

APPROXIMATION TO THE FRACTIONAL BROWNIAN SHEET FROM STOCHASTIC INTEGRALS OF POWER FUNCTION

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Abstract: In this paper, we study an approximation of the fractional Brownian sheet. By using the Wiener integrals, we obtain the approximation by stochastic integrals of power function.

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1 Introduction

The self-similarity has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing and finance. The best known and most widely used process that exhibits the self-similarity property is the fractional Brownian motion (fBm in short). The fBm with Hurst index $H \in (0, 1)$ is a zero mean Gaussian process $\{B^H(t), t \geq 0\}$ with $B^H(0) = 0$ and covariance

$$E[B^H(t)B^H(s)] = \frac{1}{2}[t^{2H} + s^{2H} - |t - s|^{2H}]$$

for all $s, t \geq 0$. Some surveys about the fBm could be found in Biagini et al. [4], Chen and Xiao [5], Mishura [10], Nualart [11], Wang and Wang [15], Yan [16] and the references therein.

By Decreusefond and Üstünel [6], B^H has the following integral representation with respect to the standard Brownian motion B ($H > \frac{1}{2}$):

$$B^H(t) = \int_0^t K_H(t, s)dB(s), \quad t \geq 0,$$

where the kernel K_H is given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t |u - s|^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

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with the normalizing constant $c_H > 0$ given by

$$c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^{\frac{1}{2}}.$$

Many authors studied the approximation of the fBm. For example, Delgado and Jolis [7] proved that B^H can be approximated in law by means of some processes constructed from the standard Poisson process. In Li and Dai [8], a special approximation to the one-parameter fractional Brownian motion is constructed using a two-parameter Poisson process. Mishura and Banna [12] found an approximation of fractional Brownian motion by Wiener integrals.

On the other hand, many authors proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm were introduced such as fractional Brownian sheet.

Recall that the fractional Brownian sheet can also be defined by a Wiener integral with respect to the Brownian sheet $\{B(t, s), (t, s) \in [0, T] \times [0, S]\}$ (see, for example, Bardina et al. [1])

$$W^{\alpha, \beta}(t, s) = \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) B(du, dv), \quad (1.1)$$

where $\alpha, \beta \in (\frac{1}{2}, 1)$, and the kernels K_α, K_β are defined above. Note that this process is a two-parameters centered Gaussian process, starting from $(0, 0)$, and its covariance is given by

$$E[W^{\alpha, \beta}(t, s) W^{\alpha, \beta}(t', s')] = \frac{1}{2} [t^{2\alpha} + t'^{2\alpha} - |t' - t|^{2\alpha}] \cdot \frac{1}{2} [s^{2\alpha} + s'^{2\alpha} - |s' - s|^{2\alpha}].$$

It was proved in Bardina et al. [1] that the fractional Brownian sheet can be weakly approximated by discrete processes constructed from the Poisson process in the space of continuous functions. Tudor [9] generalized this approximation in the Besov space. Wang et al. [13, 14] showed that the fractional Brownian sheet can be approximated in distribution by the random walks and martingale differences sequence in the Skorohod space, respectively. We refer to Bardina and Florit [3], Bardina and Jolis [2], and the references therein for more information about weak approximation for the fractional Brownian sheet and multidimensional parameter process.

Motivated by all above results, in this paper, we will consider the approximation of the fractional Brownian sheet with $\alpha, \beta \in (\frac{1}{2}, 1)$ from Wiener integrals.

More precisely, we consider the following problem. Let $T > 0, S > 0$ be two fixed number and consider the plane $[0, T] \times [0, S]$. Now, let the mapping $a : [0, T] \times [0, S] \rightarrow \mathbb{R}$ be a nonrandom measurable function of the square integral space $L_2([0, T] \times [0, S])$, that is, $a(t, s)$ is a function such that the stochastic integral $\int_0^t \int_0^s a(u, v) B(du, dv)$, $(t, s) \in [0, T] \times [0, S]$ is well defined with respect to the Brownian sheet $\{B(t, s), (t, s) \in [0, T] \times [0, S]\}$. The problem is to find

$$\min_{a \in L_2([0, T] \times [0, S])} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} E \left[W^{\alpha, \beta}(t, s) - \int_0^t \int_0^s a(u, v) B(du, dv) \right]^2.$$

The paper is organized as follows. In Section 2, we obtain an approximation of a fractional Brownian sheet by power function with a positive index. In Section 3, we construct an approximation of a fractional Brownian sheet by power function with a negative index, i.e., $a(t, s) = kt^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}$, where $k > 0$, $\alpha, \beta \in (\frac{1}{2}, 1)$, and find the point where the function attains its the minimum value.

2 An Approximation of A Fractional Brownian Sheet by Power Function with A Positive Index

Let $W^{\alpha, \beta} = \{W^{\alpha, \beta}(t, s), (t, s) \in \mathbb{R}_+^2\}$ be a fractional Brownian sheet with Hurst index $\alpha, \beta \in (\frac{1}{2}, 1)$, the number $T, S > 0$ be fixed, $a(t, s) \in L_2([0, T] \times [0, S])$ is a measurable function. $M(t, s), (t, s) \in [0, T] \times [0, S]$ is a square integrable martingale which have the form

$$M(t, s) = \int_0^t \int_0^s a(u, v) B(du, dv).$$

In this section, we will evaluate

$$\min_{a \in A} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} E [W^{\alpha, \beta}(t, s) - M(t, s)]^2, \quad (2.1)$$

where $A \subset L_2([0, T] \times [0, S])$ is some class of functions.

Lemma 2.1 If the Lebasgue measure of the set $A = \{(t, s) \in [0, T] \times [0, S] : a(t, s) < 0\}$ is positive, then we cannot attained the minimum in (2.1) at a function $a \in A$.

Proof

$$\begin{aligned} & E [W^{\alpha, \beta}(t, s) - M(t, s)]^2 \\ &= \int_0^t \int_0^s K_\alpha^2(t, u) K_\beta^2(s, v) dudv - 2 \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) a(u, v) dvdu \\ &\quad + \int_0^t \int_0^s a^2(u, v) dvdu \\ &= t^{2\alpha} s^{2\beta} - 2 \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) a(u, v) dvdu + \int_0^t \int_0^s a^2(u, v) dvdu. \end{aligned} \quad (2.2)$$

This makes it clear that if one changes $a(u, v)$ for $-a(u, v)$, at the points (u, v) where $a(u, v) < 0$, then the right hand side of (2.2) does not increase. This completes the proof.

Theorem 2.2 Among all function $a \in L_2([0, T] \times [0, S])$ such that $a(t, s)t^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}$ is nondecreasing with respect to t and s , then the minimum in (2.1) is attained at the function $a(t, s) = c_\alpha c_\beta t^{\alpha-\frac{1}{2}} s^{\beta-\frac{1}{2}}$, where c_α, c_β is given by $c_\alpha = \sqrt{\frac{2\alpha\Gamma(\frac{3}{2}-\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2-2\alpha)}}$, $c_\beta = \sqrt{\frac{2\beta\Gamma(\frac{3}{2}-\beta)}{\Gamma(\beta+\frac{1}{2})\Gamma(2-2\beta)}}$, $\alpha, \beta \in (\frac{1}{2}, 1)$.

Proof Let $\varphi(t, s)$ be the right hand of the equation (2.2), that is,

$$\varphi(t, s) = t^{2\alpha} s^{2\beta} - 2 \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) a(u, v) dvdu + \int_0^t \int_0^s a^2(u, v) dvdu. \quad (2.3)$$

Partial differentiating the right hand of (2.3) with respect to t , we get

$$\begin{aligned} \frac{\partial \varphi}{\partial t} = & -2\left(\alpha - \frac{1}{2}\right)c_\alpha t^{\alpha-\frac{1}{2}} \int_0^t u^{\frac{1}{2}-\alpha}(t-u)^{\alpha-\frac{3}{2}} \left[\int_0^s K_\beta(s,v)a(u,v)dv \right] du \\ & + 2\alpha t^{2\alpha-1}s^{2\beta} + \int_0^s a^2(t,v)dv. \end{aligned} \quad (2.4)$$

Next, partial differentiating the right hand of (2.4) with respect to s , we get

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t \partial s} = & -2\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)c_\alpha c_\beta t^{\alpha-\frac{1}{2}}s^{\beta-\frac{1}{2}} \int_0^t u^{\frac{1}{2}-\alpha}(t-u)^{\alpha-\frac{3}{2}} \\ & \times \left[\int_0^s v^{\frac{1}{2}-\beta}(s-v)^{\beta-\frac{3}{2}}a(u,v)dv \right] du + 2\alpha t^{2\alpha-1}2\beta s^{2\beta-1} + a^2(t,s). \end{aligned} \quad (2.5)$$

Changing the variable $u = tx$, $v = sy$ in the integral, we obtain

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t \partial s} = & -2\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)c_\alpha c_\beta t^{\alpha-\frac{1}{2}}s^{\beta-\frac{1}{2}} \int_0^1 x^{\frac{1}{2}-\alpha}(1-x)^{\alpha-\frac{3}{2}} \\ & \times \left[\int_0^1 y^{\frac{1}{2}-\beta}(1-y)^{\beta-\frac{3}{2}}a(tx,sy)dy \right] dx + 2\alpha t^{2\alpha-1}2\beta s^{2\beta-1} + a^2(t,s). \end{aligned} \quad (2.6)$$

Let $t^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}a(t,s) = b(t,s)$, then $a(t,s) = t^{\alpha-\frac{1}{2}}s^{\beta-\frac{1}{2}}b(t,s)$, and $\frac{\partial^2 \varphi}{\partial t \partial s}$ becomes of the form

$$\frac{\partial^2 \varphi}{\partial t \partial s} = t^{2\alpha-1}s^{2\beta-1} \cdot \psi(t,s), \quad (2.7)$$

where

$$\begin{aligned} \psi(t,s) = & 2\alpha \cdot 2\beta - 2\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)c_\alpha c_\beta \int_0^1 (1-x)^{\alpha-\frac{3}{2}} \\ & \times \int_0^1 (1-y)^{\beta-\frac{3}{2}}b(tx,sy)dydx + b^2(t,s). \end{aligned} \quad (2.8)$$

Similarly, if we differentiate (2.3) with respect to s and then t , the equation (2.5) can also be attained, because the variable s and the variable t are symmetry in (2.5). So we don't need to consider the precedence of the partial differential about t and s .

If the function $b(t,s)$ is nondecreasing with respect to t and s , i.e., $b(tx,sy) \leq b(t,sy) \leq b(t,s)$, where $x, y \in (0,1)$. Thus

$$\begin{aligned} \psi(t,s) \geq & 2\alpha \cdot 2\beta - 2\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)c_\alpha c_\beta \int_0^1 (1-x)^{\alpha-\frac{3}{2}} \\ & \int_0^1 (1-y)^{\beta-\frac{3}{2}}dydx b(t,s) + b^2(t,s) \end{aligned} \quad (2.9)$$

or

$$\psi(t,s) \geq 2\alpha \cdot 2\beta - 2c_\alpha c_\beta b(t,s) + b^2(t,s).$$

Next, we consider the discriminant of the quadratic polynomial $x^2 - 2c_\alpha c_\beta x + 2\alpha \cdot 2\beta$, so the discriminant is represented as follows:

$$\frac{D}{4} = c_\alpha^2 c_\beta^2 - 2\alpha \cdot 2\beta.$$

The bound $c_H^2 < 2H$ is easy, since

$$\Gamma\left(\frac{3}{2} - H\right) < \Gamma\left(H + \frac{1}{2}\right)\Gamma(2 - 2H), H \in \left(\frac{1}{2}, 1\right). \quad (2.10)$$

Thus the discriminant D is negative, whence $\psi(t, s) \geq 0$, and the minimal value of $\psi(t, s)$ is attained at $b(t, s) = c_\alpha c_\beta$.

Now, we show that the $b(t, s) = c_\alpha c_\beta$ can also make $\varphi(t, s)$ a minimal value. Further, we obtain

$$a(t, s) = c_\alpha c_\beta t^{\alpha - \frac{1}{2}} s^{\beta - \frac{1}{2}}.$$

Following the assumption of the $b(t, s)$, we have the form of the function $a(t, s) = t^{\alpha - \frac{1}{2}} s^{\beta - \frac{1}{2}} b(t, s)$. Because $b(t, s) = c_\alpha c_\beta$ is a constant, so we let $a(t, s) = kt^{\alpha - \frac{1}{2}} s^{\beta - \frac{1}{2}}$, and substituting it to (2.3).

$$\begin{aligned} \varphi(t, s) &= t^{2\alpha} s^{2\beta} - 2 \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) a(u, v) dv du + \int_0^t \int_0^s a^2(u, v) dv du \\ &= t^{2\alpha} s^{2\beta} - 2k \frac{t^{2\alpha} s^{2\beta}}{2\alpha 2\beta} c_\alpha c_\beta + k^2 \frac{t^{2\alpha} s^{2\beta}}{2\alpha 2\beta} \\ &= \frac{t^{2\alpha} s^{2\beta}}{2\alpha 2\beta} [k^2 - 2k c_\alpha c_\beta + 2\alpha 2\beta], \end{aligned} \quad (2.11)$$

since

$$\begin{aligned} \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) a(u, v) dv du &= k \int_0^t K_\alpha(t, u) u^{\alpha - \frac{1}{2}} du \int_0^s K_\beta(s, v) v^{\alpha - \frac{1}{2}} dv \\ &= k \frac{t^{2\alpha} s^{2\beta}}{2\alpha 2\beta} c_\alpha c_\beta \end{aligned}$$

and

$$\int_0^t \int_0^s a^2(u, v) dv du = k^2 \int_0^t \int_0^s u^{2\alpha - 1} v^{2\beta - 1} dv du = k^2 \frac{t^{2\alpha} s^{2\beta}}{2\alpha 2\beta}.$$

Hence, when $k = c_\alpha c_\beta$, $\varphi(t, s)$ have a minimal value $2\alpha \cdot 2\beta - c_\alpha^2 c_\beta^2$. Thus the minimum among all $a(t, s)$ such that $b(t, s) = a(t, s) t^{\frac{1}{2} - \alpha} s^{\frac{1}{2} - \beta}$ is nondecreasing is attained at $b(t, s) = c_\alpha c_\beta$.

3 An Approximation of a Fractional Brownian Sheet by Power Function with A Negative Index

From Lemma 2.1, we obtain that the square integral function $a(t, s)$ is positive, and we get an approximation of a fractional Brownian sheet by power function with a positive index in Theorem 2.2. In this section, we try to construct an approximation of a fractional

Brownian sheet by power function with a negative index, that is, $a(t, s) = kt^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}$, where $k > 0$, $\alpha, \beta \in (\frac{1}{2}, 1)$. In fact, if $k \leq 0$, then $a(t, s) \leq 0$, while the kernel of a fractional Brownian sheet is positive number. So, it is unreasonable to use $a(t, s) \leq 0$ to approximate the kernel of a fractional Brownian sheet.

Let

$$f(t, s, k) = E \left[W^{\alpha, \beta}(t, s) - k \int_0^t \int_0^s u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} B(du, dv) \right]^2,$$

then we need to evaluate

$$\min_{a \in A} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} E \left[W^{\alpha, \beta}(t, s) - \int_0^t \int_0^s a(u, v) B(du, dv) \right]^2 = \min_{k \in \mathbb{R}_+} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k), \quad (3.1)$$

where $A = \{a(t, s) = kt^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}, k > 0\} \subset L_2([0, T] \times [0, S])$ is some class of functions.

Lemma 3.1 (1) The function $f(t, s, k)$ admits the following representation:

$$f(t, s, k) = t^{2\alpha} s^{2\beta} - 8k \frac{\alpha\beta}{c_\alpha c_\beta} ts + k^2 \frac{t^{2-2\alpha}}{2-2\alpha} \cdot \frac{s^{2-2\beta}}{2-2\beta}.$$

(2) For all $k \in \mathbb{R}_+$,

$$\max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k) = f(T, S, k).$$

Proof By the straightforward calculations, we have

$$\begin{aligned} f(t, s, k) &= t^{2\alpha} s^{2\beta} - 2k E \left[W^{\alpha, \beta}(t, s) \int_0^t \int_0^s u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} B(du, dv) \right] \\ &\quad + k^2 \int_0^t \int_0^s u^{1-2\alpha} v^{1-2\beta} dv du \\ &= t^{2\alpha} s^{2\beta} - 2k E \left[W^{\alpha, \beta}(t, s) \int_0^t \int_0^s u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} B(du, dv) \right] \\ &\quad + k^2 \frac{t^{2-2\alpha}}{2-2\alpha} \cdot \frac{s^{2-2\beta}}{2-2\beta}. \end{aligned}$$

According to representation (1.1), we have

$$\begin{aligned} &E \left[W^{\alpha, \beta}(t, s) \int_0^t \int_0^s u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} B(du, dv) \right] \\ &= \int_0^t \int_0^s K_\alpha(t, u) K_\beta(s, v) u^{\frac{1}{2}-\alpha} v^{\frac{1}{2}-\beta} dv du \\ &= \int_0^t K_\alpha(t, u) u^{\frac{1}{2}-\alpha} du \int_0^s K_\beta(s, v) v^{\frac{1}{2}-\beta} dv \\ &= 4 \frac{\alpha\beta}{c_\alpha c_\beta} ts, \end{aligned} \quad (3.2)$$

since

$$\begin{aligned}
 \int_0^t K_\alpha(t, u) u^{\frac{1}{2}-\alpha} du &= \int_0^t u^{\frac{1}{2}-\alpha} \left(\alpha - \frac{1}{2}\right) c_\alpha u^{\frac{1}{2}-\alpha} \int_t^u x^{\alpha-\frac{1}{2}} (x-u)^{\alpha-\frac{3}{2}} dx du \\
 &= \left(\alpha - \frac{1}{2}\right) c_\alpha \int_0^t x^{\alpha-\frac{1}{2}} \left[\int_0^x u^{1-2\alpha} (x-u)^{\alpha-\frac{3}{2}} du \right] dx \\
 &= \left(\alpha - \frac{1}{2}\right) c_\alpha \int_0^t x^{\alpha-\frac{1}{2}} \left[\int_0^1 (xs)^{1-2\alpha} (x-xs)^{\alpha-\frac{3}{2}} x ds \right] dx \\
 &= \left(\alpha - \frac{1}{2}\right) c_\alpha t \int_0^1 s^{1-2\alpha} (1-s)^{\alpha-\frac{3}{2}} ds \\
 &= \left(\alpha - \frac{1}{2}\right) c_\alpha t B\left(2-2\alpha, \alpha - \frac{1}{2}\right) \\
 &= \left(\alpha - \frac{1}{2}\right) c_\alpha t \frac{\Gamma(2-2\alpha)\Gamma(\alpha - \frac{1}{2})}{\Gamma(\frac{3}{2}-\alpha)} \\
 &= c_\alpha t \frac{\Gamma(2-2\alpha)\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{3}{2}-\alpha)} = 2 \frac{\alpha}{c_\alpha} t
 \end{aligned} \tag{3.3}$$

and

$$\int_0^s K_\beta(s, v) v^{\frac{1}{2}-\beta} dv = 2 \frac{\beta}{c_\beta} s.$$

This completes the proof of assertion (1).

Assertion (2). Differentiating the function f with respect to t , we have

$$\frac{\partial f}{\partial t} = t^{2\alpha-1} \left[2\alpha s^{2\beta} - 8kt^{1-2\alpha} \frac{\alpha\beta}{c_\alpha c_\beta} s + k^2 (t^{1-2\alpha})^2 \frac{s^{2-2\beta}}{2-2\beta} \right].$$

Let $x = kt^{1-2\alpha}$, we consider the discriminant

$$\frac{D}{4} = s^2 \left[\left(\frac{2\alpha}{c_\alpha}\right)^2 \cdot \left(\frac{2\beta}{c_\beta}\right)^2 - \frac{2\alpha}{2-2\beta} \right] = s^2 \frac{2\alpha(2\alpha \cdot (2\beta)^2(2-2\beta) - c_\alpha^2 c_\beta^2)}{c_\alpha^2 c_\beta^2 (2-2\beta)} \tag{3.4}$$

of the follow equation

$$x^2 \frac{s^{2-2\beta}}{2-2\beta} - 8 \frac{\alpha\beta}{c_\alpha c_\beta} sx + 2\alpha s^{2\beta} = 0. \tag{3.5}$$

Because the denominator $c_\alpha^2 c_\beta^2 (2-2\beta)$ of the right hand of equation (3.4) is positive. So we need to consider the numerator. We have

$$2\alpha \cdot (2\beta)^2 (2-2\beta) - c_\alpha^2 c_\beta^2 < 0,$$

since $(2\beta)^2 (2-2\beta) < c_\beta^2 < 2\beta$, and the distance between $(2\beta)^2 (2-2\beta)$ and c_β^2 is longer than the distance of c_β^2 and 2β . Hence $\frac{D}{4} < 0$. So the roots of equation (3.5) is not exist with respect t in $\mathbb{R}_+$. We obtain that $\frac{\partial f}{\partial t}$ is positive. Hence, $f(t, s, k)$ is nondecreasing for all t . Similarly, we also obtain that $f(t, s, k)$ is nondecreasing for all s . Following the above discussion, we get

$$\max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k) = f(T, S, k).$$

Now, from Lemma 3.1, we easily obtain the following main result.

Theorem 3.2 Let $A = \{a(t, s) = kt^{\frac{1}{2}-\alpha}s^{\frac{1}{2}-\beta}, k > 0, \alpha, \beta \in (\frac{1}{2}, 1)\}$, then

$$\min_{a \in A} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} E \left[W^{\alpha, \beta}(t, s) - \int_0^t \int_0^s a(u, v) B(du, dv) \right]^2 = f(T, S, k^*),$$

where $k^* = \frac{4(2-2\alpha)(2-2\beta) \cdot \alpha \cdot \beta}{T^{1-2\alpha} S^{1-2\beta} c_\alpha c_\beta}$.

Proof First of all, we calculate the value of the constant k which makes $\max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k)$ a minimal value.

Following assertion (2) of Lemma 3.1, we have

$$\max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k) = f(T, S, k).$$

So we need evaluate the k such that the minimum of $f(T, S, k)$ can be attained at the k in the next work, that is,

$$\min_{k \in \mathbb{R}_+} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k) = \min_{k \in \mathbb{R}_+} f(T, S, k).$$

Now, differentiating $f(T, S, k)$ with respect to k ,

$$\frac{\partial f}{\partial k} = -8 \frac{\alpha \beta}{c_\alpha c_\beta} TS + 2k \frac{T^{2-2\alpha}}{2-2\alpha} \cdot \frac{S^{2-2\beta}}{2-2\beta}.$$

Then we have

$$k^* = \frac{4(2-2\alpha)(2-2\beta) \cdot \alpha \cdot \beta}{T^{1-2\alpha} S^{1-2\beta} c_\alpha c_\beta},$$

which makes the derivative $\frac{\partial f}{\partial k}$ is zero. So, if $k > k^*$, then $\frac{\partial f}{\partial k} > 0$, that is, $f(T, S, k)$ is increasing; if $k < k^*$, then $\frac{\partial f}{\partial k} < 0$, that is, $f(T, S, k)$ is decreasing. Thus the minimum of $f(T, S, k)$ is attained at $k = k^*$. Hence

$$\min_{k \in \mathbb{R}_+} \max_{\substack{0 \leq t \leq T \\ 0 \leq s \leq S}} f(t, s, k) = f(T, S, k^*).$$

This completes the proof.

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分数布朗单的幂函数随机积分逼近

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摘要: 本文研究了分数布朗单的逼近问题. 利用Wiener积分, 得到了分数布朗单的幂函数型随机积分逼近.

关键词: 分数布朗单; 随机积分; 幂函数

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