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# A CLASS OF DUALLY FLAT SPHERICALLY SYMMETRIC FINSLER METRICS

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**Abstract:** This paper investigates the construction of dually flat Finsler metrics. By analysing the solution of the spherically symmetric dually flat equation, we construct new examples of dually flat Finsler metrics, obtain necessary and sufficient conditions of the solution to be dually flat.

**Keywords:** dually flat; Finsler metric; spherically symmetric

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## 1 Introduction

The notion of dually flat Riemannian metrics was initially introduced by Amari and Nagaoka [1] when they studied information geometry in 2000. A Finsler metric $F = F(x, y)$ on an $m$-dimensional manifold $M$ is called locally dually flat if at every point there is a coordinate system $(x^i)$ in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a local scalar function on the tangent bundle $TM$ of $M$. Such a coordinate system is called an adapted coordinate system. Subsequently, without the quadratic restriction, the notion of dually flatness was extend to Finsler metrics by Shen when he studied Finsler information geometry [2]. In [2], Shen proved that a Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^m$ is dually flat if and only if it satisfies the following equations

$$(F^2)_{x^i y^j} y^i = 2(F^2)_{x^j}. \tag{1.1}$$

In this case, $H = -\frac{1}{6}[F^2]_{x^l} y^l$. The dually flatness of Randers metrics was studied by Cheng et al. [3]. Xia gave a characterization of locally dually flat $(\alpha, \beta)$-metrics on an $m$-dimensional manifold $M$ $(m \geq 3)$ [4]. Li found a new method to construct locally dually flat Finsler metrics by using a projectively flat Finsler metric under the condition that the projective

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factor is also a Finsler metric [5]. Huang and Mo manufactured new examples of dually flat spherically symmetric Finsler metrics [6]. From the relation between the sprays of two dually flat and conformally flat $(\alpha,\beta)$-metrics, Cheng obtained that locally dually flat and conformally flat Randers metrics are Minkowskian [7]. By using a new kind of deformation technique, Yu constructed many non-trivial explicit dually flat general $(\alpha,\beta)$-metrics and showed us that the dual flatness of an $(\alpha,\beta)$-metric always arises from that of some Riemannian metric in dimensional $m\geq 3$ [8–9].

On the other hand, the study of spherically symmetric Finsler metrics attracted a lot of attention. Many known Finsler metrics are spherically symmetric [5–6, 8]. A Finsler metric $F$ is said to be spherically symmetric (orthogonally invariant in an alternative terminology in [10]) if $F$ satisfies

$$F(Ax,Ay)=F(x,y) \tag{1.2}$$

for all $A\in O(m)$, equivalently, if the orthogonal group $O(m)$ acts as isometrics of $F$. Such metrics were first introduced by Rutz [11].

It was pointed out in [10] that a Finsler metric $F$ on $\mathbb{B}^m(\mu)$ is a spherically symmetric if and only if there is a function $\phi:[0,\mu)\times\mathbb{R}\to\mathbb{R}$ such that

$$F(x,y)=\mid y\mid \phi(\mid x\mid,\frac{\langle x,y\rangle}{\mid y\mid}), \tag{1.3}$$

where $(x,y)\in T\mathbb{R}^m(\mu)\backslash\{0\}$, $\mid\cdot\mid$ and $\langle\cdot,\cdot\rangle$ denote the standard Euclidean norm and inner product respectively. The spherically symmetric Finsler metric of form (1.3) can be rewritten as the following form [6]

$$F=\mid y\mid\sqrt{f(\frac{\mid x\mid^2}{2},\frac{\langle x,y\rangle}{\mid y\mid})}. \tag{1.4}$$

Spherically symmetric Finsler metrics are the simplest and most important general $(\alpha,\beta)$-metrics [12]. Mo, Zhou and Zhu classified the projective spherically symmetric Finsler metrics with constant flag curvature in [13–15]. A lot of spherically symmetric Finsler metrics with nice curvature properties was investigated by Mo, Huang et al. [10, 13–16].

An important example of non-Riemmannian dually flat Finsler metrics is the Funk metric

$$\Theta=\frac{\sqrt{(1-\mid x\mid^2)\mid y\mid^2+\langle x,y\rangle^2}}{1-\mid x\mid^2}+\frac{\langle x,y\rangle}{1-\mid x\mid^2} \tag{1.5}$$

on the unit ball $\mathbb{B}^m(\mu)$, where $y\in T_x\mathbb{B}^m\subset\mathbb{R}^m$. Huang and Mo in [6] decomposed the Funk metric $\Theta$ in the form

$$\Theta=\sqrt{\Theta_1^2+\Theta_2^2},$$

where

$$\Theta_1=\mid y\mid\sqrt{g(t)+g'(t)s^2},\ \ \Theta_2=\mid y\mid[h(t)s^2+\frac{1}{6}h'(t)s^4]^{\frac{1}{4}},$$

where

$$g(t)=\frac{1}{1-2t},\ \ h(t)=g(t)^3,\ \ t=\frac{\mid x\mid^2}{2},\ \ s=\frac{\langle x,y\rangle}{\mid y\mid},$$

here $\Theta_1$ and $\Theta_2$ satisfy (1.1) by straightforward calculations. It's easy to see that if $\Theta_1$ and $\Theta_2$ satisfy (1.1) then $\sqrt{a\Theta_1^2 + b\Theta_2^2}$ is also a solution of (1.1) where $a, b$ are non-negative constants. After noting this interesting fact, the two authors discussed the solution of dually flat Eq.(1.1) in the following forms

$$F(x,y) = \mid y \mid \sqrt{\sum_{j=0}^{l} f_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}}$$

and

$$F(x,y) = \mid y \mid [\sum_{j=0}^{l} f_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}]^{\frac{1}{4}}.$$

On the other hand, there is a new example of non-Riemmannian dually flat Finsler metrics given in [5, 8],

$$F = \sqrt{\frac{(\sqrt{(1- \mid x \mid^2) \mid y \mid^2 + \langle x,y\rangle^2} + \langle x,y\rangle)^3}{(1- \mid x \mid^2)^3\sqrt{(1- \mid x \mid^2) \mid y \mid^2 + \langle x,y\rangle^2}}} \tag{1.6}$$

on the unit ball $\mathbb{B}^m(\mu)$, where $y \in T_x\mathbb{B}^m \subset \mathbb{R}^m$. The metric $F$ can be expressed in the form

$$F = \sqrt{F_1^2 + F_2^2},$$

where

$$F_1 = \mid y \mid \sqrt{f(t) + f'(t)s^2}, \quad F_2 = (1 - 2t + s^2)^{-\frac{1}{4}}\sqrt{g(t)s + h(t)s^3},$$

where

$$f(t) = \frac{1}{(1-2t)^2}, \quad g(t) = 3f(t), \quad h(t) = \frac{1}{6}g'(t) + \frac{2}{3}\frac{1}{1-2t}g(t),$$
$$t = \frac{\mid x \mid^2}{2}, \quad s = \frac{\langle x,y\rangle}{\mid y \mid}.$$

We can verify that $F_1$ and $F_2$ satisfy (1.1) by direct calculations.

Inspired by the results achieved in [6], the fundamental property of the dually flat eq.(1,1) and the metric given in (1.6), in this papar, we try to find the solution of the dually flat eq.(1.1) in the following forms

$$F(x,y) = \mid y \mid \sum_{r\in N-\{0,1\}} [\sum_{j=0}^{l} f_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}]^{\frac{1}{2r}}$$

and

$$F(x,y) = \mid y \mid \sum_{r\in N^*} (1 - 2t + s^2)^{-\frac{1}{2r}}\sqrt{\sum_{j=0}^{l} f_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}}.$$

By the solutions we find, a lot of new dually flat Finsler metrics can be constructed. Through caculations, we have the following conclusions.

**Theorem 1.1** Let $f(t,s)$ be a function defined by

$$\begin{aligned} f(t,s) = \quad & g(t) + h(t)s + g'(t)s^2 + \tfrac{1}{6}h'(t)s^3 + \sum_{j=2}^{n}(-1)^{j-1}\tfrac{(2j-3)!!}{(2j+1)!}h^{(j)}(t)s^{2j+1} \\ & + b\sum_{r\geq 2}(\lambda(t) + \lambda'(t)s^2 + \tfrac{r-1}{2r}\tfrac{(\lambda'(t))^2}{\lambda(t)}s^4)^{\frac{1}{r}}, n \in N^*, r \in N, \end{aligned}$$

where $b$ is a constant and $g(t)$ and $\lambda(t)$ are any differentiable functions. $h(t)$ is an any polynomial function of $N$ degree where $N \leq n$ and $h^{(j)}$ denotes the $j$-order derivative for $h(t)$. Then the following spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$,

$$F = |y|\sqrt{f(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|})}$$

is dually flat if and only if $r = 2$.

**Theorem 1.2** Let $f(t,s)$ be a function defined by

$$\begin{aligned} f(t,s) = \quad & g(t) + h(t)s + g'(t)s^2 + \tfrac{1}{6}h'(t)s^3 + \sum_{j=2}^{n}(-1)^{j-1}\tfrac{(2j-3)!!}{(2j+1)!}h^{(j)}(t)s^{2j+1} \\ & + b\sum_{r}(1-2t+s^2)^{-\frac{1}{r}}[\lambda(t)s + (\tfrac{1}{6}\lambda'(t) + \tfrac{4}{3}\tfrac{1}{(1-2t)r}\lambda(t))s^3], n \in N^*, r \in N^*, \end{aligned}$$

where $b$ is a constant and $g(t)$ and $\lambda(t)$ are any differentiable functions, $h(t)$ is an any polynomial function of $N$ degree where $N \leq n$ and $h^{(j)}$ denotes the $j$-order derivative for $h(t)$. Then the following spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$,

$$F = |y|\sqrt{f(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|})}$$

is dually flat if and only if $r = 2$, at this time,

$$\lambda(t) = \frac{C_1}{t - \frac{1}{2}} + \frac{C_2}{(t - \frac{1}{2})^2},$$

where $C_1, C_2$ are constants.

**Remark 1** Let us take a look at a special case $b = 1$, $C_1 = 0$, $C_2 = 3$, setting $g(t) = \frac{1}{(1-2t)^2}$, $h(t) = 0$, the metric in Theorem 1.2 is given by

$$F = \sqrt{\frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^3}{(1-|x|^2)^3\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}}.$$

It is also obtained by Li [5] and Yu [8] in other different ways.

## 2 Proof of Theorem 1.1

**Lemma 2.1** [6] $F = |y|\sqrt{f(\frac{|x|^2}{2}, \frac{\langle x, y\rangle}{|y|})}$ is a solution of (1.1) if and only if $f$ satisfies

$$sf_{ts} + f_{ss} - 2f_t = 0, \tag{2.1}$$

where $t = \frac{|x|^2}{2}$ and $s = \frac{\langle x,y\rangle}{|y|}$.

The solution $f$ of (2.1) where $f = f(t,s)$ given by $f(t,s) = \sum\limits_{j=0}^{l} f_j(t)s^j$ and $f(t,s) = \sqrt{\sum\limits_{j=0}^{l} f_j(t)s^j}$ was discussed in [6]. Meanwhile, the following propositions were obtained.

**Proposition 2.1** $F = \mid y \mid \sqrt{f(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})}$ in the form

$$f(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid}) = \sum_{j=0}^{n} f_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}$$

is a solution of the dually flat eq.(2.1) if and only if $f(t,s)$ satisfies

$$f(t,s) = g(t) + h(t)s + g'(t)s^2 + \frac{1}{6}h'(t)s^3 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j+1)!}h^{(j)}(t)s^{2j+1}$$

and

$$h^{(n)}(t) = \text{constant}.$$

**Proposition 2.2** We have the following solutions of (2.1),

$$F = \mid y \mid \sqrt{f(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid})}, \quad f(t,s) = \sqrt{\frac{c_2 s^2}{(c+t)^3} - \frac{c_2 s^4}{2(c+t)^4}};$$

$$F = \mid y \mid \sqrt{f(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid})}, \quad f(t,s) = \sqrt{\lambda(t) + \lambda'(t)s^2 + \frac{(\lambda'(t))^2}{4\lambda(t)}s^4},$$

where $\lambda(t)$ is an any differentiable function.

Now let us consider the solution given by

$$f(t,s) = (\sum_{j=0}^{l} f_j(t)s^j)^{\frac{1}{r}}, \quad f_l \neq 0, \quad r \in N - \{0,1\}.$$

By a direct calculation,

$$rf^{r-1}f_t = \sum_{j=0}^{l} f_j'(t)s^j, \tag{2.2}$$

$$rf^{r-1}f_s = \sum_{j=0}^{l} jf_j(t)s^{j-1}, \tag{2.3}$$

$$r(r-1)f^{r-2}f_sf_t + rf^{r-1}f_{ts} = \sum_{j=0}^{l} jf_j'(t)s^{j-1}. \tag{2.4}$$

Putting together (2.2), (2.3), (2.4), we have

$$\begin{aligned} rf^{2r-1}f_{ts} &= f^r\sum_{j=0}^{l} jf_j'(t)s^{j-1} - (1-\tfrac{1}{r})(\sum_{j=0}^{l} f_j'(t)s^j)(\sum_{i=0}^{l} if_i(t)s^{i-1}) \\ &= (\sum_{i=0}^{l} f_i(t)s^i)(\sum_{j=0}^{l} jf_j'(t)s^{j-1}) - (1-\tfrac{1}{r})(\sum_{j=0}^{l} f_j'(t)s^j)(\sum_{i=0}^{l} if_i(t)s^{i-1}) \\ &= \sum_{k=1}^{2l}\sum_{i+j=k}[j-(1-\tfrac{1}{r})i]f_i(t)f_j'(t)s^{k-1}, \end{aligned} \tag{2.5}$$

here we use of the following lemma.

**Lemma 2.2** We have the following equations

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j = \sum_{k=1}^{2m}\sum_{i+j=k} a_ib_j,$$

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j - \sum_{i=1}^{m} c_i \sum_{j=1}^{m} d_j = \sum_{k=1}^{2m}\sum_{i+j=k} a_ib_j.$$

Differentiating (2.3), we get

$$r(r-1)f^{r-2}(f_s)^2 + rf^{r-1}f_{ss} = \sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2}.$$

Similarity, by using Lemma 2.2, we have

$$\begin{aligned} rf^{2r-1}f_{ss} &= f^r\sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2} - r(r-1)f^{2r-2}(f_s)^2 \\ &= (\sum_{i=0}^{l} f_i(t)s^i)(\sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2}) - (1-\tfrac{1}{r})(\sum_{i=0}^{l} if_i(t)s^{i-1})(\sum_{j=0}^{l} jf_j(t)s^{j-1}) \\ &= \sum_{k=2}^{2l}\sum_{i+j=k} j[j-1-(1-\tfrac{1}{r})i]f_i(t)f_j(t)s^{k-2} \\ &= \sum_{k=0}^{2l-2}\sum_{i+j=k}(j+1)[j-(1-\tfrac{1}{r})(i+1)]f_{i+1}(t)f_{j+1}(t)s^k. \end{aligned} \tag{2.6}$$

By using (2.2) and Lemma 2.2, we obtain

$$\begin{aligned} rf^{2r-1}f_t &= f^r\sum_{j=0}^{l} f_j'(t)s^j \\ &= (\sum_{i=0}^{l} f_i(t)s^i)(\sum_{j=0}^{l} f_j'(t)s^j) \\ &= \sum_{k=0}^{2l}\sum_{i+j=k} f_i(t)f_j'(t)s^k. \end{aligned} \tag{2.7}$$

Putting together (2.5), (2.6), (2.7), we have the following

$$
\begin{aligned}
rf^{2r-1}(sf_{ts}+f_{ss}-2f_t) = & \sum_{k=0}^{2l-2}\sum_{i+j=k}[j-(1-\tfrac{1}{r})i-2]f_i(t)f'_j(t)s^k \\
& +\sum_{k=0}^{2l-2}\sum_{i+j=k}(j+1)[j-(1-\tfrac{1}{r})(i+1)]f_{i+1}(t)f_{j+1}(t)s^k \\
& +\sum_{k=2l-1}^{2l}\sum_{i+j=k}[j-(1-\tfrac{1}{r})i-2]f_i(t)f'_j(t)s^k.
\end{aligned}
\tag{2.8}
$$

As $F=|y|f(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})$ on $\mathbb{B}^m(\mu)$ is dually flat, by using (2.1), we obtain the following equations

$$
\begin{cases}
[j-(1-\frac{1}{r})i-2]f_i(t)f'_j(t)+(j+1)[j-(1-\frac{1}{r})(i+1)]f_{i+1}(t)f_{j+1}(t)=0, \\
k=0,1,\cdots,2l-2, \\
(j-i+\frac{i}{r}-2)f_i(t)f'_j(t)=0, \quad k=2l-1,2l.
\end{cases}
$$

Let us focus on a special case $l=4$ and $f_1(t)=f_3(t)=0$, then

$$f_0(t)f'_0(t)-f_0(t)f_2(t)=0, \tag{2.9}$$

$$6f_0(t)f_4(t)+(-1+\frac{2}{r})f_2^2(t)+(-2+\frac{1}{r})f_2(t)f'_0(t)=0, \tag{2.10}$$

$$
\begin{aligned}
&2f_0(t)f'_4(t)+(-2+\frac{16}{r})f_2(t)f_4(t)\\
&+(-2+\frac{2}{r})f_2(t)f'_2(t)+(-6+\frac{4}{r})f_4(t)f'_0(t)=0,
\end{aligned}
\tag{2.11}
$$

$$\frac{2}{r}f_2(t)f'_4(t)+(-4+\frac{16}{r})f_4^2(t)+(-4+\frac{4}{r})f_4(t)f'_2(t)=0. \tag{2.12}$$

From (2.9), we know

$$f_0(t)=0 \tag{2.13}$$

or

$$f_0(t)\neq 0, \quad f'_0(t)=f_2(t). \tag{2.14}$$

**Case 1** Plugging (2.14) to (2.10) we get

$$f_4(t)=\frac{r-1}{2r}\frac{(f'_0(t))^2}{f_0(t)}. \tag{2.15}$$

Substituting (2.14), (2.15) into (2.11) yields

$$\frac{r-1}{r}(\frac{2}{r}-1)\frac{(f'_0(t))^3}{f_0(t)}=0. \tag{2.16}$$

If $f'_0(t)=0$, $f(t,s)=0$. As $f_0(t)\neq 0$, $f'_0(t)\neq 0$ and $r\neq 1$, we obtain

$$r=2. \tag{2.17}$$

Putting (2.14), (2.15), (2.17) into (2.12), the equality holds. Then

$$f(t,s)=\sqrt{\lambda_0(t)+\lambda_0'(t)s^2+\frac{(\lambda_0'(t))^2}{4\lambda_0(t)}s^4}. \tag{2.18}$$

**Case 2** Plugging (2.13) to (2.10) we know

$$(\frac{2}{r}-1)f_2^2(t)=0. \tag{2.19}$$

If $r=2$, the results are the same as Mo's in [9]. If $r\neq 2$, $r\neq 4$, then $f(t,s)=0$. If $r=4$, $f_0=f_2=0$, $f_4$ is an arbitrary function.

Combine Propositions 2.1, 2.2, (2.17), (2.18) and the fundamental property of the dually flat eq.(1.1), Theorem 1.1 can be achieved.

## 3 Proof of Theorem 1.2

In this section, we are going to construct more dually flat Finsler metrics. Consider the spherically symmetric Finsler metric $F=\mid y\mid\sqrt{f(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})}$ on $\mathbb{B}^m(\mu)$ where $f=f(t,s)$ is given by

$$f(t,s)=(1-2t+s^2)^{-\frac{1}{r}}(\sum_{j=0}^{l}f_j(t)s^j),\quad r\in N^*. \tag{3.1}$$

Suppose that $g(t,s)=(1-2t+s^2)^{-\frac{1}{r}}$, (3.1) can be written as

$$f(t,s)=g(t,s)(\sum_{j=0}^{l}f_j(t)s^j).$$

Thus

$$g_t:=\frac{\partial g}{\partial t}=\frac{2}{r}g^{r+1},\quad g_s:=\frac{\partial g}{\partial s}=-\frac{2}{r}g^{r+1}s. \tag{3.2}$$

Differentiating (3.1), by using (3.2), we get

$$f_t(t,s)=\frac{2}{r}g^{r+1}(\sum_{j=0}^{l}f_j(t)s^j)+g(\sum_{j=0}^{l}f_j'(t)s^j), \tag{3.3}$$

$$f_s(t,s)=-\frac{2}{r}g^{r+1}s(\sum_{j=0}^{l}f_j(t)s^j)+g(\sum_{j=0}^{l}jf_j(t)s^{j-1}), \tag{3.4}$$

$$\begin{aligned}f_{ts}(t,s)=&-\frac{2}{r}g^{r+1}(\sum_{j=0}^{l}f_j'(t)s^{j+1})+g(\sum_{j=0}^{l}jf_j'(t)s^{j-1})\\&-\frac{4}{r^2}(r+1)g^{2r+1}(\sum_{j=0}^{l}f_j(t)s^{j+1})+\frac{2}{r}g^{r+1}(\sum_{j=0}^{l}jf_j(t)s^{j-1}),\end{aligned} \tag{3.5}$$

$$\begin{aligned}f_{ss}(t,s)=&\frac{4}{r^2}(r+1)g^{2r+1}(\sum_{j=0}^{l}f_j(t)s^{j+2})-\frac{2}{r}g^{r+1}(\sum_{j=0}^{l}(j+1)f_j(t)s^j)\\&-\frac{2}{r}g^{r+1}(\sum_{j=0}^{l}jf_j(t)s^j)+g[\sum_{j=0}^{l}j(j-1)f_j(t)s^{j-2}].\end{aligned} \tag{3.6}$$

Plugging (3.3), (3.5), (3.6) into the dually flat eq.(2.1) we get the following

$$-\frac{2}{r}g^r(\sum_{j=0}^{l} f_j'(t)s^{j+2}) - \frac{2}{r}g^r[\sum_{j=0}^{l}(j+3)f_j(t)s^j] + \sum_{j=0}^{l}(j-2)f_j'(t)s^j + \sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2} = 0.$$

Multiplying $g^{-r}$ on the above equation, then

$$\begin{aligned} 0 = & -\tfrac{2}{r}[\sum_{j=0}^{l}(j+3)f_j(t)s^j] - \tfrac{2}{r}(\sum_{j=0}^{l} f_j'(t)s^{j+2}) + g^{-r}[\sum_{j=0}^{l}(j-2)f_j'(t)s^j] \\ & +g^{-r}[\sum_{j=0}^{l} j(j-1)f_j(t)s^{j-2}] \\ = & [\sum_{j=0}^{l}(j^2 - j(1+\tfrac{2}{r}) - \tfrac{6}{r})f_j(t)s^j] + \sum_{j=0}^{l}(j-2-\tfrac{2}{r})f_j'(t)s^{j+2} + \sum_{j=0}^{l}(j-2)(1-2t)f_j'(t)s^j \\ & + \sum_{j=0}^{l} j(j-1)(1-2t)f_j(t)s^{j-2} \\ = & \sum_{j=0}^{l}[j^2 - (1+\tfrac{2}{r})j - \tfrac{6}{r}]f_j(t)s^j + \sum_{j=2}^{l+2}(j-4-\tfrac{2}{r})f_{j-2}'(t)s^j + \sum_{j=0}^{l}(j-2)(1-2t)f_j'(t)s^j \\ & + \sum_{j=0}^{l-2}(j+2)(j+1)(1-2t)f_{j+2}(t)s^j. \end{aligned} \tag{3.7}$$

From (3.7), we obtain the following equations

$$\begin{aligned} & [j^2 - (1+\frac{2}{r})j - \frac{6}{r}]f_j(t) + (j-4-\frac{2}{r})f_{j-2}'(t) + (j-2)(1-2t)f_j'(t) \\ & +(j+2)(j+1)(1-2t)f_{j+2}(t) = 0, \quad j = 2, \cdots, l-2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} & [j^2 - (1+\frac{2}{r})j - \frac{6}{r}]f_j(t) + (j-2)(1-2t)f_j'(t) \\ & +(j+2)(j+1)(1-2t)f_{j+2}(t) = 0, \quad j = 0, 1, \end{aligned} \tag{3.9}$$

$$\begin{aligned} & [j^2 - (1+\frac{2}{r})j - \frac{6}{r}]f_j(t) + (j-4-\frac{2}{r})f_{j-2}'(t) \\ & +(j-2)(1-2t)f_j'(t) = 0, \quad j = l-1, l, \end{aligned} \tag{3.10}$$

$$(j-4-\frac{2}{r})f_{j-2}'(t) = 0, \quad j = l+1, l+2. \tag{3.11}$$

Let us take a look at a special case $l = 4$, $f_2(t) = f_4(t) = 0$, then

$$(-2-\frac{2}{r})f_0'(t) = 0, \tag{3.12}$$

$$-\frac{6}{r}f_0(t) - 2(1-2t)f_0'(t) = 0, \tag{3.13}$$

$$-\frac{8}{r}f_1(t) - (1-2t)f_1'(t) + 6(1-2t)f_3(t) = 0, \tag{3.14}$$

$$(6-\frac{12}{r})f_3(t) + (-1-\frac{2}{r})f_1'(t) + (1-2t)f_3'(t) = 0, \tag{3.15}$$

$$(1-\frac{2}{r})f_3'(t) = 0. \tag{3.16}$$

From (3.12), we know that

$$f_0'(t) = 0. \tag{3.17}$$

Plugging (3.17) into (3.13),

$$f_0(t) = 0. \tag{3.18}$$

From (3.14), we obtain

$$f_3(t) = \frac{1}{6}f_1'(t) + \frac{4}{3}\frac{1}{(1-2t)r}f_1(t). \tag{3.19}$$

Differentiating (3.19),

$$f_3'(t) = \frac{1}{6}f_1''(t) + \frac{8}{3}\frac{f_1(t)}{(1-2t)^2 r} + \frac{4}{3}\frac{f_1'(t)}{(1-2t)r}. \tag{3.20}$$

Substituting (3.19), (3.20) into (3.15) yields

$$f_1(t) = C_1(t-\frac{1}{2})^{\frac{-8+r+\sqrt{160-80r+r^2}}{2r}} + C_2(t-\frac{1}{2})^{\frac{-8+r-\sqrt{160-80r+r^2}}{2r}}, \tag{3.21}$$

where $C_1, C_2$ are constants. Plugging (3.20) into (3.16), if $r \neq 2$,

$$f_1(t) = C_3(t-\frac{1}{2}) + C_4(t-\frac{1}{2})^{\frac{4}{r}}, \tag{3.22}$$

where $C_3, C_4$ are constants. Obviously, $f_1(t)$ in (3.21) and $f_1(t)$ in (3.22) are not the same. Thus

$$r = 2. \tag{3.23}$$

Meanwhile, substituting (3.23) into (3.21), we obtain

$$f_1(t) = C_1(t-\frac{1}{2})^{-1} + C_2(t-\frac{1}{2})^{-2}.$$

Though the above analysis, we get the following proposition.

**Proposition 3.1** We have the following solutions of (2.1),

$$F = \mid y \mid \sqrt{f(\frac{\mid x \mid^2}{2}, \frac{\langle x, y\rangle}{\mid y \mid})}, \quad f(t,s) = \frac{1}{\sqrt{1-2t+s^2}}[\lambda(t)s + (\frac{1}{6}\lambda'(t) + \frac{2}{3}\frac{1}{1-2t}\lambda(t))s^3],$$

where $\lambda(t)$ satisfies

$$\lambda(t) = C_1(t-\frac{1}{2})^{-1} + C_2(t-\frac{1}{2})^{-2}.$$

Combine Propositions 2.1, 3.1, (3.23) and the fundamental property of the dually flat eq.(1.1), Theorem 1.2 can be achieved.

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# 一类对偶平坦的球对称的芬斯勒度量

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**摘要:** 本文研究了对偶平坦的芬斯勒度量的构造问题. 通过分析球对称的对偶平坦的芬斯勒度量的方程的解, 我们构造了一类新的对偶平坦的芬斯勒度量, 并得到了球对称的芬斯勒度量成为对偶平坦的充分必要条件.

**关键词:** 对偶平坦; 芬斯勒度量; 球对称

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