right\}d\theta\leq 2.\end{aligned}$$

所以 $J_1=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Re}\frac{zf'(z)}{f(z)}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\leq J_{11}+J_{12}\leq 3$.

(2) 记 $\mathrm{Re}\frac{zf'(z)}{f(z)}=u(re^{\mathrm{i}\theta})$, $\mathrm{Im}\frac{zf'(z)}{f(z)}=v(re^{\mathrm{i}\theta})$, 由柯西 – 黎曼条件可知对于 $\frac{1}{2}\leq r<1$, 有

$$v(re^{\mathrm{i}\theta})-v(\frac{1}{2}e^{\mathrm{i}\theta})=\int_{\frac{1}{2}}^{r}\frac{\partial v(re^{\mathrm{i}\theta})}{\partial r}dr=-\int_{\frac{1}{2}}^{r}\frac{1}{r}\frac{\partial u(re^{\mathrm{i}\theta})}{\partial\theta}dr,$$

则

$$J_2\leq\frac{1}{2\pi}\left|\int_0^{2\pi}v(\frac{1}{2}e^{\mathrm{i}\theta})e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\int_{\frac{1}{2}}^{r}\frac{1}{r}\frac{\partial u(re^{\mathrm{i}\theta})}{\partial\theta}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}drd\theta\right|=J_{21}+J_{22}.$$

而

$$J_{21}\leq\frac{1}{2\pi}\int_0^{2\pi}\max_{\theta\in[0,2\pi]}\left|v(\frac{1}{2}e^{\mathrm{i}\theta})\right|d\theta\leq\max_{\theta\in[0,2\pi]}\left|\frac{\frac{1}{2}f'(\frac{1}{2}e^{\mathrm{i}\theta})}{f(\frac{1}{2}e^{\mathrm{i}\theta})}\right|\leq\frac{1+\frac{1}{2}}{1-\frac{1}{2}}=3.$$

由分部积分和引理 2.3 可知

$$J_{22}=\frac{\alpha}{2\pi}\left|\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\frac{1}{r}u(re^{\mathrm{i}\theta})e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\left(\frac{\partial}{\partial\theta}\arg\frac{f(z)}{z}-\frac{\partial}{\partial\theta}\arg\frac{g(z)}{z}\right)d\theta dr\right|.$$

由引理 2.2 和 $\frac{1}{2}\leq r<1$ 可知

$$\begin{aligned}J_{22}&\leq\frac{\alpha}{\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\mathrm{Re}\frac{zf'(z)}{f(z)}\right|\left|\mathrm{Re}\frac{zf'(z)}{f(z)}-\mathrm{Re}\frac{zg'(z)}{g(z)}\right|d\theta dr\\&\leq 2\alpha\left(\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2d\theta dr+\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|\left|\frac{zg'(z)}{g(z)}\right|d\theta dr\right).\end{aligned}$$

由引理 2.4 和 Schwarz 不等式可知

$$\begin{aligned}J_{22}\leq&2\alpha\left(1+2\log\frac{1}{1-r}\right)+2\alpha\left(\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_{0}^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^{2}d\theta dr\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_{0}^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^{2}d\theta dr\right)^{\frac{1}{2}}\\\leq&4\alpha\left(1+2\log\frac{1}{1-r}\right).\end{aligned}$$

所以

$$J_{2}=\frac{1}{2\pi}\left|\int_{0}^{2\pi}\mathrm{Im}\frac{zf'(z)}{f(z)}e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|\leq J_{21}+J_{21}\leq 3+4\alpha\left(1+2\log\frac{1}{1-r}\right).$$

(3) 由 (2) 式可知

$$\frac{zf'(z)}{f(z)}=1+z\left(\log\frac{f(z)}{z}\right)'=1+\sum_{k=1}^{\infty}2kb_{k}z^{k}.\tag{2.1}$$

所以

$$\begin{aligned}\frac{zf'(z)}{f(z)}e^{\mathrm{i}n\theta}=&e^{in\theta}\left(1+\sum_{k=1}^{\infty}2kb_{k}z^{k}\right)=e^{\mathrm{i}n\theta}+\sum_{k=1}^{\infty}2kb_{k}r^{k}e^{\mathrm{i}(n+k)\theta}\\=&\frac{1}{\mathrm{i}}\frac{\partial}{\partial\theta}\left(\frac{e^{in\theta}}{n}+\sum_{k=1}^{\infty}\frac{2kb_{k}r^{k}e^{\mathrm{i}(n+k)\theta}}{n+k}\right)=\frac{1}{\mathrm{i}}\frac{\partial}{\partial\theta}F(z).\end{aligned}$$

由引理 2.2, 引理 2.3 和分部积分可得

$$\begin{aligned}J_{3}=&\frac{1}{2\pi}\left|\int_{0}^{2\pi}\frac{1}{i}\frac{\partial}{\partial\theta}F(z)e^{2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|=\frac{\alpha}{\pi}\left|\int_{0}^{2\pi}F(z)e^{2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\frac{\partial}{\partial\theta}\arg\frac{f(z)}{g(z)}d\theta\right|\\=&\frac{\alpha}{\pi}\left|\int_{0}^{2\pi}F(z)e^{2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\left(\mathrm{Re}\frac{zf'(z)}{f(z)}-\mathrm{Re}\frac{zg'(z)}{g(z)}\right)d\theta\right|\\\leq&\frac{\alpha}{\pi}\left|\int_{0}^{2\pi}\left(|F(z)||\frac{zf'(z)}{f(z)}|+|F(z)||\frac{zg'(z)}{g(z)}|\right)d\theta\right|\\\leq&2\alpha\left(\frac{1}{2\pi}\int_{0}^{2\pi}|F(z)|^{2}d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{0}^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^{2}d\theta\right)^{\frac{1}{2}}\\&+2\alpha\left(\frac{1}{2\pi}\int_{0}^{2\pi}|F(z)|^{2}d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{0}^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^{2}d\theta\right)^{\frac{1}{2}}.\end{aligned}$$

由引理 2.6 可知

$$\frac{1}{2\pi}\int_{0}^{2\pi}|F(z)|^{2}d\theta=\frac{1}{n^{2}}+4\sum_{k=1}^{\infty}\frac{k^{2}|b_{k}|^{2}r^{2k}}{(n+k)^{2}}\leq\frac{1}{n^{2}}+\frac{4}{n}\sum_{k=1}^{\infty}k|b_{k}|^{2}r^{2k}\leq\frac{1}{n^{2}}+\frac{4}{n}\log\frac{1}{1-r}.$$

所以由引理 2.4 可知

$$J_{3}\leq 4\alpha\left(\frac{1}{n^{2}}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}.$$

引理 2.8 设 $f(z)\in B_\alpha(C,D)$ $(\alpha>0)$, 则对 $z=re^{\mathrm{i}\theta}$, $0\le r<1$, 有

$$\frac{1-|D|r}{1+|C|r}\frac{1-r}{1+r}\le\left|\left(\frac{g(z)}{f(z)}\right)^\alpha\right|\le\frac{1+|D|r}{1-|C|r}\frac{1+r}{1-r}.$$

证 设 $f(z)\in B_\alpha(C,D)$, 则存在 $g(z)\in S^*$ 使得

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha\prec\frac{1+Cz}{1+Dz}.$$

由从属关系定义可知, 存在 Schwarz 函数 $\omega(z)$, 使得

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha=\frac{1+C\omega(z)}{1+D\omega(z)}.$$

经简单计算有

$$\left(\frac{g(z)}{f(z)}\right)^\alpha=\frac{1+D\omega(z)}{1+C\omega(z)}\frac{zf'(z)}{f(z)}.$$

因为 $|\omega(z)|\le|z|$, 所以

$$\frac{1-|D||z|}{1+|C||z|}\left|\frac{zf'(z)}{f(z)}\right|\le\left|\left(\frac{g(z)}{f(z)}\right)^\alpha\right|\le\frac{1+|D||z|}{1-|C||z|}\left|\frac{zf'(z)}{f(z)}\right|.$$

又因为

$$\frac{1-r}{1+r}\le\left|\frac{zf'(z)}{f(z)}\right|\le\frac{1+r}{1-r},$$

所以

$$\frac{1-|D|r}{1+|C|r}\frac{1-r}{1+r}\le\left|\left(\frac{g(z)}{f(z)}\right)^\alpha\right|\le\frac{1+|D|r}{1-|C|r}\frac{1+r}{1-r}.$$

3 主要结论

定理 3.1 设 $f(z)\in B_\alpha(C,D)$, 则对 $n\ge 2$,

$$|b_n|\le A\frac{1}{n}\log n+B\frac{1}{n}+16\frac{1-C}{1-D}\frac{(1+|D|)n-|D|}{(1-|C|)n+|C|},$$

其中 $A=32\alpha+8\alpha\left(\frac{1}{4(\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}$, $B=24+16\alpha$.

证 设 $f(z)\in B_\alpha(C,D)$, 则存在 $g(z)\in S^*$, 使得 $\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha\in P(C,D)$. 由引理 2.1 可知

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha-\frac{1-C}{1-D}\right\}>0.$$

记 $q(z)=\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha-\frac{1-C}{1-D}$, 则 $\mathrm{Re}q(z)>0$. 由 (2.1) 式可知对 $z=re^{\mathrm{i}\theta}$,

$$2nb_n=\frac{1}{2\pi\mathrm{i}}\oint_{|z|=r}\frac{zf'(z)}{f(z)}z^{-n-1}dz=\frac{1}{2\pi}\int_0^{2\pi}\frac{zf'(z)}{f(z)}r^{-n}e^{-\mathrm{i}n\theta}d\theta,$$

因此

$$\begin{aligned}|2nb_nr^n| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{-\mathrm{i}n\theta}d\theta\right| \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi}\left(\left(\frac{g(z)}{f(z)}\right)^{\alpha}q(z)e^{-\mathrm{i}n\theta}+\left(\frac{g(z)}{f(z)}\right)^{\alpha}\frac{1-C}{1-D}e^{-\mathrm{i}n\theta}\right)d\theta\right|.\end{aligned}$$

由于 $q(z)=2\mathrm{Re}q(z)-\overline{q(z)}$, 所以

$$\begin{aligned}|2nb_nr^n| \leq &\frac{1}{2\pi}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}2\mathrm{Re}q(z)e^{-\mathrm{i}n\theta}d\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\overline{q(z)}e^{-\mathrm{i}n\theta}d\theta\right| \\ &+\frac{1}{2\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-\mathrm{i}n\theta}d\theta\right| = I_1+I_2+I_3.\end{aligned}$$

因为 $\mathrm{Re}q(z)>0$, 所以

$$\begin{aligned}I_1 \leq &\frac{1}{\pi}\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|\mathrm{Re}q(z)d\theta \leq \frac{1}{\pi}\left|\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|q(z)d\theta\right| \\ \leq &\frac{1}{\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|\left(\frac{f(z)}{g(z)}\right)^{\alpha}d\theta\right|+\frac{1}{\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|d\theta\right| \\ \leq &\frac{1}{\pi}\left|\int_0^{2\pi}\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\frac{1}{\pi}\left|\int_0^{2\pi}\mathrm{Im}\left(\frac{zf'(z)}{f(z)}\right)e^{\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right| \\ &+\frac{1}{\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|d\theta\right|.\end{aligned}$$

由引理 2.7 可知

$$\begin{aligned}I_1 \leq &12+8\alpha+16\alpha\log\frac{1}{1-r}+\frac{1}{\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|d\theta\right|, \\ I_2 \leq &\frac{1}{2\pi}\left|\int_0^{2\pi}\overline{\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-\mathrm{i}n\theta}d\theta\right|+\frac{1}{2\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-\mathrm{i}n\theta}d\theta\right| \\ = &\frac{1}{2\pi}\left|\int_0^{2\pi}\left(\frac{zf'(z)}{f(z)}\right)e^{2\mathrm{i}\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}e^{\mathrm{i}n\theta}d\theta\right|+\frac{1}{2\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-\mathrm{i}n\theta}d\theta\right|.\end{aligned}$$

故由引理 2.7 可知

$$I_2 \leq 4\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}+\frac{1}{2\pi}\frac{1-C}{1-D}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-\mathrm{i}n\theta}d\theta\right|.$$

由引理 2.8 可知

$$I_3 \leq \frac{1}{2\pi}\frac{1-C}{1-D}\int_0^{2\pi}\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|d\theta \leq \frac{1-C}{1-D}\frac{1+|D|r}{1-|C|r}\frac{1+r}{1-r},$$

所以

$$\begin{aligned}|b_n| \leq &\frac{1}{n}r^{-n}\left(6+4\alpha+8\alpha\log\frac{1}{1-r}\right)+\frac{2}{n}r^{-n}\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}} \\ &+\frac{2}{n}r^{-n}\frac{1-C}{1-D}\frac{1+|D|r}{1-|C|r}\frac{1+r}{1-r}.\end{aligned}$$

设 $r=1-\frac{1}{n}$, 则

$$\begin{aligned}|b_n|\leq&(6+4\alpha)\frac{1}{n}\left(1-\frac{1}{n}\right)^{-n}+8\alpha\frac{1}{n}\left(1-\frac{1}{n}\right)^{-n}\log n\\&+\frac{2}{n}\left(1-\frac{1}{n}\right)^{-n}\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log n+\frac{4}{n}\log\frac{1}{1-\sqrt{1-\frac{1}{n}}}+16\log n\log\frac{1}{1-\sqrt{1-\frac{1}{n}}}\right)^{\frac{1}{2}}\\&+\frac{2}{n}\left(1-\frac{1}{n}\right)^{-n}\frac{1-C}{1-D}\frac{(1+|D|)n-|D|}{(1-|C|)n+|C|}(2n-1).\end{aligned}$$

因为 $(1-\frac{1}{n})^{-n}\leq 4$, $\frac{1}{n}\leq\frac{\log n}{n\log 2}$ $(n\geq 2)$, $\log\frac{1}{1-\sqrt{1-\frac{1}{n}}}\leq\log 2n\leq 2\log n$, 所以

$$\begin{aligned}|b_n|\leq&(6+4\alpha)\frac{1}{n}\cdot 4+8\alpha\frac{1}{n}\cdot 4\cdot\log n\\&+\frac{8\alpha}{n}\left(\frac{(\log n)^2}{n^2(\log 2)^2}+4\frac{1}{n\log 2}(\log n)^2+8\frac{1}{n\log 2}(\log n)^2+32(\log n)^2\right)^{\frac{1}{2}}\\&+16\frac{1-C}{1-D}\frac{(1+|D|)n-|D|}{(1-|C|)n+|C|}\\=&32\alpha\frac{1}{n}\log n+8\alpha\left(\frac{1}{n^2(\log 2)^2}+\frac{12}{n\log 2}+32\right)^{\frac{1}{2}}\frac{1}{n}\log n\\&+(24+16\alpha)\frac{1}{n}+16\frac{1-C}{1-D}\frac{(1+|D|)n-|D|}{(1-|C|)n+|C|}\\\leq&A\frac{1}{n}\log n+B\frac{1}{n}+16\frac{1-C}{1-D}\frac{(1+|D|)n-|D|}{(1-|C|)n+|C|},\end{aligned}$$

其中 $A=32\alpha+8\alpha\left(\frac{1}{4(\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}$, $B=24+16\alpha$.

因为当 $C=1,D=-1$ 时, $B_\alpha(1,-1)=B_\alpha$, 可得

推论 3.2 [9] 设 $f(z)\in B_\alpha$, 则对 $n\geq 2$,

$$|b_n|\leq E\left(1+\alpha\right)\frac{1}{n}\log n,$$

其中 E 是绝对常数.

证 由定理 3.1 可知, 当 $C=1,D=-1$ 时,

$$\begin{aligned}|b_n|&\leq A\frac{1}{n}\log n+B\frac{1}{n}\leq A\frac{1}{n}\log n+B\frac{1}{n}\frac{\log n}{\log 2}=\left(A+B\frac{1}{\log 2}\right)\frac{1}{n}\log n\\&=\left(32\alpha+8\alpha\left(\frac{1}{4(\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}+(24+16\alpha)\frac{1}{\log 2}\right)\frac{1}{n}\log n\\&=\left\{\frac{24}{\log 2}+\left(32+8\left(\frac{1}{4(\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}+\frac{16}{\log 2}\right)\alpha\right\}\frac{1}{n}\log n.\end{aligned}$$

记 $E=32+8\left(\frac{1}{4(\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}+\frac{16}{\log 2}$, 则

$$|b_n|\leq E\left(\frac{24}{\log 2}\frac{1}{E}+\alpha\right)\frac{1}{n}\log n\leq E\left(1+\alpha\right)\frac{1}{n}\log n.$$

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THE LOGARITHMIC COEFFICIENTS OF BAZILEVIČ FUNCTIONS

NIU Xiao-meng, LI Shu-hai

(*School of Mathematics and Statistics, Chifeng University, Chifeng 024000, China*)

Abstract: In this paper, we discuss the logarithmic coefficients of Bazilevič functions $B_\alpha(C,D)$. By using a nonnegative function and estimate the integration of model of a complex function, we obtain the logarithmic coefficients of $B_\alpha(C,D)$, which generalize some known results.

Keywords: univalent functions; logarithmic coefficients; Bazilevič functions

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