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THE EXISTENCE OF POSITIVE SOLUTIONS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE CONDITIONS

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Abstract: In this paper, we investigate the impulsive fractional differential equation with boundary value conditions. By using the theory of Kuratowski measure of noncompactness and Sadovskii' fixed point theorem, we obtain the existence of positive solution for the impulsive fractional differential equations, which generalize the results of previous literatures.

Keywords: fractional differential equations; impulsive fractional differential equations; measure of noncompactness; α-contraction

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1 Introduction

In the past few decades, fractional differential equations arise in many engineering and scientific disciplines, such as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. Because of this, the investigation of the theory of fractional differential equation attracted many researchers attention.

In [4], Ahmad and Sivasundaram studied the solution of a nonlinear impulsive fractional differential equation with integral boundary conditions given by

$$\begin{cases} {}^cD_t^q x(t) = f(t, x(t)), & t \in J = [0,1] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), & \Delta x'(t_k) = J_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ \alpha x(0) + \beta x'(0) = \displaystyle\int_0^1 g_1(s)ds, & \alpha x(1) + \beta x'(1) = \displaystyle\int_0^1 g_2(s)ds, \end{cases}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order $q \in (1,2)$. The authors investigate the existence of the solution for the equation by applying contraction mapping principle and Krasnoselskii's fixed point theorem.

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In [5], Nieto and Pimentel studied the positive solutions of a fractional thermostat model of the following

$$\begin{cases} -{}^cD^{\alpha}u(t) = f(t,u(t)), \ \ t \in [0,1], \\ u'(0) = 0, \ \ \beta{}^cD^{\alpha}u(1) + u(\eta) = 0, \end{cases}$$

where $\alpha \in (1,2]$, $\beta > 0$, $0 < \eta \leq 1$ are given numbers. Based on the known Guo-Krasnoselskii fixed point theorem on cones, the authors proved the existence of positive solutons for the fractional order thermostat model.

In [6], Zhao etc. investigated the existence of positive solutions for the nonlinear fractional differential equation with boundary value problem

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) = \lambda f(t,u(t)), \ \ 0 < t < 1, \\ u(0) + u'(0) = 0, \ \ u(1) + u'(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative. By using the properties of the Green function and Guo-Krasnoselskii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation with boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

A lot of scholars were engaged in the research about the positive solution of fractional differential equations (see [5–20]). To the best of our knowledge, there is few result about the positive solutions for nonlinear impulsive fractional differential equations with boundary value conditions so far.

Motivated by the above articles, in this paper, we will consider the positive solution of the following impulsive fractional differential equation with boundary value conditions

$$\begin{cases} {}^cD^q_t u(t) + f(t,u(t)) = 0, & t \in J = [0,1] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta u(t_k) = -I_k(u(t_k^-)), \ \ \Delta u'(t_k) = -J_k(u(t_k^-)), & k = 1,2,\cdots,m, \\ u'(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where ${}^cD^q_t$ is the Caputo fractional derivative of order $q \in (1,2)$ with the lower limit zero. $u(t_k^+) = \lim\limits_{\varepsilon \to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim\limits_{\varepsilon \to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = 1,2,\cdots,m$ for $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\mathbb{R}_+ = [0,\infty)$.

2 Preliminaries and Lemmas

Let E be a real Banach space and P be a cone inwhich defined a partial ordering in E by $x \leq y$ if and only if $y - x \in P$, P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P, P is called solid if its interior P is nonempty. If $x \leq y$ and $x \neq y$, we write $x < y$. If P is solid and $y - x \in \dot{P}$, we write $x << y$. For details on cone theory, see [1].

Let $PC[J,E]=\{x:J\to E,\ x\in C((t_k,t_{k+1}],E)$, and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k^-)=x(t_k),\ k=1,2,\cdots,m\}$ and $PC^1[J,E]=\{x:J\to E,\ x\in C^1((t_k,t_{k+1}],E)$, and there exist $x'(t_k^-)$ and $x'(t_k^+)$ with $x'(t_k^-)=x'(t_k),\ k=1,2,\cdots,m\}$. Obviously, $PC[J,E]$ is a Banach space with the norm $\|x\|_{PC}=\sup\limits_{t\in J}\|x(t)\|$, and $PC^1[J,E]$ is also a Banach space with the norm $\|x\|_{PC^1}=\max\{\|x\|_{PC},\|x'\|_{PC}\}$. Let $K_{PC^1}=\{x\in PC^1[J,E]:x(t)\geq\theta\}$, evidently, K_{PC^1} is a cone of $PC^1[J,E]$.

A map $u\in PC^1[J,E]$ is called a nonnegative solution of BVP (1.1) if $u\geq\theta$ for $t\in J$ and $u(t)$ satisfies BVP (1.1). A map $u\in PC^1[J,E]$ is called a positive solution of BVP (1.1) if it is a nonnegative solution of BVP (1.1) and $u(t)\neq\theta$.

Let α, α_{PC^1} be the Kuratowski measure of non-compactness in E and $PC^1[J,E]$, respectively. For details on the definition and properties of the measure of non-compactness, the reader is referred to [2].

As the main application of this paper, we fist give the definition of α-contraction and the related lemma to be used to prove our main result.

Definition 2.1 (see [3]) Let X be a Banach space. If there exists a positive constant $k<1$ satisfying $\alpha(Q(K))\leq k\alpha(K)$ for any bounded closed subset $K\subseteq W$, then the map $Q:W\subset X\to X$ is called an α-contraction, where $\alpha(\cdot)$ is the Kuratowski measure of non-compactness.

Lemma 2.1 (see [3]) If $W\subset X$ is bounded closed and convex, the continuous map $Q:W\to W$ is an α-contraction, then the map Q has at least one fixed point in W.

Lemma 2.2 (see [20]) If $V\subset PC^1[J,E]$ is bounded and the elements of V' are equicontinuous on each (t_k,t_{k+1}) $(k=1,2,\cdots,m)$, then

$$\alpha_{PC^1}(V)=\max\{\sup_{t\in J}\alpha(V(t)),\sup_{t\in J}\alpha(V'(t))\}.$$

Lemma 2.3 (see [20]) Let H be a countable set of strongly measurable function $x:J\to E$ such that there exists an $M\in L[J,\mathbb{R}_+]$ such that $\|x\|\leq M(t)$ a.e. $t\in J$ for all $x\in H$. Then $\alpha(H(t))\in L[J,\mathbb{R}_+]$ and

$$\alpha\Big(\int_J x(t)dt:x\in H\Big)\leq 2\int_J\alpha(H(t))dt.$$

Lemma 2.4 For a linear function $g\in C[0,1]$, a function u is a solution of the following impulsive fractional differential equation with boundary value conditions

$$\begin{cases} {}^cD_t^q u(t)+g(t)=0,\ t\in J=[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ \Delta u(t_k)=-y_k,\ \Delta u'(t_k)=-\overline{y_k},\\ u'(0)=u(1)=0\end{cases}\tag{2.1}$$

if and only if u satisfies the integral equation

$$u(t)=\int_0^1 G(t,s)g(s)ds+(1-t)\sum_{i=1}^{k}\overline{y_i}+\sum_{i=k+1}^{m}\overline{y_i}(1-t_i)+\sum_{i=k+1}^{m}y_i\ \text{ for } t\in(t_k,t_{k+1}),\tag{2.2}$$

where

$$G(t,s)=\begin{cases}\dfrac{(1-s)^{q-1}-(t-s)^{q-1}}{\Gamma(q)}, & 0\le s\le t\le 1,\\ \dfrac{(1-s)^{q-1}}{\Gamma(q)}, & 0\le t\le s\le 1.\end{cases}$$

Proof A general solution u of equation (2.1) on each interval (t_k,t_{k+1}) $(k=0,1,2,\cdots,m)$ can be given by

$$u(t)=-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds+a_kt+b_k.$$

It is known that

$$u'(t)=-\frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}g(s)ds+a_k,\quad t\in(t_k,t_{k+1}).$$

According to impulsive condition of system (2.1), we get

$$\begin{cases}a_k=a_{k-1}-\overline{y_k},\\ b_k=b_{k-1}-y_k+\overline{y_k}t_k\end{cases}$$

for $k=1,2,\cdots,m$, then we can obtain the following relations

$$\begin{cases}a_k=a_0-\displaystyle\sum_{i=1}^k\overline{y_i},\\ b_k=b_0-\displaystyle\sum_{i=1}^k(y_i-\overline{y_i}t_i)\end{cases}$$

with $u'(0)=a_0$ and $u(1)=-\frac{1}{\Gamma(q)}\int_0^1(1-s)^{q-1}g(s)ds+a_m+b_m$. So by boundary value conditions, we have

$$\begin{cases}a_0=0,\\ b_0=\displaystyle\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds+\sum_{i=1}^m y_i+\sum_{i=1}^m\overline{y_i}(1-t_i),\end{cases}$$

which implies that

$$a_kt+b_k=\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds+(1-t)\sum_{i=1}^k\overline{y_i}+\sum_{i=k+1}^m\overline{y_i}(1-t_i)+\sum_{i=k+1}^m y_i.$$

Thus we get (2.2) considering the above equations.

On the contrary, if u is a solution of (2.2), then a q order fractional differentiation of (2.2) yields

$${}^cD_t^q u(t)=-g(t)\ \text{ for }\ t\in(t_k,t_{k+1})$$

and we can get

$$u'(t) = -\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} g(s)ds - \sum_{i=1}^{k} \overline{y_i}.$$

Clearly, for $k = 1, 2, \cdots, m$, we have

$$\begin{aligned} &\Delta u(t_k) = -y_k,\ \Delta u'(t_k) = -\overline{y_k}, \\ &u'(0) = u(1) = 0. \end{aligned}$$

This completes the proof.

3 Main Results

We shall reduce BVP (1.1) to an integral equation in E. To this end, we first consider operator T defined by the following, for $t \in (t_k, t_{k+1})$ $(k = 0, 1, \cdots, m)$,

$$\begin{aligned} (Tu)(t) = &\int_0^1 G(t,s) f(s, u(s))ds + (1-t)\sum_{i=1}^{k} J_i(u(t_i^-)) \\ &+ \sum_{i=k+1}^{m} J_i(u(t_i^-))(1-t_i) + \sum_{i=k+1}^{m} I_i(u(t_i^-)). \end{aligned} \tag{3.1}$$

Hereafter, we write $Q = \{x \in K_{PC^1} : \|x\|_{PC^1} \le R\}$. Then Q is a bounded closed and convex subset of $PC^1[J, E]$.

We will list the following assumptions, which will stand throughout this paper.

(H1) $f \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$, there exist $a, b, c \in L[J, \mathbb{R}_+]$ and $h \in C[\mathbb{R}_+, \mathbb{R}_+]$ such that

$$\|f(t,x)\| \le a(t) + b(t)h(\|x\|), \quad \forall t \in J, \quad x \in \mathbb{R}_+$$

and

$$\frac{\|f(t,x)\|}{c(t)\|x\|} \to 0 \ \text{ as } x \in \mathbb{R}_+,\ \|x\| \to \infty,\ \text{uniformly for } t \in J,$$

where

$$\int_0^1 a(t)dt = a^* < \infty, \quad \int_0^1 b(t)dt = b^* < \infty, \quad \int_0^1 c(t)dt = c^* < \infty.$$

(H2) $I_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ and there exist $F_1 \in C[\mathbb{R}_+, \mathbb{R}_+]$ and constants η_{1k}, γ_{1k} such that

$$\|I_k(x)\| \le \eta_{1k} F_1(\|x\|), \quad \forall\, x \in \mathbb{R}_+$$

and

$$\frac{\|I_k(x)\|}{\gamma_{1k}\|x\|} \to 0 \text{ as } x \in \mathbb{R}_+,\ \|x\| \to \infty, \quad \text{uniformly for } k = 1, 2, \cdots, m.$$

We write

$$\eta_1^* = \sum_{k=1}^{m} \eta_{1k} < \infty, \quad \gamma_1^* = \sum_{k=1}^{m} \gamma_{1k} < \infty.$$

(H3) $J_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ and there exist $F_2 \in C[\mathbb{R}_+, \mathbb{R}_+]$ and constants η_{2k}, γ_{2k} such that

$$\|J_k(x)\| \le \eta_{2k} F_2(\|x\|), \ \ \forall\, x \in \mathbb{R}_+$$

and
$$\frac{\|J_k(x)\|}{\gamma_{2k}\|x\|} \to 0 \ \text{ as } \ x \in \mathbb{R}_+, \ \|x\| \to \infty, \ \ \text{uniformly for } k = 1, 2, \cdots, m.$$

We write
$$\eta_2^* = \sum_{k=1}^{m} \eta_{2k} < \infty, \ \ \gamma_2^* = \sum_{k=1}^{m} \gamma_{2k} < \infty.$$

(H4) For any $t \in J$ and bounded sets $V \subset PC^1[J, E]$, there exist positive numbers l, d_k, $f_k (k = 1, 2, \cdots, m)$ such that

$$\alpha(f(t, V(t))) \le l\alpha(V(t)), \ \ \alpha(I_k(V(t))) \le d_k \alpha(V(t)), \ \ \alpha(J_k(V(t))) \le f_k \alpha(V(t)).$$

Theorem 3.1 If conditions (H1)–(H3) are satisfied, then operator T is a continuous operator form Q into Q.

Proof Let
$$\varepsilon_0 = \frac{\Gamma(q)}{4c^*},$$
by (H1), there exist a $r > 0$ such that

$$\|f(t, x)\| \le \varepsilon_0 c(t)\|x\|, \ \ \forall t \in J, \ x \in \mathbb{R}_+, \ \|x\| > r$$

and
$$\|f(t, x)\| \le a(t) + M_0 b(t) \text{ for } t \in J, \ x \in \mathbb{R}_+, \ \|x\| \le r,$$
where
$$M_0 = \max\{h(\|x\|) : \|x\| \le r\}.$$

Hence we get

$$\|f(t, x)\| \le \varepsilon_0 c(t)\|x\| + a(t) + M_0 b(t), \ \ \forall t \in J, \ x \in \mathbb{R}_+. \tag{3.2}$$

Let
$$\varepsilon_1 = \max\{\frac{1}{8\gamma_1^*}, \frac{1}{8\gamma_2^*}\},$$
we see that by (H2)–(H3), for $k = 1, 2, \cdots, m$, there exist a $r_1 > 0$, such that

$$\|I_k(x)\| \le \varepsilon_1 \gamma_{1k}\|x\|, \ \ \forall x \in \mathbb{R}_+, \ \|x\| > r_1,$$
$$\|J_k(x)\| \le \varepsilon_1 \gamma_{2k}\|x\|, \ \ \forall x \in \mathbb{R}_+, \ \|x\| > r_1,$$

and

$$\|I_k(x)\| \le \eta_{1k} M_1, \ \ \forall x \in \mathbb{R}_+, \ \|x\| \le r_1,$$
$$\|J_k(x)\| \le \eta_{2k} M_2, \ \ \forall x \in \mathbb{R}_+, \ \|x\| \le r_1,$$

where

$$M_1 = \max\{F_1(\|x\|) : \|x\| \leq r_1\},$$
$$M_2 = \max\{F_2(\|x\|) : \|x\| \leq r_1\}.$$

Then $\forall x \in \mathbb{R}_+$, we have

$$\|I_k(x)\| \leq \varepsilon_1\gamma_{1k}\|x\| + \eta_{1k}M_1, \tag{3.3}$$
$$\|J_k(x)\| \leq \varepsilon_1\gamma_{2k}\|x\| + \eta_{2k}M_2. \tag{3.4}$$

Define

$$R \geq 2(\frac{a^* + b^*M_0}{\Gamma(q)} + \eta_1^*M_1 + \eta_2^*M_2).$$

For $u \in Q$, $t \in (t_k, t_{k+1})$ $(k = 1, 2, \cdots, m)$, we have

$$\int_0^1 G(t,s)f(s,u(s))ds \geq 0 \text{ as } G(t,s) \geq 0, f(s,u(s)) \geq 0, \ \forall\, t,s \in [0,1].$$

By (H2)–(H3), we have

$$(1-t)\sum_{i=1}^{k} J_i(u(t_i^-)) \geq 0, \ \sum_{i=k+1}^{m} J_i(u(t_i^-))(1-t_i) \geq 0, \ \sum_{i=k+1}^{m} I_i(u(t_i^-)) \geq 0.$$

So

$$(Tu)(t) \geq 0 \text{ for } u \in Q, \ t \in (t_k, t_{k+1}) \ (k = 0, 1, \cdots, m), \tag{3.5}$$

$$\begin{aligned}
\|(Tu)(t)\| &\leq \int_0^1 G(t,s)\|f(s,u(s))\|ds + (1-t)\sum_{i=1}^{k}\|J_i(u(t_i^-))\| \\
&\quad + \sum_{i=k+1}^{m}\|J_i(u(t_i^-))\|(1-t_i) + \sum_{i=1}^{k}\|I_i(u(t_i^-))\| \\
&\leq \int_0^1 G(t,s)(\varepsilon_0 c(s)\|u\| + a(s) + M_0 b(s))ds \\
&\quad + \sum_{i=1}^{m}(\varepsilon_1\gamma_{2i}\|u\| + \eta_{2i}M_2) + \sum_{i=1}^{m}(\varepsilon_1\gamma_{1i}\|u\| + \eta_{1i}M_1) \\
&\leq \frac{\varepsilon_0 c^*\|u\| + a^* + M_0 b^*}{\Gamma(q)} + \varepsilon_1(\gamma_1^* + \gamma_2^*)\|u\| + \eta_1^*M_1 + \eta_2^*M_2 \\
&\leq \frac{1}{2}\|u\| + \frac{a^* + M_0 b^*}{\Gamma(q)} + \eta_1^*M_1 + \eta_2^*M_2 \\
&\leq \frac{1}{2}\|u\|_{PC^1} + \frac{a^* + b^*M_0}{\Gamma(q)} + \eta_1^*M_1 + \eta_2^*M_2 \\
&\leq R.
\end{aligned} \tag{3.6}$$

Differentiating (3.1), we get

$$(Tu)'(t)=\int_0^1\frac{\partial G(t,s)}{\partial t}f(s,u(s))ds-\sum_{i=1}^{k}J_i(u(t_i^-)),\tag{3.7}$$

where

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}-\dfrac{(t-s)^{q-2}}{\Gamma(q-1)}, & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1.\end{cases}$$

By assumption (H1), we obtain

$$a(t)\le a^*,\quad b(t)\le b^*,\quad c(t)\le c^*,\quad \text{as } a(t)>0,\quad b(t)>0,\quad c(t)>0\quad \text{for } t\in[0,1].$$

Thus by (3.2), we also have

$$\|f(t,u)\|\le \varepsilon_0c^*\|u\|+a^*+M_0b^*,\quad \forall\, t\in J,\ u\in\mathbb{R}_+.$$

Then we can get

$$\begin{aligned}\|(Tu)'(t)\| &\le \int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}\|f(s,u(s))\|ds+\|\sum_{i=1}^{m}J_i(u(t_i^-))\|\\ &\le \sup_{0\le s\le t}\|f(s,u(s))\|\int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}ds+\sum_{i=1}^{m}\|J_i(u(t_i^-))\|\\ &\le \frac{(\varepsilon_0c^*\|u\|+a^*+M_0b^*)t^{q-1}}{\Gamma(q)}+\sum_{i=1}^{m}(\varepsilon_1\gamma_{2k}\|u\|+\eta_{2k}M_2)\\ &\le \frac{\varepsilon_0c^*\|u\|+a^*+M_0b^*}{\Gamma(q)}+\varepsilon_1\gamma_2^*\|u\|+\eta_2^*M_2\\ &\le \frac{3}{8}\|u\|_{PC^1}+\frac{a^*+M_0b^*}{\Gamma(q)}+\eta_2^*M_2\\ &\le \frac{1}{2}R+\frac{1}{2}R=R.\end{aligned}\tag{3.8}$$

So by (3.6), (3.7) and (3.8), we obtain $Tu\in Q$. Thus we have proved that T maps Q into Q.

Finally, we show that T is continuous. Let $u_n,\overline{u}\in Q$, $\|u_n-\overline{u}\|_{PC^1}\to 0(n\to\infty)$. It is easy to get

$$\begin{aligned}\|Tu_n-T\overline{u}\|_{PC^1} &\le \int_0^1 G(t,s)\|f(s,u_n(s))-f(s,\overline{u}(s))\|ds\\ &+ \sum_{i=1}^{m}\|J_i(u_n(t_i^-))-J_i(\overline{u}(t_i^-))\|+\sum_{i=1}^{m}\|I_i(u_n(t_i^-))-I_i(\overline{u}(t_i^-))\|\\ &\le \frac{1}{\Gamma(q)}\int_0^1\|f(s,u_n(s))-f(s,\overline{u})\|ds+\sum_{i=1}^{m}\|J_i(u_n(t_i^-))-J_i(\overline{u}(t_i^-))\|\\ &+ \sum_{i=1}^{m}\|I_i(u_n(t_i^-))-I_i(\overline{u}(t_i^-))\|.\end{aligned}\tag{3.9}$$

It is clear that

$$f(t,u_n(t)) \to f(t,\overline{u}(t)) \ \text{ as } \ n \to \infty, \ \ \forall t \in J, \tag{3.10}$$

and by (3.2),

$$\|f(t,u_n(t)) - f(t,\overline{u}(t))\| \le 2R\varepsilon_0 c(t) + 2a(t) + 2M_0 b(t) = \sigma(t) \in L[J,\mathbb{R}_+]. \tag{3.11}$$

By (3.10) and (3.11) and the dominated convergence theorem, we obtain that

$$\lim_{n\to\infty} \int_0^1 \|f(s,u_n(s)) - f(s,\overline{u}(s))\| ds = 0. \tag{3.12}$$

Obviously, for $i = 1, 2, \cdots, m$,

$$\begin{aligned} &I_i(u_n(t_i^-)) \to I_i(\overline{u}(t_i^-)) \ \text{ as } \ n \to \infty, \\ &J_i(u_n(t_i^-)) \to J_i(\overline{u}(t_i^-)) \ \text{ as } \ n \to \infty. \end{aligned}$$

So

$$\lim_{n\to\infty} (\sum_{i=1}^{m} \|I_i(u_n(t_i^-)) - I_i(\overline{u}(t_i^-))\|) = 0, \tag{3.13}$$

$$\lim_{n\to\infty} (\sum_{i=1}^{m} \|J_i(u_n(t_i^-)) - J_i(\overline{u}(t_i^-))\|) = 0. \tag{3.14}$$

Following (3.12), (3.13) and (3.14), we obtain that $\|Tu_n - T\overline{u}\|_{PC^1} \to 0$ as $n \to \infty$, and the continuity of T is proved.

Theorem 3.2 Assumes that conditions (H1)–(H4) are satisfied, if $\dfrac{2l}{\Gamma(q)} + \sum_{i=1}^{m}(d_i + f_i) < 1$, then BVP(1.1) has at least one positive solution on Q.

Proof Define $\Omega = [0,1] \times B_R$ and $f_{\max} = \sup_{(t,x)\in\Omega} \{\|f(t,x)\|\}$. For $u \in Q$, $t_k < t_1 < t_2 < t_{k+1}$, by (3.2),(3.4) and (3.7), we get

$$\begin{aligned} \|(Tu)'(t_2) - (Tu)'(t_1)\| =& \|\int_0^{t_2} \frac{(t_2-s)^{q-2}}{\Gamma(q-1)} f(s,u(s))ds - \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} f(s,u(s))ds\| \\ \le& \int_0^{t_1} \frac{(t_1-s)^{q-2} - (t_2-s)^{q-2}}{\Gamma(q-1)} \|f(s,u(s))\| ds \\ &+ \int_{t_1}^{t_2} \frac{(t_2-s)^{q-2}}{\Gamma(q-1)} \|f(s,u(s))\| ds \\ \le& \frac{[2(t_2-t_1)^{q-1} + t_2^{q-1} - t_1^{q-1}] f_{\max}}{\Gamma(q)}. \end{aligned}$$

Consequently,

$$\lim_{t_1 \to t_2} \|(Tu)'(t_2) - (Tu)'(t_1)\| = 0,$$

which implies that operator T' is equicontinuous on each (t_k, t_{k+1}) $(k = 1, 2, \cdots, m)$.

By Lemma 2.2, for any bounded and closed subset $V \subset Q$ we have

$$\alpha_{PC^1}(TV) = \max\{\sup_{t\in J} \alpha(TV)(t), \sup_{t\in J} \alpha(TV)'(t)\}.$$

It follows from Lemma 2.3 that

$$\begin{aligned}
\alpha((TV)(t)) &\le 2l \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \alpha(V(s))ds + \sum_{i=1}^{m} (f_i + d_i)\alpha(V(t_i)) \\
&\le [\frac{2l}{\Gamma(q+1)} + \sum_{i=1}^{m} (d_i + f_i)]\alpha_{PC^1}(V), \\
\alpha((TV)'(t)) &\le 2l \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \alpha(V(s))ds + \sum_{i=1}^{m} f_i \alpha(V(t_i)) \\
&\le (\frac{2l}{\Gamma(q)} + \sum_{i=1}^{m} f_i)\alpha_{PC^1}(V).
\end{aligned}$$

Therefore

$$\alpha_{PC^1}(TV) \le (\frac{2l}{\Gamma(q)} + \sum_{i=1}^{m} (f_i + d_i))\alpha_{PC^1}(V).$$

Then operator T is a α-contraction as $\dfrac{2l}{\Gamma(q)} + \sum_{i=1}^{m} (d_i + f_i) < 1$. By Lemma 2.1, we obtain that operator T has at least one fixed points on Q. Given that $Tu \ge 0$ for $u \in Q$, we learn that problem (1.1) has at least one positive solution.

4 An Example

Consider the following fractional differential equation with boundary value conditions

$$\begin{cases} {}^cD^q x(t) + \dfrac{\ln(1+x(t))}{5+t} = 0, \quad t \in (0,1) \setminus \{\dfrac{1}{2}\}, \\ \Delta x(\dfrac{1}{2}) = -\dfrac{\sin(x(\frac{1}{2}^-))}{5 + x(\frac{1}{2}^-)}, \ \Delta x'(\dfrac{1}{2}) = -\dfrac{1}{5}\sin(x(\dfrac{1}{2}^-)), \\ x'(0) = x(1) = 0. \end{cases} \tag{4.1}$$

Conclusion BVP (4.1) has at least one positive solution on [0,1].

Proof Let $E = \mathbb{R}$ and $P = \mathbb{R}_+$, $\mathbb{R}_+$ denotes the set of all nonnegative numbers. It is clear, P is a normal and solid cone in E. In this situation, $m = 1$, $t_1 = \frac{1}{2}$,

$$f(t,x) = \frac{\ln(1+x)}{5+t}, \quad \forall t \in J, \ x \ge 0 \tag{4.2}$$

and

$$I_1(x) = \frac{\sin x}{5+x}, \quad J_1(x) = \frac{\sin x}{5}. \tag{4.3}$$

Obviously, $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, I_1, $J_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$. By a direct computation about (4.2), we have

$$\|f(t,x)\| \le \frac{2\ln(1+\|x\|)}{5+t}, \quad \forall t \in J, x \le 0.$$

So (H1) is satisfied for $a(t) = 0$, $b(t) = c(t) = \frac{1}{5+t}$, $h(x) = 2\ln(1+x)$.

On the other hand, by (4.3), we have that

$$\|I_1(x)\| \le \frac{\|x\|}{5}, \quad \|J_1(x)\| \le \frac{\|x\|}{5},$$

which imply that condition (H2) and (H3) are satisfied for $F_1(x) = F_2(x) = x$ and $\eta_{11} = \eta_{21} = \gamma_{11} = \gamma_{21} = \frac{1}{5}$.

For $t \in J$, $x_1, x_2 \in \mathbb{R}_+$, by (4.2), we have

$$\begin{aligned}
|f(t,x_1) - f(t,x_2)| &= \frac{1}{5+t}|\ln(1+x_1) - \ln(1+x_2)| \\
&\le \frac{|x_1 - x_2|}{5(1+\xi)} \le \frac{|x_1 - x_2|}{5}, \\
|I_1(x_1) - I_1(x_2)| &= |\frac{\sin x_1}{5+x_1} - \frac{\sin x_2}{5+x_2}| \\
&\le \frac{\cos\delta}{5}|x_1 - x_2| \le \frac{|x_1 - x_2|}{5}, \\
|J_1(x_1) - J_2(x_2)| &= |\frac{\sin x_1}{5} - \frac{\sin x_2}{5}| \\
&\le \frac{\cos\zeta}{5}|x_1 - x_2| \le \frac{|x_1 - x_2|}{5},
\end{aligned}$$

where ξ, δ, ζ are all between x_1 and x_2, and clearly $l = \frac{1}{5}$, $d_1 = f_1 = \frac{1}{5}$, which mean that (H4) is satisfied. Then

$$\frac{2l}{\Gamma(q)} + \sum_{i=1}^{m}(d_i + f_i) = \frac{2}{5\Gamma(q)} + 2 \times \frac{1}{5} < 1,$$

It is not difficult to see that the condition of Theorem 3.2 are satisfied. Hence, boundary value problem (4.1) has at least one positive solution on [0,1].

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带有边界值问题的脉冲分数阶微分方程正解的存在性

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摘要: 本文研究了具有边界值条件的脉冲分数阶微分方程. 利用Kuratowski非紧性测度理论和Sadovskii不动点定理, 得到了脉冲分数阶微分方程正解的存在性的结果, 推广了已有文献的结论.

关键词: 分数阶微分方程; 脉冲分数阶微分方程; 非紧性测度; α-压缩

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