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THE MILITARU-STEFAN LIFTING THEOREM OVER WEAK HOPF ALGEBRAS

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Abstract: The paper is concerned with extension modules for weak Hopf-Galois extensions. By using faithfully flat weak Hopf-Galois extension theory, we investigative the Militaru-Stefan lifting theorem over weak Hopf algebras, which extends the corresponding result given in [10]. Moreover, we characterizer weak stable modules by a weak cleft extension of endomorphism rings of induced modules.

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1 Introduction and Preliminaries

Let H be a Hopf algebra, A a faithfully flat Hopf-Galois extension over its subalgebra of coinvariants B and M a B-module. Generalizing a result due to Dade [7] on strongly graded rings, Militaru and Stefan checked the following classical result: the B-action on M can be extended to an A-action if and only if there exists a total integral and algebra map $\phi : H \to \mathrm{END}_A(M \otimes_B A)$, where $\mathrm{END}_A(M \otimes_B A)$, consisting of the rational space of $\mathrm{End}_A(M \otimes_B A)$, was introduced by Ulbrich [17]. Moreover, Caenepeel also studied and obtained this result using isomorphisms of small categories in [4].

The purpose of the present paper is to investigate the above result in the case of weak Hopf algebras. But this is not a direct promotion, we give a new simple proof.

Weak bialgebras (or weak Hopf algebras), as a generalization of ordinary bialgebras (or Hopf algebras) and groupoid algebras, were introduced by Böhm and Szlachányi in [3] (see also their joint work with Nill in [2]). The main difference between ordinary and weak Hopf algebras comes from the fact that the comultiplication of the latter is no longer required to preserve the unit (equivalently, the counit is not required to be an algebra homomorphism). Consequently, there are two canonical subalgebras (H^L and H^R) playing the role of "non-commutative bases" in a weak Hopf algebra H. Moreover, the well known examples of weak

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Hopf algebras are groupoid algebras, face algebras and generalized Kac algebras (see [8, 20]). The main motivation for studying weak Hopf algebras comes from quantum field theory and operator algebras. It turned out that many results of classical Hopf algebra theory can be generalized to weak Hopf algebras.

This paper is organized as follows. In Section 1, we recall some basic definitions and give a summary of the fundamental properties concerning weak Hopf algebras. In Section 2, based on the work of [19], we obtain the main result of this paper by a new method, that is, the Militaru-Stefan lifting theorem over weak Hopf algebras. As an application, we check that if A/B is a weak right H-Galois extension, then the weak smash product $\mathrm{End}_B(M)\#H$ is isomorphic to $\mathrm{END}_A(M\otimes_B A)$ as an algebra for any $M\in\mathcal{M}_A$, which extends Theorem 2.3 in [18], given for a finite dimensional Hopf algebra. Moreover, for any B-module M, we prove that there exists a one-to-one correspondence between all A-isomorphism classes of extensions of M to a right A-module and the conjugation classes of total integrals and algebra maps $t: H\to \mathrm{END}_A(M\otimes_B A)$. In Section 3, under the condition "faithfully flat weak Hopf-Galois extensions", we mainly prove that a right B-module M is weak H-stable if and only if $\mathrm{END}_A(M\otimes_B A)/\mathrm{End}_B(M)$ is a weak cleft extension, which generalizes Theorem 3.6 in [15].

We always work over a fixed field k and follow Montgomery's book [11] for terminologies on algebras, coalgebras and comodules, but omit the usual summation indices and summation symbols.

In what follows, we recall some concepts and results used in this paper.

Definition 1.1 [2] Let H be both an algebra and a coalgebra. If H satisfies conditions (1.1)–(1.3) below, then it is called a weak bialgebra. If it satisfies conditions (1.1)–(1.4) below, then it is called a weak Hopf algebra with antipode S.

For any $x,y,z\in H$,

$$\Delta(xy)=\Delta(x)\Delta(y), \tag{1.1}$$

$$\Delta^2(1)=(\Delta(1)\otimes 1)(1\otimes\Delta(1));\quad \Delta^2(1)=(1\otimes\Delta(1))(\Delta(1_H)\otimes 1), \tag{1.2}$$

where $\Delta^2=(\Delta\otimes id)\Delta$.

$$\varepsilon(xyz)=\varepsilon(xy_1)\varepsilon(y_2z);\quad \varepsilon(xyz)=\varepsilon(xy_2)\varepsilon(y_1z), \tag{1.3}$$

$$x_1S(x_2)=\varepsilon(1_1x)1_2;\quad S(x_1)x_2=1_1\varepsilon(x1_2);\quad S(x_1)x_2S(x_3)=S(x), \tag{1.4}$$

where $\Delta(1)=1_1\otimes 1_2$.

For any weak bialgebra H, define the maps $\sqcap^L,\sqcap^R: H\to H$ by the formulas

$$\sqcap^L(h)=\varepsilon(1_1h)1_2,\quad \sqcap^R(h)=1_1\varepsilon(h1_2).$$

We have that $H^L=\mathrm{Im}(\sqcap^L)$ and $H^R=\mathrm{Im}(\sqcap^R)$ (see [2, 5]).

By [2], the antipode S of a weak Hopf algebra H is anti-multiplicative and anti-comultiplicative, that is, for any $h,g\in H$,

$$S(hg)=S(g)S(h), S(h)_1\otimes S(h)_2=S(h_2)\otimes S(h_1).$$

The unit and counit are S-invariants, that is, $S(1_H) = 1_H$, $\varepsilon \circ S = \varepsilon$.

H is always considered as a weak Hopf algebra. The following results (W1) – (W9) are given in [2]. For any $x \in H^L$, $y \in H^R$ and $h, g \in H$,

$$\Delta(1_H) = 1_1 \otimes 1_2 \in H^R \otimes H^L, \quad xy = yx; \tag{W1}$$

$$\Delta(x) = 1_1 x \otimes 1_2, \quad \Delta(y) = 1_1 \otimes y 1_2; \tag{W2}$$

$$xS(1_1) \otimes 1_2 = S(1_1) \otimes 1_2 x, \quad y 1_1 \otimes S(1_2) = 1_1 \otimes S(1_2) y; \tag{W3}$$

$$\sqcap^L \circ \sqcap^L = \sqcap^L, \quad \sqcap^R \circ \sqcap^R = \sqcap^R; \tag{W4}$$

$$S \circ \sqcap^R = \sqcap^L \circ S = \sqcap^L \circ \sqcap^R, \quad S \circ \sqcap^L = \sqcap^R \circ S = \sqcap^R \circ \sqcap^L; \tag{$W5$}$$

$$\varepsilon(h \sqcap^L (g)) = \varepsilon(hg), \quad \varepsilon(\sqcap^R(h) g) = \varepsilon(hg); \tag{W6}$$

$$h_1 \otimes \sqcap^R(h_2) = h 1_1 \otimes S(1_2), \quad \sqcap^L(h_1) \otimes h_2 = S(1_1) \otimes 1_2 h; \tag{W7}$$

$$\sqcap^R(h_1) \otimes h_2 = 1_1 \otimes h 1_2, \quad h_1 \otimes \sqcap^L(h_2) = 1_1 h \otimes 1_2; \tag{W8}$$

$$h_1 \sqcap^R (g) \otimes h_2 = h_1 \otimes h_2 S \circ \sqcap^R(g). \tag{W9}$$

Let H be a weak Hopf algebra with bijective antipode S. Then it is clear that S^{-1} is anti-multiplicative and anti-comultiplicative such that

$$\varepsilon S^{-1} = \varepsilon, S^{-1}(1_H) = 1_H; \tag{W10}$$

$$S^{-1}(h_2) h_1 = \sqcap^L(S^{-1}(h)) = 1_2 \varepsilon(h 1_1), \quad h_2 S^{-1}(h_1) = \sqcap^R(S^{-1}(h)) = 1_1 \varepsilon(1_2 h); \tag{W11}$$

$$S^{-1} \circ \sqcap^R = \sqcap^L \circ S^{-1}, \quad S^{-1} \circ \sqcap^L = \sqcap^R \circ S^{-1}. \tag{W12}$$

The following results (W13) – (W14) are given in [12].

$$S^2|_{H^L} = id_{H^L}, S^2|_{H^R} = id_{H^R}. \tag{W13}$$

If the antipode S is bijective, then for any $h \in H$,

$$\sqcap^L(h_1) \otimes h_2 = S^{-1}(1_1) \otimes 1_2 h. \tag{W14}$$

Definition 1.2 [5] Let H be a weak bialgebra, and A a right H-comodule, which is also an algebra, such that

$$\rho_A(ab) = \rho_A(a)\rho_A(b), \tag{1.5}$$

$$\rho_A(1_A)(a \otimes 1_H) = (id \otimes \sqcap^L)\rho_A(a) \tag{1.6}$$

for any $a, b \in A$. In this case we call A a weak right H-comodule algebra.

Definition 1.3 [5] Let H be a weak Hopf algebra and A a weak right H-comodule algebra. If M is both a right A-module and a right H-comodule such that for any $m \in M, a \in A$,

$$\rho_M(m \cdot a) = m_{(0)} \cdot a_{(0)} \otimes m_{(1)} a_{(1)}, \tag{1.7}$$

then M is called a weak right (A, H)-Hopf module.

Similarly, we can define the weak left right (A, H)-Hopf modules. We denote by $\mathcal{M}_A^H$ the category of weak right (A, H)-Hopf modules, and right A-linear H-colinear maps, and ${}_A\mathcal{M}^H$ the category of weak left right (A, H)-Hopf modules, and left A-linear right H-colinear maps.

Definition 1.4 [12] Let H be a weak bialgebra. The algebra A is called a weak left H-module algebra if A is a left H-module via $h \otimes a \mapsto h \cdot a$ such that for any $a, b \in A$ and $h \in H$,

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \tag{1.8}$$

$$h \cdot 1_A = \sqcap^L(h) \cdot 1_A. \tag{1.9}$$

Definition 1.5 [12] Let H be a weak Hopf algebra and A a weak left H-module algebra. A weak smash product $A\#H$ of A with H is defined on a k-vector space $A \otimes_{H^L} H$, where H is a left H^L-module via its multiplication and A is a right H^L-module via

$$a \cdot x = S^{-1}(x) \cdot a = a(x \cdot 1_A), a \in A, x \in H^L.$$

Its multiplication is given by the familiar formula: for any $a, b \in A$ and $h, g \in H$,

$$(a\#h)(b\#g) = a(h_1 \cdot b)\#h_2 g. \tag{1.10}$$

Then by [12], $A\#H$ is an associative algebra with unit $1_A\#1_H$.

Definition 1.6 [1] Let H be a weak Hopf algebra and A a weak right H-comodule algebra. A map $\phi : H \to A$ is called a total integral if ϕ is a right H-comodule map and $\phi(1_H) = 1_A$.

2 The Militaru-Stefan Lifting Theorem

In this section, we always assume that H is a weak Hopf algebra with bijective antipode S and A a weak right H-comodule algebra.

Denote $B = A^{coH} = \{a \in A | \rho(a) = a_{(0)} \otimes \sqcap^L(a_{(1)})\}$. Then by [9, 23], we know that B is a subalgebra of A, $M^{coH} = \{m \in M | \rho(m) = m_{(0)} \otimes \sqcap^L(m_{(1)})\}$ is a right B-submodule of M for any $M \in \mathcal{M}_A^H$. Set

$$A \boxtimes H = (A \otimes H)\rho(1_A) = \{a1_{(0)} \otimes h1_{(1)} | a \in A, h \in H\},$$
$$N \boxtimes H = (N \otimes H)\rho(1_A) = \{n \cdot 1_{(0)} \otimes h1_{(1)} | n \in N, h \in H\}$$

for any $N \in \mathcal{M}_A$. Then by [19], $N \boxtimes H \in \mathcal{M}_A^H$, whose action and coaction are given by

$$(n \boxtimes h) \cdot a = n \cdot a_{(0)} \boxtimes ha_{(1)}, \rho(n \boxtimes h) = n \boxtimes h_1 \otimes h_2. \tag{2.1}$$

Definition 2.1 [9] If the given map

$$\beta : A \otimes_B A \to A \boxtimes H, a \otimes_B b \mapsto ab_{(0)} \boxtimes b_{(1)}$$

is a bijection, we say that A/B is a weak right H-Galois extension, where A is a left and right B-module via its multiplication.

We will write for any $h \in H$, $\beta^{-1}(1_A \boxtimes h) = h^{[1]} \otimes_B h^{[2]} \in A \otimes_B A$.

Lemma 2.2 Let $N \in \mathcal{M}_A$. If A/B is a weak right H-Galois extension, then $N \boxtimes H \cong N \otimes_B A$ as weak right (A, H)-Hopf modules, where the A-action and H-coaction on $N \otimes_B A$ are given by

$$(n \otimes_B a) \cdot b = n \otimes_B ab, \quad \rho(n \otimes_B a) = n \otimes_B a_{(0)} \otimes a_{(1)} \tag{2.2}$$

for any $a, b \in A, n \in N$.

Proof Define a map φ to be the composite

$$N \boxtimes H \xrightarrow{\cong} N \otimes_A A \boxtimes H \xrightarrow{id_N \otimes_A \beta^{-1}} N \otimes_A A \otimes_B A \xrightarrow{\cong} N \otimes_B A\ ,$$

that is, $\varphi(n \boxtimes h) = n \cdot h^{[1]} \otimes_B h^{[2]}$. This implies φ is a bijection. Additionally, by Lemma 2.2 in [13], we can easily check that φ is both a right A-module map and a right H-comodule map. Thus $N \boxtimes H \cong N \otimes_B A$ as weak right (A, H)-Hopf modules.

Lemma 2.3 The following assertions are equivalent.

(1) There exists a total integral and algebra map $\phi : H \to A$.

(2) $B\#H \cong A$ as weak right H-comodule algebras.

If these assertions hold then B is a weak left H-module algebra via the adjoint action $h \cdot b = \phi(h_1)b\phi(S(h_2))$.

Proof Define a map $\tau : H \to B\#H, h \mapsto 1\#h$. For any $h, g \in H$, $(1\#h)(1\#g) = 1\#hg$. This implies that τ is an algebra map. Obviously, τ is a total integral. Hence the map $\phi = \lambda \circ \tau : H \to A$ is also a total integral and algebra map, where the map $\lambda : B\#H \to A$ is an isomorphism of right H-comodule algebras.

Conversely, assume that there exists a total integral and algebra map $\phi : H \to A$. Then B is a weak left H-module algebra via the adjoint action $h \cdot b = \phi(h_1)b\phi(S(h_2))$.

In fact, since ϕ is a right H-comodule map, $(\phi \otimes id_H)\Delta(1_H) = \rho_A\phi(1_H)$, that is, $\phi(1_1) \otimes 1_2 = 1_{(0)} \otimes 1_{(1)}$. Hence

$$\begin{aligned} 1_H \cdot b &= \phi(1_1)b\phi(S(1_2)) = 1_{(0)}b\phi(S(1_{(1)})) \\ &= b1_{(0)}\phi(S(1_{(1)})) = b\phi(1_1)\phi(S(1_2)) \\ &= b\phi(1_1S(1_2)) = b. \end{aligned}$$

In view of Theorem 3.3 in [22], we know that the rest is true.

Take $M, N \in \mathcal{M}_A^H$. Consider $\rho(f) \in \mathrm{Hom}_A(M, N \otimes H)$ as

$$\rho(f)(m) = f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)}S(m_{(1)}) \tag{2.3}$$

for any $f \in \mathrm{Hom}_A(M, N), m \in M$, where the A-action on $N \otimes H$ is induced by the A-action on N. Then by [19], $\rho(f)$ is right A-linear. In addition, by [19], we know that $\mathrm{Hom}_A(M, N)$ becomes a right H^R-module via

$$(f \leftarrow y)(m) = f(y \bullet m) \tag{2.4}$$

for any $f \in \mathrm{Hom}_A(M,N)$ and $y \in H^R$, where M is a left H^R-module via

$$y \bullet m = m_{(0)}\varepsilon(ym_{(1)}). \tag{2.5}$$

Recall from [19], we say that a map $f \in \mathrm{Hom}_A(M,N)$ is rational if there is an element $f_i \otimes f_j \in \mathrm{Hom}_A(M,N) \otimes H$ such that

$$(f_i \leftharpoonup 1_1)(m) \otimes f_j 1_2 = f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)} S(m_{(1)}) \tag{2.6}$$

for any $m \in M$, where $\Delta(1_H) = 1_1 \otimes 1_2$. Set $\mathrm{HOM}_A(M,N) = \{f \in \mathrm{Hom}_A(M,N) | f$ is rational$\}$. Then by (2.3) and (2.6), for any $f \in \mathrm{HOM}_A(M,N)$,

$$\rho(f) = (f_i \leftharpoonup 1_1) \otimes f_j 1_2. \tag{2.7}$$

By [19], we know that $\mathrm{HOM}_A(M,N)$ is a right H-comodule via (2.7), $\mathrm{END}_A(M) = \mathrm{HOM}_A(M,M)$ is a weak right H-comodule algebra, $\mathrm{END}_A(M)^{coH} = \mathrm{End}_A^H(M)$, and (2.6) is equivalent to that

$$\rho(f(m)) = f_{(0)}(m_{(0)}) \otimes f_{(1)} m_{(1)} \tag{2.8}$$

for any $m \in M$ and $f \in \mathrm{HOM}_A(M,N)$.

From (2.8), for any $M \in \mathcal{M}_A^H$, we can easily check that $M \in {}_{\mathrm{END}_A(M)}\mathcal{M}^H$ the category of weak left right $(\mathrm{END}_A(M), H)$-Hopf modules, and left $\mathrm{END}_A(M)$-linear right H-colinear maps, where M is a left $\mathrm{END}_A(M)$-module via $f \cdot m = f(m)$ for any $f \in END_A(M), m \in M$.

Let $M \in \mathcal{M}_A^H$. Consider the induction functor $- \otimes_{H^R} M$ and the functor $\mathrm{HOM}_A(M,-)$ between $\mathcal{M}^H$ and $\mathcal{M}_A^H$:

$$\begin{aligned} - \otimes_{H^R} M : \quad & \mathcal{M}^H \to \mathcal{M}_A^H, \quad P \mapsto P \otimes_{H^R} M, \\ \mathrm{HOM}_A(M,-) : \quad & \mathcal{M}_A^H \to \mathcal{M}^H, \quad N \mapsto \mathrm{HOM}_A(M,N), \end{aligned}$$

where for a right H-comodule P, it is a right H^R-module via $p \cdot y = p_{(0)}\varepsilon(p_{(1)}y)$ for any $p \in P, y \in H^R$, M is a left H^R-module via (2.5), and the A-action and H-coaction on $P \otimes_{H^R} M$ are given by

$$(p \otimes_{H^R} m) \cdot a = p \otimes_{H^R} m \cdot a, \quad \rho(p \otimes_{H^R} m) = p_{(0)} \otimes_{H^R} m_{(0)} \otimes p_{(1)} m_{(1)}. \tag{2.9}$$

With notation as above, the following assertion holds.

Lemma 2.4 Let $M \in \mathcal{M}_A^H$. Then $(- \otimes_{H^R} M, \mathrm{HOM}_A(M,-))$ is an adjoint pair.

Proof To show that $(- \otimes_{H^R} M, \mathrm{HOM}_A(M,-))$ is an adjoint pair, it suffices to prove that $\mathrm{Hom}^H(P, \mathrm{HOM}_A(M,N)) \cong \mathrm{Hom}_A^H(P \otimes_{H^R} M, N)$ for any $P \in \mathcal{M}^H, M, N \in \mathcal{M}_A^H$.

Define a map $F : \mathrm{Hom}_A^H(P \otimes_{H^R} M, N) \to \mathrm{Hom}^H(P, \mathrm{HOM}_A(M,N))$ by

$$F(f)(p)(m) = f(p \otimes_{H^R} m).$$

The map F is well defined. In fact, for any $f \in \mathrm{Hom}_A^H(P \otimes_{H^R} M, N), p \in P, m \in M$,

$$\begin{aligned}
\rho(F(f)(p))(m) &= F(f)(p)(m_{(0)})_{(0)} \otimes F(f)(p)(m_{(0)})_{(1)} S(m_{(1)}) \\
&= f(p \otimes_{H^R} m_{(0)})_{(0)} \otimes f(p \otimes_{H^R} m_{(0)})_{(1)} S(m_{(1)}) \\
&= f(p_{(0)} \otimes_{H^R} m_{(0)}) \otimes p_{(1)} m_{(1)1} S(m_{(1)2}) \\
&= f(p_{(0)} \otimes_{H^R} m_{(0)}) \otimes p_{(1)} \varepsilon(1_1 m_{(1)}) 1_2 \\
&= F(f)(p_{(0)})(m_{(0)}) \otimes p_{(1)} \varepsilon(1_1 m_{(1)}) 1_2 \\
&= F(f)(p_{(0)})(1_1 \bullet m) \otimes p_{(1)} 1_2 \\
&= (F(f)(p_{(0)}) \leftarrow 1_1)(m) \otimes p_{(1)} 1_2,
\end{aligned}$$

that is, $\rho(F(f)(p)) = F(f)(p_{(0)}) \leftarrow 1_1 \otimes p_{(1)} 1_2$. The right A-linearity of f implies that $F(f)(p)$ is also a right A-linear map. Hence $F(f)(p) \in \mathrm{HOM}_A(M, N)$. Moreover, in the light of the right H-colinearity of f, we can easily show that $F(f)$ is also a right H-colinear map.

Now, we define a map $G : \mathrm{Hom}^H(P, \mathrm{HOM}_A(M, N)) \to \mathrm{Hom}_A^H(P \otimes_{H^R} M, N)$ by

$$G(T)(p \otimes_{H^R} m) = T(p)(m).$$

Obviously, G is well defined, and F is a bijection with inverse G. Hence $\mathrm{Hom}^H(P, \mathrm{HOM}_A(M, N)) \cong \mathrm{Hom}_A^H(P \otimes_{H^R} M, N)$.

Consider H as a right H-comodule via its comultiplication, hence by (2.9), $H \otimes_{H^R} M \in \mathcal{M}_A^H$. Then the following assertion holds.

Lemma 2.5 Let $M \in \mathcal{M}_A^H$. Then $H \otimes_{H^R} M \cong M \boxtimes H$ as weak right (A, H)-Hopf modules, where $M \boxtimes H$ is a weak right (A, H)-Hopf module via (2.1).

Proof Define a map

$$\delta : H \otimes_{H^R} M \to M \boxtimes H, h \otimes_{H^R} m \mapsto m_{(0)} \boxtimes h m_{(1)}.$$

Using (W2), we can check that δ is well defined. It is easy to see that δ is both a right A-module map and a right H-comodule map.

In what follows, we show that δ is a bijection with inverse

$$\gamma : M \boxtimes H \to H \otimes_{H^R} M, m \boxtimes h \mapsto h S^{-1}(m_{(1)}) \otimes_{H^R} m_{(0)}.$$

The map γ is well defined, since for any $m \in M, h \in H, y \in H^R$,

$$\begin{aligned}
h S^{-1}(m_{(1)}) \otimes y \bullet m_{(0)} &= h S^{-1}(m_{(1)2}) \otimes m_{(0)} \varepsilon(y m_{(1)1}) \\
&\overset{(W6)}{=} h S^{-1}(m_{(1)2}) \otimes m_{(0)} \varepsilon(y \sqcap^L (m_{(1)1})) \\
&\overset{(W14)}{=} h S^{-1}(1_2 m_{(1)}) \otimes m_{(0)} \varepsilon(y S^{-1}(1_1)) \\
&= h S^{-1}(m_{(1)}) 1_1 \otimes m_{(0)} \varepsilon(y 1_2) \\
&\overset{(W4)}{=} h S^{-1}(m_{(1)}) y \otimes m_{(0)},
\end{aligned}$$

that is, $\mathrm{Im}\gamma \subseteq H \otimes_{H^R} M$.

Now we calculate that

$$\begin{aligned}\gamma\delta(h\otimes_{H^R} m) &= \gamma(m_{(0)}\boxtimes hm_{(1)}) = hm_{(1)2}S^{-1}(m_{(1)1})\otimes_{H^R} m_{(0)}\\ &\overset{(W11)}{=} h1_1\varepsilon(1_2m_{(1)})\otimes_{H^R} m_{(0)} = h\otimes_{H^R} 1_1\bullet m_{(0)}\varepsilon(1_2m_{(1)})\\ &= h\otimes_{H^R} m_{(0)}\varepsilon(1_1m_{(1)1})\varepsilon(1_2m_{(1)2}) = h\otimes_{H^R} m\end{aligned}$$

for any $h\otimes_{H^R} m\in H\otimes_{H^R} M$, and

$$\begin{aligned}\delta\gamma(m\boxtimes h) &= \delta(hS^{-1}(m_{(1)})\otimes_{H^R} m_{(0)}) = m_{(0)}\boxtimes hS^{-1}(m_{(1)2})m_{(1)1}\\ &= m_{(0)}\boxtimes h\sqcap^L S^{-1}(m_{(1)}) \overset{(W12)}{=} m_{(0)}\boxtimes hS^{-1}\sqcap^R(m_{(1)})\\ &= m_{(0)}\cdot 1_{(0)}\boxtimes hS^{-1}(S(1_{(1)})) = m\boxtimes h,\end{aligned}$$

where the fifth equality follows by the fact that $m_{(0)}\otimes\sqcap^R(m_{(1)}) = m\cdot 1_{(0)}\otimes S(1_{(1)})$ for any $m\in M$ (see [19]). Therefore $H\otimes_{H^R} M\cong M\boxtimes H$ as weak right (A,H)-Hopf modules.

In what follows, we obtain the Militaru-Stefan lifting theorem over weak Hopf algebras, which extends Theorem 2.3 in [10].

Theorem 2.6 Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Assume that $(M,\prec)$ is a right B-module. Then the following assertions are equivalent.

(1) M can be extended to a right A-module.

(2) There exists a total integral and algebra map $\phi: H\to \mathrm{END}_A(M\otimes_B A)$, where $M\otimes_B A$ is a weak right (A,H)-Hopf module via (2.2).

(3) There is a weak left H-module algebra structure on $\mathrm{End}_B(M)$ such that

$$\mathrm{End}_B(M)\#H\cong \mathrm{END}_A(M\otimes_B A)$$

as weak right H-comodule algebras.

Proof (1) $\Leftrightarrow$ (2) Since A/B is a weak right H-Galois extension and A is faithfully flat as a left B-module, the functor $-\otimes_B A$ is an equivalence between $\mathcal{M}_B$ and $\mathcal{M}_A^H$ according to [6]. Hence we have a sequence of isomorphisms:

$$\begin{aligned}\mathrm{Hom}^H(H,\mathrm{END}_A(M\otimes_B A)) &\cong \mathrm{Hom}_A^H(H\otimes_{H^R}(M\otimes_B A), M\otimes_B A)\\ &\cong \mathrm{Hom}_A^H((M\otimes_B A)\boxtimes H, M\otimes_B A)\\ &\cong \mathrm{Hom}_A^H(M\otimes_B A\otimes_B A, M\otimes_B A)\\ &\cong \mathrm{Hom}_B(M\otimes_B A, M),\end{aligned}$$

where the first isomorphism follows by Lemma 2.4, the second one by Lemma 2.5 and the third one by Lemma 2.2. This resulting isomorphism relates the desired A-action $\leftharpoonup$ on M to the multiplicative total integral ϕ on $\mathrm{END}_A(M\otimes_B A)$.

In fact, the associativity and unitality of the action $\leftharpoonup$ are equivalent to the multiplicativity and unitality of ϕ, respectively. Indeed, there are further similar isomorphisms:

$$\mathrm{Hom}^H(H\otimes_{H^R} H,\mathrm{END}_A(M\otimes_B A))\cong \mathrm{Hom}_B(M\otimes_B A\otimes_B A, M)$$

and

$$\mathrm{Hom}^H(k, \mathrm{END}_A(M \otimes_B A)) \cong \mathrm{End}_B(M).$$

They relate, respectively,

$$H \otimes_{H^R} H \xrightarrow{\text{multiplication}} H \xrightarrow{\phi} \mathrm{END}_A(M \otimes_B A)$$

with

$$M \otimes_B A \otimes_B A \xrightarrow{id_M \otimes_B \text{multiplication}} M \otimes_B A \xrightarrow{\leftharpoonup} M$$

and

$$H \otimes_{H^R} H \xrightarrow{\phi \otimes_{H^R} \phi} \mathrm{END}_A(M \otimes_B A) \otimes_{H^R} \mathrm{END}_A(M \otimes_B A) \xrightarrow{\text{multiplication}} \mathrm{END}_A(M \otimes_B A)$$

with

$$M \otimes_B A \otimes_B A \xrightarrow{\leftharpoonup \otimes_B id_A} M \otimes_B A \xrightarrow{\leftharpoonup} M$$

while

$$k \xrightarrow{\text{unit}} H \xrightarrow{\phi} \mathrm{END}_A(M \otimes_B A)$$

with $(-) \leftharpoonup 1_A : M \to M$; furthermore the unit of $\mathrm{END}_A(M \otimes_B A)$ with the identity map on M. So (1) $\Leftrightarrow$ (2) holds.

(2)$\Leftrightarrow$ (3) Since A/B is a weak right H-Galois extension and A is faithfully flat as a left B-module, the functor $- \otimes_B A$ is an equivalence between $\mathcal{M}_B$ and $\mathcal{M}_A^H$ according to [6], hence

$$\mathrm{END}_A(M \otimes_B A)^{coH} = \mathrm{End}_A^H(M \otimes_B A) \cong \mathrm{End}_B(M).$$

So by Lemma 2.3, (2)$\Leftrightarrow$ (3) holds.

The following conclusion extends Theorem 3.5 in [16].

Proposition 2.7 Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Assume that $(M, \prec)$ is a right B-module. Then the following assertions are equivalent.

(1) $\iota : M \to M \otimes_B A, m \mapsto m \otimes_B 1_A$ is a B-split monomorphism.

(2) $\mathrm{END}_A(M \otimes_B A)$ is a relative injective H-comodule.

Proof We only sketch the proof. This result can be derived from the isomorphism $\mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A)) \cong \mathrm{Hom}_B(M \otimes_B A, M)$ together with Theorem 1.7 in [1] and the observation in the proof of Theorem 2.6 about the simultaneous unitality of the corresponding morphisms $\kappa \in \mathrm{Hom}_B(M \otimes_B A, M)$ and $\phi \in \mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A))$.

Remark (1) Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Assume that $(M, \leftharpoonup)$ is a right A-module. Then $(M, \leftharpoonup)$ is also a right B-module, which can be extended to a right A-module. Therefore, by Theorem 2.6, $\mathrm{End}_B(M)\#H \cong \mathrm{END}_A(M \otimes_B A)$ as weak right H-comodule algebras, which extends Theorem 2.3 in [18], given for a finite dimensional Hopf algebra.

(2) By [6, 21], we know that H is a weak right H-Galois extension of H^L, hence, by (1), $\mathrm{End}_{H^L}(H)\#H \cong \mathrm{END}_H(H \otimes_{H^L} H)$ as algebras. In particular, if H is a finite dimensional

weak Hopf algebra, then by Corollary 3.4 in [12], we have $H\#H^* \cong End_{H^L}(H)$ as algebras. Then there exists an algebra isomorphism $(H\#H^*)\#H \cong \mathrm{END}_H(H \otimes_{H^L} H)$.

Set

$$\Omega_E = \{\phi \in \mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A)) | \phi \text{ is an algebra map}\}.$$

For any $\phi_1, \phi_2 \in \Omega_E$, if there exists $\psi \in \mathrm{Aut}_B(M)$ such that

$$\phi_2(h) = (\psi \otimes_B id_A) \circ \phi_1(h) \circ (\psi^{-1} \otimes_B id_A) \tag{2.10}$$

for any $h \in H$, we say that ϕ_1, ϕ_2 are conjugate, denoted by $\phi_1 \sim \phi_2$. It is obvious that $\sim$ is an equivalence relation on Ω_E. We denote by $\overline{\Omega}_E$ the quotient set of Ω_E relative to this equivalence relation $\sim$.

With notation as above, the following assertion holds.

Theorem 2.8 Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Consider M as a right B-module. Then there is a bijection between all A-isomorphism classes of extensions of M to a right A-module and $\overline{\Omega}_E$.

Proof By the proof of Theorem 2.6, we know that $\mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A)) \cong \mathrm{Hom}_B(M \otimes_B A, M)$. This isomorphism relates

$$(\psi \otimes_B id_A) \circ \phi(-) \circ (\psi^{-1} \otimes_B id_A) : H \to \mathrm{END}_A(M \otimes_B A)$$

with the map

$$f : M \otimes_B A \to M, m \otimes_B a \mapsto \psi(\psi^{-1}(m) \leftharpoonup a),$$

where $\psi \in \mathrm{Aut}_B(M)$. Therefore, the bijection between Ω_E and the set of extensions of M, induces a bijection between $\overline{\Omega}_E$ and the set of A-isomorphism classes of extensions of M.

Recall from Remark 2.8(1) in [19], we know that $\mathrm{END}_A(A) \cong A$ as weak right H-comodule algebras. Hence $\mathrm{END}_A(B \otimes_B A) \cong \mathrm{END}_A(A) \cong A$ as weak right H-comodule algebras. Let $M = B$, then $\Omega_E = \Omega_A = \{\phi \in \mathrm{Hom}^H(H, A) | \phi \text{ is an algebra map}\}$. At the same time, it is easy to see that equation (2.10) is replaced by the equation

$$\phi_2(h) = b\phi_1(h)b^{-1}, \tag{2.11}$$

where $b \in U(B) = \{b \in B | b \text{ is invertible}\}$. That is, for any $\phi_1, \phi_2 \in \Omega_A$, ϕ_1, ϕ_2 are conjugate if there exists $b \in U(B)$ such that for any $h \in H$, (2.11) holds. Denote by $\overline{\Omega}_A$ the quotient set of Ω_A relative to this conjugate relation. Then by Theorem 2.6 and Theorem 2.8, the following assertion holds.

Corollary 2.9 Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Consider M as a right B-module. Then the following assertions are equivalent.

(1) B can be extended to a right A-module.

(2) $\Omega_A \neq \emptyset$.

(3) There exists a weak left H-module algebra structure on B such that $B\#H \cong A$ as weak right H-comodule algebras.

Furthermore, there exists a one-to-one correspondence between the set of isomorphism classes of extensions of B and $\overline{\Omega}_A$.

3 Weak Stable Modules

In this section, we always assume that H is a weak Hopf algebra with bijective antipode S, A a weak right H-comodule algebra and $B = A^{coH}$.

Definition 3.1 If there exists a right H-comodule map $\phi : H \to A$, called a weak cleaving map, and a map $\psi : H \to A$ that satisfy the following conditions

(1) $\psi(h_1)\phi(h_2) = 1_{(0)}\varepsilon(h1_{(1)})$,

(2) $\psi(h_2)_{(0)} \otimes h_1\psi(h_2)_{(1)} = \psi(h)1_{(0)} \otimes 1_{(1)}$

for any $h \in H$. Then we say that A/B is a weak cleft extension (see [14]).

Definition 3.2 Let M be both a right B-module and a left H^L-module. M is called weak H-stable if $M \otimes_B A$ and $H \otimes_{H^L} M$ are isomorphic as right H-comodules and right B-modules, where H is a right H^L-module via

$$h \cdot x = S(x)h \tag{3.1}$$

for any $h \in H, x \in H^L$, and the actions and coactions are given by

$$(m \otimes_B a) \cdot b = m \otimes_B ab, \quad \rho(m \otimes_B a) = m \otimes_B a_{(0)} \otimes a_{(1)},$$
$$(h \otimes_{H^L} m) \cdot b = h \otimes_{H^L} m \cdot b, \quad \rho(h \otimes_{H^L} m) = h_2 \otimes_{H^L} m \otimes S^{-1}(h_1)$$

for any $b \in B, m \otimes_B a \in M \otimes_B A, h \otimes_{H^L} m \in H \otimes_{H^L} M$.

Lemma 3.3 Let $M \in \mathcal{M}_A^H$. Then $H \otimes_{H^L} M$ is a weak right (A, H)-Hopf module, where H is a right H^L-module as in (3.1), M is a left H^L-module via $x \cdot m = m_{(0)}\varepsilon(m_{(1)}S(x))$ for any $x \in H^L, m \in M$, and the A-action and H-coaction on $H \otimes_{H^L} M$ are given by

$$(h \otimes_{H^L} m) \cdot a = S(a_{(1)})h \otimes_{H^L} m \cdot a_{(0)}, \quad \rho(h \otimes_{H^L} m) = h_2 \otimes_{H^L} m \otimes S^{-1}(h_1)$$

for any $h \otimes_{H^L} m \in H \otimes_{H^L} M, a \in A$.

Proof The A-action on $H \otimes_{H^L} M$ is well defined, since for any $x \in H^L, a \in A, h \otimes_{H^L} m \in H \otimes_{H^L} M$,

$$\begin{aligned}
(h \otimes_{H^L} x \cdot m) \cdot a &= S(a_{(1)})h \otimes_{H^L} m_{(0)} \cdot a_{(0)}\varepsilon(m_{(1)}S(x)) \\
&\overset{(W6)}{=} S(a_{(1)})h \otimes_{H^L} m_{(0)} \cdot a_{(0)}\varepsilon(\sqcap^R(m_{(1)})S(x)) \\
&= S(a_{(1)})h \otimes_{H^L} m_{(0)} \cdot 1_{(0)}a_{(0)}\varepsilon(S(1_{(1)})S(x)) \\
&= S(a_{(1)})h \otimes_{H^L} m_{(0)} \cdot 1_{(0)}a_{(0)}\varepsilon(x1_{(1)}) \\
&\overset{(1.6)}{=} S(a_{(1)2})h \otimes_{H^L} m_{(0)} \cdot a_{(0)}\varepsilon(x \sqcap^L (a_{(1)1})) \\
&\overset{(W7)}{=} S(1_2a_{(1)})h \otimes_{H^L} m_{(0)} \cdot a_{(0)}\varepsilon(xS(1_1)) \\
&\overset{(W3)}{=} S(xa_{(1)})h \otimes_{H^L} m_{(0)} \cdot a_{(0)} = (h \cdot x \otimes_{H^L} m) \cdot a,
\end{aligned}$$

where the third equality follows by the fact that $m_{(0)}\otimes\sqcap^R(m_{(1)})=m\cdot 1_{(0)}\otimes S(1_{(1)})$ for any $m\in M$. Using (W2) and the fact that $S(H^L)\subseteq H^R$, we can easily show that the H-coaction on $H\otimes_{H^L}M$ is also well defined. What is more, it is easy to see that $H\otimes_{H^L}M$ is a weak right (A,H)-Hopf module.

Let $M\in\mathcal{M}_A^H$. By Lemma 2.5, we know that $H\otimes_{H^R}M$ is a weak right (A,H)-Hopf module. In view of Lemma 3.3, we obtain the following result.

Lemma 3.4 Let $M\in\mathcal{M}_A^H$. Then $H\otimes_{H^R}M\cong H\otimes_{H^L}M$ as weak right (A,H)-Hopf modules.

Proof We first have a well defined map

$$\theta: H\otimes_{H^L}M\to H\otimes_{H^R}M, h\otimes_{H^L}m\mapsto S^{-1}(m_{(1)}h)\otimes_{H^R}m_{(0)}.$$

In fact, for any $h\otimes_{H^L}m\in H\otimes_{H^L}M, x\in H^L$,

$$\begin{aligned}
\theta(h\otimes_{H^L}x\cdot m) &= S^{-1}(m_{(1)1}h)\otimes_{H^R}m_{(0)}\varepsilon(m_{(1)2}S(x))\\
&= S^{-1}(m_{(1)1}h)\otimes_{H^R}m_{(0)}\varepsilon(\sqcap^R(m_{(1)2})S(x))\\
&\overset{(W7)}{=} S^{-1}(m_{(1)}1_1h)\otimes_{H^R}m_{(0)}\varepsilon(S(1_2)S(x))\\
&= S^{-1}(m_{(1)}S(x)1_1h)\otimes_{H^R}m_{(0)}\varepsilon(S(1_2))\\
&= S^{-1}(m_{(1)}S(x)h)\otimes_{H^R}m_{(0)}\\
&= \theta(h\cdot x\otimes_{H^L}m),
\end{aligned}$$

where the fourth equality follows by (W3) and the fact that $S(H^L)\subseteq H^R$. And for any $h\in H, m\in M, y\in H^R$,

$$\begin{aligned}
S^{-1}(m_{(1)}h)\otimes y\cdot m_{(0)} &= S^{-1}(m_{(1)2}h)\otimes m_{(0)}\varepsilon(ym_{(1)1})\\
&\overset{(W6)}{=} S^{-1}(m_{(1)2}h)\otimes m_{(0)}\varepsilon(y\sqcap^L(m_{(1)1}))\\
&\overset{(W14)}{=} S^{-1}(1_2m_{(1)}h)\otimes m_{(0)}\varepsilon(yS^{-1}(1_1))\\
&= S^{-1}(m_{(1)}h)1_1\otimes m_{(0)}\varepsilon(y1_2)\\
&= S^{-1}(m_{(1)}h)y\otimes m_{(0)},
\end{aligned}$$

that is, $\mathrm{Im}\theta\subseteq H\otimes_{H^R}M$. Moreover, from (1.6), we can easily show that θ is a right A-module map, and θ is a right H-comodule map, because

$$\begin{aligned}
&\theta(h\otimes_{H^L}m)_{(0)}\otimes\theta(h\otimes_{H^L}m)_{(1)}\\
=\ & S^{-1}(m_{(1)2}h)_1\otimes_{H^R}m_{(0)}\otimes S^{-1}(m_{(1)2}h)_2m_{(1)1}\\
=\ & S^{-1}(h_2)S^{-1}(m_{(1)3})\otimes_{H^R}m_{(0)}\otimes S^{-1}(h_1)S^{-1}(m_{(1)2})m_{(1)1}\\
=\ & S^{-1}(h_2)S^{-1}(m_{(1)2})\otimes_{H^R}m_{(0)}\otimes S^{-1}(h_1)S^{-1}\sqcap^R(m_{(1)1})\\
\overset{(W8)}{=}\ & S^{-1}(h_2)S^{-1}(m_{(1)}1_2)\otimes_{H^R}m_{(0)}\otimes S^{-1}(h_1)S^{-1}(1_1)\\
=\ & S^{-1}(1_2h_2)S^{-1}(m_{(1)})\otimes_{H^R}m_{(0)}\otimes S^{-1}(1_1h_1)\\
=\ & S^{-1}(h_2)S^{-1}(m_{(1)})\otimes_{H^R}m_{(0)}\otimes S^{-1}(h_1)\\
=\ & \theta(h_2\otimes_{H^L}m)\otimes S^{-1}(h_1).
\end{aligned}$$

Next, we show that θ is a bijection with inverse

$$\vartheta : H \otimes_{H^R} M \to H \otimes_{H^L} M, h \otimes_{H^R} m \mapsto S(hm_{(1)}) \otimes_{H^L} m_{(0)}.$$

The map ϑ is well defined, since for any $y \in H^R, h \otimes_{H^L} m \in H \otimes_{H^L} M$,

$$\begin{aligned}
\vartheta(h \otimes_{H^R} y \cdot m) &= S(hm_{(1)1}) \otimes_{H^L} m_{(0)}\varepsilon(ym_{(1)2}) = S(hm_{(1)1}) \otimes_{H^L} m_{(0)}\varepsilon(y \sqcap^L (m_{(1)2})) \\
&= S(h1_1 m_{(1)}) \otimes_{H^L} m_{(0)}\varepsilon(y1_2) = S(hym_{(1)}) \otimes_{H^L} m_{(0)} \\
&= \vartheta(hy \otimes_{H^R} m),
\end{aligned}$$

and for any $h \in H, m \in M, x \in H^L$,

$$\begin{aligned}
S(hm_{(1)}) \otimes x \cdot m_{(0)} &= S(hm_{(1)2}) \otimes m_{(0)}\varepsilon(m_{(1)1}S(x)) = S(hm_{(1)2}S^2(x)) \otimes m_{(0)}\varepsilon(m_{(1)1}) \\
&\overset{\text{(W13)}}{=} S(hm_{(1)}x) \otimes m_{(0)} = S(x)S(hm_{(1)}) \otimes m_{(0)} \\
&= S(hm_{(1)}) \cdot x \otimes m_{(0)},
\end{aligned}$$

where the second equality follows by (W9) and the fact that $S(H^L) \subseteq H^R$. This implies $\mathrm{Im}\vartheta \subseteq H \otimes_{H^L} M$.

Now we calculate that

$$\begin{aligned}
\vartheta\theta(h \otimes_{H^L} m) &= S(m_{(1)1})m_{(1)2}h \otimes_{H^L} m_{(0)} = 1_1 h \otimes_{H^L} m_{(0)}\varepsilon(m_{(1)}1_2) \\
&= S(1_2)h \otimes_{H^L} m_{(0)}\varepsilon(m_{(1)}S(1_1)) = h \otimes_{H^L} 1_2 \cdot m_{(0)}\varepsilon(m_{(1)}S(1_1)) \\
&= h \otimes_{H^L} m_{(0)}\varepsilon(m_{(1)1}S(1_2))\varepsilon(m_{(1)2}S(1_1)) \\
&= h \otimes_{H^L} m_{(0)}\varepsilon(m_{(1)1}1_1)\varepsilon(m_{(1)2}1_2) = h \otimes_{H^L} m, \\
\theta\vartheta(h \otimes_{H^R} m) &= \theta(S(hm_{(1)}) \otimes_{H^L} m_{(0)}) = hm_{(1)2}S^{-1}(m_{(1)1}) \otimes_{H^R} m_{(0)} \\
&= h1_1 \otimes_{H^R} m_{(0)}\varepsilon(1_2 m_{(1)}) = h \otimes_{H^R} 1_1 \cdot m_{(0)}\varepsilon(1_2 m_{(1)}) \\
&= h \otimes_{H^R} m_{(0)}\varepsilon(1_1 m_{(1)1})\varepsilon(1_2 m_{(1)2}) = h \otimes_{H^R} m,
\end{aligned}$$

that is, θ is a bijection with inverse ϑ.

Therefore, $H \otimes_{H^R} M \cong H \otimes_{H^L} M$ as weak right (A, H)-Hopf modules.

With notation as above, we obtain the following result which extends Theorem 3.6 in [15].

Theorem 3.5 Let A/B be a weak right H-Galois extension and A faithfully flat as a left B-module. Let M be both a right B-module and a left H^L-module. Then the following assertions are equivalent.

(1) M is weak H-stable.

(2) $\mathrm{END}_A(M \otimes_B A)/\mathrm{End}_B(M)$ is a weak cleft extension.

Proof By the proof of Theorem 2.6, we have a sequence of isomorphisms

$$\begin{aligned}
\mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A)) &\cong \mathrm{Hom}_B(M \otimes_B A, M) \\
&\cong \mathrm{Hom}_B^H(M \otimes_B A, H \otimes_{H^L} M),
\end{aligned}$$

where the second isomorphism is given by

$$\begin{aligned}\chi(f)(m \otimes_B a) &= S(a_{(1)}) \otimes_{H^L} f(m \otimes_B a_{(0)}),\\ \chi^{-1}(g)(m \otimes_B a) &= (\varepsilon \otimes id_M) g(m \otimes_B a)\end{aligned}$$

for any $f \in \mathrm{Hom}_B(M \otimes_B A, M), g \in \mathrm{Hom}_B^H(M \otimes_B A, H \otimes_{H^L} M)$. This resulting isomorphism relates $\Phi \in \mathrm{Hom}_B^H(M \otimes_B A, H \otimes_{H^L} M)$ with $\phi \in \mathrm{Hom}^H(H, \mathrm{END}_A\ (M \otimes_B A))$ which is given by

$$\phi(h)(m \otimes_B a) = (\varepsilon \otimes id_M)\Phi(m \otimes_B h^{[1]}) \otimes_B h^{[2]}a.$$

Moreover, by Theorem 2.6 and Lemma 3.4, we have the following sequence of isomorphisms

$$\begin{aligned}\mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A)) &\cong \mathrm{Hom}_A^H(H \otimes_{H^R} (M \otimes_B A), M \otimes_B A)\\ &\cong \mathrm{Hom}_A^H(H \otimes_{H^L} (M \otimes_B A), M \otimes_B A)\\ &\cong \mathrm{Hom}_B^H(H \otimes_{H^L} M, M \otimes_B A).\end{aligned}$$

This resulting isomorphism relates $\Psi \in \mathrm{Hom}_B^H(H \otimes_{H^L} M, M \otimes_B A)$ with $\psi \circ S^{-1} \in \mathrm{Hom}^H(H, \mathrm{END}_A(M \otimes_B A))$ which is given by

$$\psi \circ S^{-1}(h)(m \otimes_B a) = \Psi(h \otimes_{H^L} m) \cdot a.$$

Therefore, ϕ and $\psi \circ S^{-1}$ satisfy conditions (1) and (2) in Definition 3.1 if and only if Φ is a bijection with inverse Ψ, that is, $\mathrm{END}_A(M \otimes_B A)/\mathrm{End}_B(M)$ is a weak cleft extension if and only if M is weak H-stable.

References

[1] Alonso Álvarez J N, Fernández Vilaboa J M, González Rodríguez R, Rodríguez Raposo A B. A Maschke type theorem for weak Hopf algebras[J]. Acta Math. Sin., 2008, 24: 2065–2080.

[2] Böhm G, Nill F, Szlachányi K. Weak Hopf algebras I. integral theory and C^*-structure[J]. J. Alg., 1999, 221: 385–438.

[3] Böhm G, Szlachányi K. A coassociative C^*-quantum group with nonintegral dimensions[J]. Lett. Math. Phys., 1996, 35: 437–456.

[4] Caenepeel S. Hopf-Galois extensions and isomorphisms of small categories[J]. Math., 2010, 52: 121–142.

[5] Caenepeel S, De Groot E. Modules over weak entwining structures[J]. Contemp. Math., 2000, 267: 31–54.

[6] Caenepeel S, De Groot E. Galois theory for weak Hopf algebras[J]. Rev. Roumaine Math. Pures Appl., 2007, 52: 51–76.

[7] Dade E C. Extending irreducible modules[J]. J. Alg., 1981, 72: 374–403.

[8] Hayashi T. Quantum group symmetry of partition functions of IRF models and its applications to Jones's index theory[J]. Comm. Math. Phys. 1993, 157: 331–345.

[9] Kadison L. Galois theory for bialgebroids, depth two and normal Hopf subalgebras[J]. Ann. Univ. Ferrara Sez. VII (N.S.), 2005, 51: 209–231.

[10] Militaru G, Stefan D. Extending modules for Hopf Galois extensions[J]. Comm. Alg., 1994, 22: 5657–5678.

[11] Montgomery S. Hopf algebras and their actions on rings[M]. Chicago: CBMS, 1993.

[12] Nikshych D. A duality theorem for quantum groupoids[J]. Contemp. Math., 2000, 267: 237–243.

[13] Niu R F, Wang Y, Zhang L Y. The structure theorem for endomorphism algebras of weak Doi-Hopf modules[J]. Acta Math. Hungar., 2010, 127: 273–290.

[14] Rodríguez Raposo A B. Crossed products for weak Hopf algebras[J]. Comm. Alg., 2009, 37: 2274–2289.

[15] Schneider H J. Representation theory of Hopf Galois extensions[J]. Israel J. Math., 1990, 72: 196–231.

[16] Schneider H J. Hopf Galois extensions, crossed products and Clifford theory[A]. Bergen J, Montgomery S. Advances in Hopf Algebras[C]. FL, USA: CRC Press, 1994: 267–298.

[17] Ulbrich L H. Smash products and comodules of linear maps[J]. Tsukuba J. Math., 1990, 14: 371–378.

[18] Van Oystaeyen F, Zhang Y H. H-module endomorphism rings[J]. J. Pure Appl. Alg., 1995, 102: 207–219.

[19] Wang Y, Zhang L Y. The structure theorem and duality theorem for endomorphism algebras of weak Hopf algebras[J]. J. Pure Appl. Alg., 2011, 215: 1133–1145.

[20] Yamanouchi T. Duality for generalized Kac algebras and a characterization of finite groupoid algebras[J]. J. Alg., 1994, 163: 9–50.

[21] Zhang L Y. Weak Galois extensions over H-module algebras[J]. J. Math., 2008, 28: 647–652.

[22] Zhang L Y. The structure theorem for weak comodule algebras[J]. Comm. Alg., 2010, 38: 254–260.

[23] Zhang L Y, Zhu S L. Fundamental theorems of weak Doi-Hopf modules and semisimple weak smash product Hopf algebras[J]. Comm. Alg., 2004, 32: 3403–3415.

弱Hopf代数上的Militaru-Stefan提升定理

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摘要: 本文研究了弱Hopf-Galois扩张的扩张模. 利用忠实平坦的弱Hopf-Galois扩张理论, 研究了弱Hopf代数上的Militaru-Stefan提升定理, 推广了文献[10]中的相应结果. 进一步地, 通过诱导模的自同态环的cleft扩张刻画了弱稳定模.

关键词: 弱Hopf代数; 弱Hopf-Galois扩张; 弱cleft扩张

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