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SOME RESULTS OF WEAKLY f-STATIONARY MAPS WITH POTENTIAL

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Abstract: In this paper, we investigate a generalized functional $\Phi_{f,H}$ related to the pullback metric. By using the stress-energy tensor, we obtain some Liouville type theorems for weakly f-stationary maps with potential under some conditions on H.

Keywords: weakly f-stationary map with potential; stress-energy tensor; Liouville type theorems

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1 Introduction

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds (M^m, g) and (N^n, h). Recently, Kawai and Nakauchi [1] introduced a functional related to the pull-back metric u^*h as follows:

$$\Phi(u) = \frac{1}{4}\int_M ||u^*h||^2 dv_g$$

(see [2–5]), where u^*h is the symmetric 2-tensor defined by

$$(u^*h)(X, Y) = h(du(X), du(Y))$$

for any vector fields X, Y on M and $||u^*h||$ is given by

$$||u^*h||^2 = \sum_{i,j=1}^{m} [h(du(e_i), du(e_j))]^2$$

with respect to a local orthonormal frame $(e_1, \cdots, e_m)$ on (M, g). The map u is stationary for Φ if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of u. Asserda [6] introduced the following functional Φ_F by

$$\Phi_F(u) = \int_M F(\frac{||u^*h||^2}{4}) dv_g,$$

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where $F:[0,\infty)\to[0,\infty)$ is a C^2 function such that $F(0)=0$ and $F'(t)>0$ on $[0,\infty)$. The map u is F-stationary for Φ if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of u. Following [6], Han and Feng in [5] introduced the following functional Φ_f by

$$\Phi_f(u)=\int_M f(x)\frac{||u^*h||^2}{4}dv_g, \tag{1.1}$$

where $f:(M,g)\to(0,+\infty)$ is a smooth function. They derived the first variation formula of Φ_f and introduced the f-stress energy tensor S_{Φ_f} associated to Φ_f. Then, by using the f-stress energy tensor, they obtained the monotonicity formula and vanishing theorems for stationary map for the functional $\Phi_f(u)$ under some conditions on f.

The theory of harmonic maps was developed by many researchers so far, and a lot of results were obtained (see [7, 8]). Lichnerowicz in [9] (also see [7]) introduced the f-harmonic maps, generalizing harmonic maps. Since then, there were many results for f-harmonic maps such as [10–14]. Ara [15] introduced the notion of F-harmonic map, which is a special f-harmonic map and also is a generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps. Since then, there were many results for F-harmonic maps such as [16–19].

On the other hand, Fardon and Ratto in [20] introduced generalized harmonic maps of a certain kind, harmonic maps with potential, which had its own mathematical and physical background, for example, the static Landu-Lifschitz equation. They discovered some properties quite different from those of ordinary harmonic maps due to the presence of the potential. After this, there were many results for harmonic map with potential such as [21, 22], p-harmonic map with potential such as [23], F-harmonic map with potential such as [24], f-harmonic map with potential such as [25] and F-stationary maps with potential such as [4].

In this paper, we generalize and unify the concept of critical point of the functional Φ. For this, we define the functional $\Phi_{f,H}$ by

$$\Phi_{f,H}(u)=\int_M [f(x)\frac{||u^*h||^2}{4}-H\circ u]dv_g, \tag{1.2}$$

where H is a smooth function on N^n. If $H=0$, then we have $\Phi_{f,H}=\Phi_f$. If $H=0$ and $f=1$, then we have $\Phi_{f,H}=\Phi$. Let

$$u_t:(M^m,g)\to(N^n,h)\,(-\epsilon<t<\epsilon)$$

be a variation of u, i.e., $u_t=\Psi(t,.)$ with $u_0=u$, where $\Psi:(-\epsilon,\epsilon)\times M\to N$ is a smooth map. Let $\Gamma_0(u^{-1}TN)$ be a subset of $\Gamma(u^{-1}TN)$ consisting of all elements with compact supports contained in the interior of M. For each $\psi\in\Gamma_0(u^{-1}TN)$, there exists a variation $u_t(x)=\exp_{u(x)}(t\psi)$ (for t small enough) of u, which has the variational field ψ. Such a variation is said to have a compact support. Let

$$D_\psi\Phi_{f,H}(u)=\frac{d\Phi_{f,H}(u_t)}{dt}|_{t=0}.$$

Definition 1.1 A smooth map u is called f-stationary map with potential H for the functional $\Phi_{f,H}(u)$, if

$$D_V\Phi_{f,H}(u)=\frac{d\Phi_{f,H}(u_t)}{dt}|_{t=0}=0$$

for $V\in\Gamma_0(u^{-1}TN)$.

It is known that $du(X)\in\Gamma(u^{-1}TN)$ for any vector field X of M. If X has a compact support which is contained in the interior of M, then $du(X)\in\Gamma_0(u^{-1}TN)$.

Definition 1.2 A smooth map u is called weakly f-stationary map with potential H for the functional $\Phi_{f,H}(u)$ if $D_{du(X)}\Phi_{f,H}(u)=0$ for all $X\in\Gamma_0(TM)$.

Remark 1.1 From Definition 1.1 and Definition 1.2, we know that f-stationary maps with potential H must be weakly f-stationary maps with potential H, that is, the weakly f-stationary maps with potential H are the generalization of the f-stationary maps with potential H.

In this paper, we investigate weakly f-stationary maps with potential H. By using the stress-energy tensor, we obtain some Liouville type theorems for weakly f-stationary maps with potential under some conditions on H.

2 Preliminaries

Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_XW={}^N\nabla_{du(X)}W$, where $X\in\Gamma(TM)$ and $W\in\Gamma(u^{-1}TN)$. We choose a local orthonormal frame field $\{e_i\}$ on M. We define the tension field $\tau_{\Phi_{f,H}}(u)$ of u by

$$\tau_{\Phi_{f,H}}(u)=-\delta(f\sigma_u)+{}^N\nabla H\circ u=\tau_{\Phi_f}(u)+{}^N\nabla H\circ u, \tag{2.1}$$

where $\sigma_u=\sum_j h(du(.),du(e_j))du(e_j)$, which was defined in [1].

Under the notation above we have the following:

Lemma 2.1 [5] (The first variation formula) Let $u:M\to N$ be a C^2 map. Then

$$\frac{d}{dt}\Phi_{f,H}(u_t)|_{t=0}=-\int_M h(\tau_{\Phi_{f,H}}(u),V)dv_g, \tag{2.2}$$

where $V=\frac{d}{dt}u_t|_{t=0}$.

Let $u:M\to N$ be a weakly f-stationary map with potential H and $X\in\Gamma_0(TM)$. Then from Lemma 2.1 and the definition of weakly f-stationary maps with potential H, we have

$$D_{du(X)}\Phi_{f,H}(u)=-\int_M h(\tau_{\Phi_{f,H}}(u),du(X))dv_g=0. \tag{2.3}$$

Recall that for a 2-tensor field $T\in\Gamma\,(T^*M\otimes T^*M)$, its divergence $\mathrm{div}T\in\Gamma\,(T^*M)$ is defined by

$$(\mathrm{div}T)(X)=\sum_{i=1}^m(\nabla_{e_i}T)(e_i,X), \tag{2.4}$$

where X is any smooth vector field on M. For two 2-tensors T_1, $T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined as follows:

$$\langle T_1, T_2 \rangle = \sum_{i,j=1}^{m} T(e_i, e_j) T_2(e_i, e_j), \tag{2.5}$$

where $\{e_i\}$ is an orthonormal basis with respect to g. For a vector field $X \in \Gamma(TM)$, we denote by θ_X its dual one form, i.e., $\theta_X(Y) = g(X, Y)$, where $Y \in \Gamma(TM)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$(\nabla\theta_X)(Y, Z) = (\nabla_Y \theta_X)(Z) = g(\nabla_Y X, Z). \tag{2.6}$$

If $X = \nabla\varphi$ is the gradient field of some C^2 function φ on M, then $\theta_X = d\varphi$ and $\nabla\theta_X = \mathrm{Hess}\varphi$.

Lemma 2.2 (see [26, 27]) Let T be a symmetric $(0,2)$-type tensor field and let X be a vector field, then

$$\mathrm{div}(i_X T) = (\mathrm{div}T)(X) + \langle T, \nabla\theta_X \rangle = (\mathrm{div}T)(X) + \frac{1}{2}\langle T, L_X g \rangle, \tag{2.7}$$

where L_X is the Lie derivative of the metric g in the direction of X. Indeed, let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame field on M. Then

$$\begin{aligned} \frac{1}{2}\langle T, L_X g \rangle &= \sum_{i,j=1}^{m} \frac{1}{2}\langle T(e_i, e_j), L_X g(e_i, e_j) \rangle \\ &= \sum_{i,j=1}^{m} T(e_i, e_j) g(\nabla_{e_i} X, e_j) = \langle T, \nabla\theta_X \rangle. \end{aligned}$$

Let D be any bounded domain of M with C^1 boundary. By using the Stokes' theorem, we immediately have the following integral formula

$$\int_{\partial D} T(X, \nu) ds_g = \int_D [\langle T, \frac{1}{2} L_X g \rangle + \mathrm{div}(T)(X)] dv_g, \tag{2.8}$$

where ν is the unit outward normal vector field along ∂D.

From equation (2.8), we have

Lemma 2.3 If X is a smooth vector field with a compact support contained in the interior of M, then

$$\int_M [\langle T, \frac{1}{2} L_X g \rangle + \mathrm{div}(T)(X)] dv_g = 0. \tag{2.9}$$

Han and Feng in [5] introduced a symmetric 2-tensor S_{Φ_f} to the functional $\Phi_f(u)$ by

$$S_{\Phi_f} = f[\frac{||u^*h||^2}{4} g - h(\sigma_u(.), du(.))], \tag{2.10}$$

which is called the f-stress-energy tensor.

Lemma 2.4 [5] Let $u : (M, g) \to (N, h)$ be a smooth map, then for all $x \in M$ and for each vector $X \in T_x M$,

$$(\mathrm{div} S_{\Phi_f})(X) = -h(\tau_{\Phi_f}(u), du(X)) + \frac{||u^*h||^2}{4} df(X), \tag{2.11}$$

where

$$\tau_{\Phi_f}(u) = f \mathrm{div}\sigma_u + \sigma_u(\mathrm{grad} f).$$

By using equations (2.3), (2.9) and (2.11), we know that if $u : M \to N$ is a weakly f-stationary map with potential H, then we have

$$\begin{aligned} 0 &= \int_M \langle S_{\Phi_f}, \frac{1}{2} L_X g\rangle dv_g - \int_M h(\tau_{\Phi_f}(u) +^N \nabla H \circ u -^N \nabla H \circ u, du(X)) dv_g \\ &\quad + \int_M \frac{||u^*h||^2}{4} df(X) dv_g \\ &= \int_M \langle S_{\Phi_f}, \frac{1}{2} L_X g\rangle dv_g + \int_M h(^N\nabla H \circ u, du(X)) dv_g + \int_M \frac{||u^*h||^2}{4} df(X) dv_g, \end{aligned}$$

i.e.,

$$0 = \int_M \langle S_{\Phi_f}, \frac{1}{2} L_X g\rangle dv_g + \int_M h(^N\nabla H \circ u, du(X)) dv_g + \int_M \frac{||u^*h||^2}{4} df(X) dv_g \tag{2.12}$$

for any $X \in \Gamma_0(TM)$.

On the other hand, we may introduce the stress-energy tensor with potential $S_{\Phi_{f,H}}$ by the following

$$S_{\Phi_{f,H}} = S_{\Phi_f} - H \circ u g = [f \frac{||u^*h||^2}{4} - H \circ u] g - f h(\sigma_u(.), du(.)). \tag{2.13}$$

Then

$$\begin{aligned} (\mathrm{div} S_{\Phi_{f,H}})(X) &= (\mathrm{div} S_{\Phi_f})(X) - (\mathrm{div}(H \circ u g))(X) \\ &= -h(\tau_{\Phi_f}(u), du(X)) + \frac{||u^*h||^2}{4} df(X) - \sum_i (\nabla_{e_i}(Hg))(e_i, X) \\ &= -h(\tau_{\Phi_f}(u), du(X)) + \frac{||u^*h||^2}{4} df(X) -^N \nabla_X H \circ u \\ &= -h(\tau_{\Phi_f}(u), du(X)) + \frac{||u^*h||^2}{4} df(X) - h(^N\nabla H, du(X)) \\ &= -h(\tau_{\Phi_{f,H}}(u), du(X)) + \frac{||u^*h||^2}{4} df(X). \end{aligned} \tag{2.14}$$

By using equations (2.3), (2.9) and (2.14), we know that if $u : M \to N$ is a weakly f-stationary map with potential H, then we have

$$\int_M [\langle S_{\Phi_{f,H}}, \frac{1}{2} L_X g\rangle + \frac{||u^*h||^2}{4} df(X)] dv_g = 0 \tag{2.15}$$

for any $X \in \Gamma_0(TM)$.

3 Liouville Type Theorems

Let (M, g_0) be a complete Riemannian manifold with a pole x_0. Denote by $r(x)$ the g_0-distance function relative to the pole x_0, that is $r(x) = \mathrm{dist}_{g_0}(x, x_0)$. Set

$$B(r) = \{x \in M^m : r(x) \leq r\}.$$

It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $\mathrm{Hess}_{g_0}(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the maximum (resp. minimal) eigenvalues of $\mathrm{Hess}_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. Let (N^n, h) be a Riemannian manifold, and H be a smooth function on N.

From now on, we suppose that $u : (M^m, g) \to (N, h)$ is an f-stationary map with potential H, where

$$g = \varphi^2 g_0,\ 0 < \varphi \in C^\infty(M).$$

Clearly the vector field $\nu = \varphi^{-1}\frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$. The following conditions that we will assume for φ are as follows:

(φ_1)

$$\frac{\partial \log \varphi}{\partial r} \geq 0.$$

(φ_2) There is a constant $C_0 > 0$ such that

$$(m-4)r\frac{\partial \log \varphi}{\partial r} + \frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq C_0.$$

Remark If $\varphi(r) = r^{\frac{1}{4}}$, conditions (φ_1) and (φ_2) turn into the following

$$(m-4)\frac{1}{4} + \frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq C_0. \tag{3.1}$$

Now we set

$$\mu = \sup_M r|\frac{\partial \log f}{\partial r}| < +\infty.$$

Theorem 3.1 Let $u : (M, \varphi^2 g_0) \to (N, h)$ be a weakly f-stationary map with potential H where $0 < \varphi \in C^\infty(M)$. If φ satisfies $(\varphi_1)(\varphi_2)$, $H \leq 0$ (or $H_{u(M)} \leq 0$), $C_0 - \mu > 0$ and

$$\int_M [f\frac{||u^*h||^2}{4} - H \circ u]dv_g < \infty,$$

then u is constant.

Proof We take

$$X = \phi(r)r\frac{\partial}{\partial r} = \frac{1}{2}\phi(r)\nabla^0 r^2,$$

where ∇^0 denotes the covariant derivative determined by g_0 and $\phi(r)$ is a nonnegative function determined later. By a direct computation, we have

$$\langle S_{\Phi_{f,H}}, \frac{1}{2}L_X g\rangle = \phi(r) r\frac{\partial \log\varphi}{\partial r}\langle S_{\Phi_{f,H}}, g\rangle + \frac{1}{2}\varphi^2\langle S_{\Phi_{f,H}}, L_{\phi(r)r\frac{\partial}{\partial r}}g_0\rangle. \tag{3.2}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $\mathrm{Hess}_{g_0}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$. Then $\{\widetilde{e_i} = \varphi^{-1}e_i\}$ is an orthonormal basis with respect to g.

Now we compute

$$\begin{aligned}
&\varphi^2\langle S_{\Phi_{f,H}}, L_{\phi(r)r\frac{\partial}{\partial r}}g_0\rangle \\
= \;& \varphi^2\sum_{i,j} S_{\Phi_{f,H}}(\widetilde{e_i},\widetilde{e_j})(L_{\phi(r)r\frac{\partial}{\partial r}}g_0)(\widetilde{e_i},\widetilde{e_j}) \\
= \;& \varphi^2\{\sum_{i,j}[f\frac{||u^*h||^2}{4} - H\circ u]g(\widetilde{e_i},\widetilde{e_j})(L_{\phi(r)r\frac{\partial}{\partial r}}g_0)(\widetilde{e_i},\widetilde{e_j}) \\
& -\sum_{i,j} fh(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))(L_{\phi(r)r\frac{\partial}{\partial r}}g_0)(\widetilde{e_i},\widetilde{e_j})\} \qquad (3.3)\\
= \;& \sum_i[f\frac{||u^*h||^2}{4} - H\circ u](L_{\phi(r)r\frac{\partial}{\partial r}}g_0)(e_i,e_i) - \sum_{i,j} fh(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))(L_{\phi(r)r\frac{\partial}{\partial r}}g_0)(e_i,e_j)\} \\
= \;& \phi(r)\sum_i[f\frac{||u^*h||^2}{4} - H\circ u]\mathrm{Hess}_{g_0}(r^2)(e_i,e_i) + 2[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) \\
& -\phi(r)\sum_{i,j} fh(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))\mathrm{Hess}_{g_0}(r^2)(e_i,e_j) - 2fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
\geq \;& \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u][2+(m-1)\lambda_{\min}] - \phi(r)f\max\{2,\lambda_{\max}\}\sum_i h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_i})) \\
& +2[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) - 2fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
= \;& \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u][2+(m-1)\lambda_{\min}] - \phi(r)f\max\{2,\lambda_{\max}\}||u^*h||^2 \\
& +2[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) - 2fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
\geq \;& \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u][2+(m-1)\lambda_{\min}] - 4\phi(r)[f\frac{||u^*h||^2}{4} - H\circ u]\max\{2,\lambda_{\max}\} \\
& +2[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) - 2fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
\geq \;& \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u][2+(m-1)\lambda_{\min} - 4\max\{2,\lambda_{\max}\}] \\
& +2[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) - 2fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})).
\end{aligned}$$

From (3.2), (2.14), (3.3), (φ_1) and (φ_2), we have

$$\begin{aligned}
\langle S_{\Phi_{f,H}}, \frac{1}{2}L_X g\rangle \;\geq\; & \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u]C_0 + [f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) \\
- \;& fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})).
\end{aligned} \tag{3.4}$$

From (3.4), we have

$$\begin{aligned}
& \langle S_{\Phi_{f,H}}, \frac{1}{2}L_X g\rangle + \frac{||u^*h||^2}{4}df(X) \\
\geq\ & \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u]C_0 + [f\frac{||u^*h||^2}{4}) - H\circ u]r\phi'(r) \\
-\ & fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) + f\frac{||u^*h||^2}{4}\phi(r)r\frac{\partial logf}{\partial r} \\
\geq\ & \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u]C_0 + 2[f\frac{||u^*h||^2}{4}) - H\circ u]r\phi'(r) \\
-\ & fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) - \phi(r)[f\frac{||u^*h||^2}{4}) - H\circ u]\mu \qquad (3.5) \\
=\ & \phi(r)[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu) + [f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r) \\
-\ & fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})).
\end{aligned}$$

For any fixed $R>0$, we take a smooth function $\phi(r)$ which takes value 1 on $B(\frac{R}{2})$, 0 outside $B(R)$ and $0\leq\phi(r)\leq 1$ on $T(R)=B(R)-B(\frac{R}{2})$. And $\phi(r)$ also satisfies the condition $|\phi'(r)|\leq\frac{C_1}{r}$ on M, where C_1 is a positive constant.

From (2.15) and (3.5), we have

$$\begin{aligned}
0\ \geq\ & \int_M [\phi(r)[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu) + [f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r)]dv_g \\
& -\int_M fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g \\
\geq\ & \int_{B(\frac{R}{2})}[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu)dv_g + \int_{T(R)}[f\frac{||u^*h||^2}{4} - H\circ u]r\phi'(r)dv_g \\
& -\int_{T(R)} fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g \\
\geq\ & \int_{B(\frac{R}{2})}[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu)dv_g - C_1\int_{T(R)}[f\frac{||u^*h||^2}{4} - H\circ u]dv_g \\
& -C_1\int_{T(R)} f||u^*h||^2dv_g \\
\geq\ & \int_{B(\frac{R}{2})}[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu)dv_g - C_1(1+4)\int_{T(R)}[f\frac{||u^*h||^2}{4} - H\circ u]dv_g \\
=\ & \int_{B(\frac{R}{2})}[f\frac{||u^*h||^2}{4} - H\circ u](C_0-\mu)dv_g - 5C_1\int_{T(R)}[f\frac{||u^*h||^2}{4} - H\circ u]dv_g. \qquad (3.6)
\end{aligned}$$

From $\int_M [f\frac{||u^*h||^2}{4} - H\circ u]dv_g<\infty$, we have

$$\lim_{R\to\infty}\int_{T(R)}[f\frac{||u^*h||^2}{4} - H\circ u]dv_g = 0. \qquad (3.7)$$

From (3.6) and (3.7), we have we have

$$0 \geq [C_0 - \mu]\int_M [f\frac{||u^*h||^2}{4} - H\circ u]dv_g \geq [C_0 - \mu]\int_M f\frac{||u^*h||^2}{4}dv_g.$$

So we know that u is a constant.

Remark Let (M^m, g) be a complete Riemannian manifold with a pole x_0. Assume that the radial curvature K_r of M satisfies the following conditions: $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha \geq \beta$ and $(m-1)\beta - 4\alpha \geq 0$. From the equation (3.1) and Lemma 4.4 in [5], we have $(m-4)\frac{1}{4} + \frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq (m-4)\frac{1}{4} + m - \frac{4\alpha}{\beta} \geq \frac{m}{4} = C_0$. Let $f(x) = f(r(x)) = r^{\frac{m}{8}}$ be a smooth function on (M^m, g), we have $\mu = \frac{m}{8}$ and $C_0 - \mu = \frac{m}{8} > 0$.

Theorem 3.2 Let $u : (M, \varphi^2 g_0) \to (N, h)$ be a weakly f-stationary map with potential H where $0 < \varphi \in C^\infty(M)$. If φ satisfies $(\varphi_1)(\varphi_2)$, $\frac{\partial H\circ u}{\partial r} \geq 0$, $C_0 - \mu > 0$ and $\int_M f\frac{||u^*h||^2}{4}dv_g < \infty$, then u is constant.

Proof By using the similar method in the proof in Theorem 3.1, we can obtain the following

$$\langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle \geq \phi(r)f\frac{||u^*h||^2}{4}C_0 + f\frac{||u^*h||^2}{4}r\phi'(r) - fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})). \tag{3.8}$$

From $\frac{\partial H\circ u}{\partial r} \geq 0$ and (3.8), we have

$$\begin{aligned} & \langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle + \frac{||u^*h||^2}{4}df(X) + h({}^N\nabla H\circ u, du(X)) \\ \geq\ & \phi(r)f\frac{||u^*h||^2}{4}C_0 + f\frac{||u^*h||^2}{4}r\phi'(r) - fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) + \frac{||u^*h||^2}{4}df(X) \\ \geq\ & \phi(r)f\frac{||u^*h||^2}{4}(C_0 - \mu) + f\frac{||u^*h||^2}{4}r\phi'(r) - fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})). \end{aligned} \tag{3.9}$$

For any fixed $R > 0$, we take a smooth function $\phi(r)$ which takes value 1 on $B(\frac{R}{2})$, 0 outside $B(R)$ and $0 \leq \phi(r) \leq 1$ on $T(R) = B(R) - B(\frac{R}{2})$. And $\phi(r)$ also satisfies the condition: $|\phi'(r)| \leq \frac{C_1}{r}$ on M, where C_1 is a positive constant.

From (2.12) and (3.9), we have

$$\begin{aligned} 0 \ \geq\ & \int_M \phi(r)[f\frac{||u^*h||^2}{4}(C_0 - \mu) + f\frac{||u^*h||^2}{4}r\phi'(r)]dv_g \\ -\ & \int_M fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g \\ \geq\ & \int_{B(\frac{R}{2})} f\frac{||u^*h||^2}{4}(C_0 - \mu)dv_g + \int_{T(R)} f\frac{||u^*h||^2}{4}r\phi'(r)dv_g \\ -\ & \int_{T(R)} fr\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g \\ \geq\ & \int_{B(\frac{R}{2})} f\frac{||u^*h||^2}{4}(C_0 - \mu)dv_g - C_2\int_{T(R)} f\frac{||u^*h||^2}{4}dv_g - C_2\int_{T(R)} f||u^*h||^2 dv_g \\ =\ & (C_0 - \mu)\int_{B(\frac{R}{2})} f\frac{||u^*h||^2}{4}dv_g - 5C_2\int_{T(R)} f\frac{||u^*h||^2}{4}dv_g. \end{aligned} \tag{3.10}$$

From $\int_M f\frac{||u^*h||^2}{4}dv_g<\infty$, we have

$$\lim_{R\to\infty}\int_{T(R)} f\frac{||u^*h||^2}{4}dv_g=0. \tag{3.11}$$

From (3.10) and (3.11), we have

$$0\geq[C_0-\mu]\int_M f\frac{||u^*h||^2}{4}dv_g.$$

So we know that u is a constant.

We say the functional $\Phi_{f,H}(u)$ (or $\Phi_f(u)$) of u is slowly divergent if there exists a positive function $\psi(r)$ with $\int_{R_0}^{\infty}\frac{dr}{r\psi(r)}=+\infty$ $(R_0>0)$, such that

$$\lim_{R\to\infty}\int_{B(R)}\frac{[f\frac{||u^*h||^2}{4}-H\circ u]}{\psi(r(x))}dv_g<\infty \quad (\text{or} \quad \lim_{R\to\infty}\int_{B(R)}\frac{f\frac{||u^*h||^2}{4}}{\psi(r(x))}dv_g<\infty). \tag{3.12}$$

Theorem 3.3 Suppose $u:(M,\varphi^2g_0)\to(N,h)$ is a smooth map which satisfies the following

$$\int_M(\operatorname{div}S_{\Phi_{f,H}})(X)dv_g=\int_M\frac{||u^*h||^2}{4}df(X)dv_g \tag{3.13}$$

for any $X\in\Gamma(TM)$. If φ satisfies $(\varphi_1)(\varphi_2)$, $H\leq0$ (or $H_{u(M)}\leq0$), $C_0-\mu>0$ and $\Phi_{f,H}(u)$ of u is slowly divergent, then u is constant.

Proof From the inequality (3.5) for $\phi(r)=1$, we have

$$\langle S_{\Phi_{f,H}},\frac{1}{2}L_Xg\rangle+\frac{||u^*h||^2}{4}df(X)\geq(C_0-\mu)[f\frac{||u^*h||^2}{4}-H\circ u]. \tag{3.14}$$

On the other hand, taking $D=B(r)$ and $T=S_{\Phi_{f,H}}$ in (2.8), we have

$$\begin{aligned}
&\int_{B(r)}\langle S_{\Phi_{f,H}},\frac{1}{2}L_Xg\rangle dv_g+\int_{B(r)}(\operatorname{div}S_{\Phi_{f,H}})(X)dv_g=\int_{\partial B(r)}S_{\Phi_{f,H}}(X,\nu)ds_g\\
&=\int_{\partial B(r)}[f\frac{||u^*h||^2}{4}-H\circ u]g(X,\nu)dv_g-\int_{\partial B(r)}fh(du(X),\sigma_u(\nu))dv_g\\
&=\int_{\partial B(r)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi^2g_0(r\frac{\partial}{\partial r},\varphi^{-1}\frac{\partial}{\partial r})dv_g-\int_{\partial B(r)}f\varphi^{-1}rh(du(\frac{\partial}{\partial r}),\sigma_u(\frac{\partial}{\partial r}))dv_g\\
&=r\int_{\partial B(r)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi dv_g-\int_{\partial B(r)}f\varphi^{-1}r\sum_i h(du(\widetilde{e_i}),du(\frac{\partial}{\partial r}))^2dv_g\\
&\leq r\int_{\partial B(r)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi dv_g.
\end{aligned} \tag{3.15}$$

Now suppose that u is a nonconstant map, so there exists a constant $R_1>0$ such that for $R\geq R_1$,

$$\int_{B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]dv_g\geq\int_{B(R)}f\frac{||u^*h||^2}{4}dv_g\geq C_3, \tag{3.16}$$

where C_3 is a positive constant.

From (3.13), we have

$$\lim_{R\to\infty}\int_{B(R)}(\mathrm{div}S_{\Phi_{f,H}})(X)dv_g=\lim_{R\to\infty}\int_{B(R)}\frac{||u^*h||^2}{4}df(X)dv_g, \tag{3.17}$$

so we know that there exists a positive constant $R_2>R_1$ such that for $R\geq R_2$, we have

$$-\frac{(C_0-\mu)C_3}{2}\leq\int_{B(R)}(\mathrm{div}S_{\Phi_{f,H}})(X)dv_g-\int_{B(R)}\frac{||u^*h||^2}{4}df(X)dv_g\leq\frac{(C_0-\mu)C_3}{2}. \tag{3.18}$$

From (3.14) (3.15) and (3.18), we have for $R>R_2$,

$$\begin{aligned}
&R\int_{\partial B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi dv_g\\
\geq\ &\int_{B(R)}<S_{\Phi_{f,H}},\frac{1}{2}L_Xg>dv_g+\int_{B(R)}(\mathrm{div}S_{\Phi_{f,H}})(X)dv_g\\
\geq\ &\int_{B(R)}[<S_{\Phi_{f,H}},\frac{1}{2}L_Xg>+\frac{||u^*h||^2}{4}df(X)]dv_g-\frac{(C_0-\mu)C_3}{2}\\
\geq\ &(C_0-\mu)\int_{B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]dv_g-\frac{(C_0-\mu)C_3}{2}\\
\geq\ &\frac{(C_0-\mu)C_3}{2}.
\end{aligned} \tag{3.19}$$

From (3.19) and $|\nabla r|=\varphi^{-1}$, we have

$$\begin{aligned}
\lim_{R\to\infty}\int_{B(R)}\frac{[f\frac{||u^*h||^2}{4}-H\circ u]}{\varphi(r(x))}dv_g &= \int_0^\infty\frac{dR}{\varphi(R)}\int_{\partial B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]/|\nabla r|ds_g\\
&=\int_0^\infty\frac{dR}{\varphi(R)}\int_{\partial B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi ds_g\\
&\geq\int_{R_2}^\infty\frac{dR}{\varphi(R)}\int_{\partial B(R)}[f\frac{||u^*h||^2}{4}-H\circ u]\varphi ds_g\\
&\geq\int_{R_2}^\infty\frac{(C_0-\mu)C_3dR}{2R\varphi(R)}=+\infty.
\end{aligned} \tag{3.20}$$

This contradicts (3.12), therefore u is a constant.

Theorem 3.4 Suppose $u:(M,\varphi^2g_0)\to(N,h)$ is a smooth map which satisfies the following

$$\int_M(\mathrm{div}S_{\Phi_f})(X)dv_g=\int_M\frac{||u^*h||^2}{4}df(X)dv_g+\int_M h(^N\nabla H\circ u,du(X))dv_g \tag{3.21}$$

for any $X\in\Gamma(TM)$. If φ satisfies $(\varphi_1)(\varphi_2)$, $\frac{\partial H\circ u}{\partial r}\geq 0$ $C_0-\mu>0$ and $\Phi_f(u)$ of u is slowly divergent (see (3.12)), then u is constant.

Proof From inequality (3.9) for $\phi(r)=1$, we have

$$\langle S_{\Phi_f},\frac{1}{2}L_Xg\rangle+\frac{||u^*h||^2}{4}df(X)+h(^N\nabla H\circ u,du(X))\geq(C_0-\mu)f\frac{||u^*h||^2}{4}. \tag{3.22}$$

On the other hand, taking $D = B(r)$ and $T = S_{\Phi_f}$ in (2.8), we have

$$\begin{aligned}
&\int_{B(r)} \langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle dv_g + \int_{B(r)} (\mathrm{div} S_{\Phi_f})(X) dv_g = \int_{\partial B(r)} S_{\Phi_f}(X,\nu) ds_g \\
= &\int_{\partial B(r)} f\frac{||u^*h||^2}{4} g(X,\nu) dv_g - \int_{\partial B(r)} f h(du(X), \sigma_u(\nu)) dv_g \\
= &\int_{\partial B(r)} f\frac{||u^*h||^2}{4} \varphi^2 g_0(r\frac{\partial}{\partial r}, \varphi^{-1}\frac{\partial}{\partial r}) dv_g - \int_{\partial B(r)} f\varphi^{-1} r h(du(\frac{\partial}{\partial r}), \sigma_u(\frac{\partial}{\partial r})) dv_g \\
= &\ r\int_{\partial B(r)} f\frac{||u^*h||^2}{4} \varphi dv_g - \int_{\partial B(r)} f\varphi^{-1} r \sum_i h(du(\widetilde{e_i}), du(\frac{\partial}{\partial r}))^2 dv_g \\
\leq &\ r\int_{\partial B(r)} f\frac{||u^*h||^2}{4} \varphi dv_g.
\end{aligned} \tag{3.23}$$

Now suppose that u is a nonconstant map, so there exists a constant $R_3 > 0$ such that for $R \geq R_3$,

$$\int_{B(R)} f\frac{||u^*h||^2}{4} dv_g \geq C_4, \tag{3.24}$$

where C_4 is a positive constant.

From (3.21), we have

$$\lim_{R\to\infty} \int_{B(R)} (\mathrm{div} S_{\Phi_{f,H}})(X) dv_g = \lim_{R\to\infty} \int_{B(R)} [\frac{||u^*h||^2}{4} df(X) + h({}^N\nabla H \circ u, du(X))] dv_g, \tag{3.25}$$

so we know that there exists a positive constant $R_4 > R_3$ such that for $R \geq R_4$, we have

$$\begin{aligned}
-\frac{(C_0-\mu)C_4}{2} \quad &\leq \quad \int_{B(R)} (\mathrm{div} S_{\Phi_{f,H}})(X) dv_g - \int_{B(R)} [\frac{||u^*h||^2}{4} df(X) + h({}^N\nabla H \circ u, du(X))] dv_g \\
&\leq \quad \frac{(C_0-\mu)C_4}{2}.
\end{aligned} \tag{3.26}$$

From (3.22), (3.23) and (3.26), we have for $R > R_4$,

$$\begin{aligned}
R\int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi dv_g \quad &\geq \quad \int_{B(R)} \langle S_{\Phi_f}, \frac{1}{2}L_X g\rangle dv_g + \int_{B(R)} (\mathrm{div} S_{\Phi_f})(X) dv_g \\
&\geq \quad \int_{B(R)} [\langle S_{\Phi_{f,H}}, \frac{1}{2}L_X g\rangle + \frac{||u^*h||^2}{4} df(X) + h({}^N\nabla H \circ u, du(X))] dv_g \\
&\quad\ -\frac{(C_0-\mu)C_4}{2} \\
&\geq \quad (C_0-\mu)\int_{B(R)} f\frac{||u^*h||^2}{4} dv_g - \frac{(C_0-\mu)C_4}{2} \\
&\geq \quad \frac{(C_0-\mu)C_4}{2}.
\end{aligned} \tag{3.27}$$

From (3.27) and $|\nabla r| = \varphi^{-1}$, we have

$$\begin{aligned} \lim_{R\to\infty} \int_{B(R)} \frac{f\frac{||u^*h||^2}{4}}{\varphi(r(x))} dv_g &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} f\frac{||u^*h||^2}{4} / |\nabla r| ds_g \\ &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g \\ &\geq \int_{R_4}^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} f\frac{||u^*h||^2}{4} \varphi ds_g \\ &\geq \int_{R_4}^\infty \frac{(C_0-\mu)C_4 dR}{2R\varphi(R)} = +\infty. \end{aligned}$$

This contradicts (3.12), therefore u is a constant.

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具有势函数的弱f-稳态映射的若干结果

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摘要: 本文研究了与拉回度量有关广义泛函$\Phi_{f,H}$. 利用应力能量张量的方法, 得到具有势函数的弱f-稳态映射的一些刘维尔型定理.

关键词: 具有势函数的弱f-稳态映射; 应力能量张量; 刘维尔型定理

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