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DRINFELD DOUBLE FOR MONOIDAL HOM-HOPF GROUP-COALGEBRAS

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Abstract: In this paper, Drinfeld double over monoidal Hom-Hopf group-coalgebras is introduced. Via the definition of crossed monoidal Hom-Hopf T-coalgebras and the definition of quasitriangular monoidal Hom-Hopf group-coalgebras, we get the result that this Drinfeld double is a quasitriangular monoidal Hom-Hopf group-coalgebra.

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1 Introduction

Braided T-categories introduced by Turaev [1] are of interest due to their applications in homotopy quantum field theories, which are generalizations of ordinary topological quantum field theories. Braided crossed categories based on a group G, is braided monoidal categories in Freyd-Yetter categories of crossed G-sets (see [2]) play a key role in the construction of these homotopy invariants. In [3], Zhou and Yang studied cotriangular weak Hopf group-coalgebras and promoted Kegel theorem on the weak Hopf group-coalgebras. Motivated by this fact, Yang [4] introduced the notion of a monoidal Hom-group-coalgebra as a development of the notion of monoidal Hom-coalgebras in sense of Caenepeel and Goyvaerts (see [5]), and as a natural generalization of the notions of both the Hom-type Hopf algebras and the Hopf group-coalgebra in [1, 6], and constructed a new kind of braided T-categories.

Starting from a finite-dimensional Hopf algebra H, Drinfeld [7] showed how to obtain a quasitriangular Hopf algebra $D(H)$, the quantum double of H. It is now very natural to ask how to construct Drinfeld quantum double for finite-type monoidal Hom-Hopf group-coalgebras. In this article, we essentially construct Drinfeld quantum double over monoidal Hom-Hopf group-coalgebras.

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This article is organized as follows. In Section 1, we recall some notions and results about monoidal Hom-Hopf group-coalgebras. In Section 2, we construct the Drinfeld quantum double over monoidal Hom-Hopf group-coalgebras and study quasitriangular monoidal Hom-Hopf group-coalgebras.

2 Preliminaries

In this section, we recall the definitions and properties of monoidal Hom-Hopf algebras and monoidal Hom-Hopf group-coalgebras. Throughout this paper, we always let G be a discrete group with a neutral element 1 and k a field. If U and V are k-spaces, $T_{U,V}$: $U \otimes V \to V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$.

Definition 2.1 (see [4]) A monoidal Hom-G-coalgebra is a family of k-spaces $C = \{(C_\alpha, \xi_{C_\alpha})\}_{\alpha \in G}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta \in G}$ and a k-linear map $\varepsilon : C_1 \to k$, such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes \xi_{C_\gamma}^{-1})\Delta_{\alpha\beta,\gamma} = (\xi_{C_\alpha}^{-1} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma} \quad \text{for any } \alpha, \beta, \gamma \in G, \tag{2.1}$$

$$(\varepsilon \otimes \xi_{C_\alpha})\Delta_{1,\alpha} = \xi_{C_\alpha}^{-1} = (\xi_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} \quad \text{for all } \alpha \in G. \tag{2.2}$$

Remark 2.2 $(C_1, \xi_{C_1}, \Delta_{1,1}, \varepsilon)$ is a monoidal Hom-coaglegbra in the sense of Caenepeel and Goyvaerts [5].

Following the Sweedler's notation for G-coalgebras, for any $\alpha, \beta \in G$ and $c \in (C_{\alpha\beta}, \xi_{C_{\alpha\beta}})$ one writes

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta.$$

The coassociativity axiom (2.1) gives that, for any $\alpha, \beta, \gamma \in G$ and $c \in (C_{\alpha\beta\gamma}, \xi_{C_{\alpha\beta\gamma}})$,

$$(c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)}) \otimes \xi_{C_\gamma}^{-1}(c_{(2,\gamma)}) = \xi_{C_\alpha}^{-1}(c_{(1,\alpha)}) \otimes (c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}). \tag{2.3}$$

Definition 2.3 (see [4]) A monoidal Hom-Hopf G-coaglebra is a monoidal Hom-G-coalgebra $H = (\{H_\alpha, \xi_{H_\alpha}\}, \Delta, \varepsilon)$ together with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in G}$ such that the following data holds:

Each H_α is a monoidal Hom-algebra with multiplication m_α and unit $1_\alpha \in H_\alpha$. (2.4)

For all $\alpha, \beta \in G$, $\Delta_{\alpha,\beta}$ and $\varepsilon : H_1 \to k$ are algebra maps. (2.5)

For $\alpha \in G$, $m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$. (2.6)

Note that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a monoidal Hom-Hopf algebra. A monoidal Hom-Hopf G-coalgebra H is termed to be of finite type if, for all $\alpha \in G$, H_α is finite-dimensional as k-vector space.

Remark 2.4 Let $H = (\{H_\alpha, \xi_{H_\alpha}\}, \Delta, \varepsilon, S)$ be a monoidal Hom-Hopf G-coalgebra. Suppose that the antipode $S = \{S_\alpha\}_{\alpha \in G}$ of H is bijective. For any $\alpha \in G$, let H_α^{op} be the opposite algebra to H_α. Then $H^{op} = \{H_\alpha^{op}\}_{\alpha \in G}$, endowed with the comultiplication and the counit of H and with the antipode $S^{op} = \{S^{op}_\alpha = S^{-1}_{\alpha^{-1}}\}_{\alpha\in G}$, is an opposite monoidal Hom-Hopf G-coalgebra of H. The coopposite monoidal Hom-G-coalgebra equipped with $H^{cop}_\alpha = H_{\alpha^{-1}}$ as an algebra and with the comultiplication $\Delta_{\alpha,\beta} = T_{C_{\beta^{-1}},C_{\alpha^{-1}}}\Delta_{\beta^{-1},\alpha^{-1}}$ and with the antipode $S^{cop} = \{S^{cop}_\alpha = S^{-1}_\alpha\}_{\alpha\in G}$.

Definition 2.5 (see [4]) A monoidal Hom-G-coalgebra $H = (\{H_\alpha, \xi_{H_\alpha}\}, \Delta, \varepsilon, S)$ is said to be a monoidal Hom-T-coalgebra provided it is endowed with a family of algebra isomorphisms $\varphi = \{\varphi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in G}$ such that each φ_β preserves the comultiplication and the counit, i.e., for all $\alpha, \beta, \gamma \in G$,

$$(\varphi_\beta \otimes \varphi_\beta) \circ \Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}} \circ \varphi_\beta, \ \ \varepsilon \circ \varphi_\beta = \varepsilon,$$

and φ is multiplicative in the sense that $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta$ for all $\alpha, \beta \in G$.

Let H be a monoidal Hom-T-coalgebra. Then one has that $\varphi_1|H_\alpha = id_{H_\alpha}$, $\varphi^{-1}_\alpha = \varphi_{\alpha^{-1}}$, for all $\alpha \in G$ and φ preserves the antipode, i.e., $\varphi_\beta \circ S_\alpha = S_{\beta\alpha\beta^{-1}} \circ \varphi_\beta$ for all $\alpha, \beta \in G$.

3 The Drinfeld Quantum Double for Monoidal Hom-Hopf T-Coalgebras

In order to construct the Drinfeld quantum double for monoidal Hom-Hopf T-coalgebras and study the definition of quasitriangular monoidal Hom-Hopf group-algebra. The following definitions are necessary.

Definition 3.1 The Duality C^*. Let $C = (\{C_\alpha, \xi_{C_\alpha}, \Delta, \varepsilon\})$ be a G-coalgebra and A an algebra with multiplication m and unit element 1_A. For any $f \in \mathrm{Hom}_k(C_\alpha, A)$ and $g \in \mathrm{Hom}_k(C_\beta, A)$, we have their convolution product by

$$(f * g)(c) = m(f \otimes g)\Delta_{\alpha,\beta}(c) = f(c_{(1,\alpha)})g(c_{(2,\beta)}) \in \mathrm{Hom}_k(C_{\alpha,\beta}, A)$$

for all $c \in C_{\alpha,\beta}$. Equations (2.1) and (2.2) will imply that k-space

$$\mathrm{Conv}(C, A) = \bigoplus_{\alpha\in G} \mathrm{Hom}_k(C_\alpha, A)$$

endowed with the convolution product $*$ and the unit element $1_A\varepsilon$, is a G-algebra, called a convolution algebra.

In particular, for $A = k$, the G-algebra $\mathrm{Con}v(C, k) = \oplus_{\alpha\in G} C^*_\alpha$ is called dual to C and is denoted by C^*.

Definition 3.2 The Mirror $\overline{H}$. Let H be a monoidal Hom-T-coalgebra. Then the notion of the mirror $\overline{H}$ of H is given by the following data.

- For any $\alpha \in G$, set $\overline{H}_\alpha = H_{\alpha^{-1}}$.
- For any $\alpha, \beta \in G$, the G-coalgebra structure is defined by

$$\begin{aligned} \overline{\Delta}_{\alpha,\beta} &= ((\varphi_\beta \otimes id_{H_{\beta^{-1}}}) \circ \Delta_{\beta^{-1}\alpha\beta,\beta^{-1}})(h) \\ &= \varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \otimes h_{(2,\beta^{-1})} \in \overline{H}_\alpha \otimes \overline{H}_\beta \end{aligned} \tag{3.1}$$

for any $h \in \overline{H_{\alpha\beta}} = H_{\beta^{-1}\alpha^{-1}}$. The counit of $\overline{H}$ is given by $\varepsilon \in H^*_1 = \overline{H}^*_1$.

- For any $\alpha \in G$, the αth component of the antipode $\overline{S}$ of $\overline{H}$ is given by $\overline{S}_\alpha = \varphi_\alpha \circ S_{\alpha^{-1}}$.

- For any $\alpha \in G$, the αth component of the crossed map $\overline{\varphi}$ of $\overline{H}$ is given by $\overline{\varphi}_\alpha = \varphi_\alpha$.

Dually, a monoidal Hom-G-algebra is a family of k-spaces $A = \{(A_\alpha, \xi_{A_\alpha})\}_{\alpha \in G}$ together with a family of k-linear maps $m = \{m_{\alpha,\beta} : A_\alpha \otimes A_\beta \to A_{\alpha\beta}\}_{\alpha,\beta \in G}$ and a k-linear map $\eta : k \to A_1$, such that m is associative in the sense that, for any $\alpha, \beta, \gamma \in G$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes \xi_{A_\gamma}) = m_{\alpha,\beta\gamma}(\xi_{A_\alpha} \otimes m_{\beta,\gamma}), \tag{3.2}$$

and for all $\alpha, \beta \in G$,

$$m_{\alpha,1}(id_{A_\alpha} \otimes \eta) = \xi_{A_\alpha} = m_{1,\alpha}(\eta \otimes id_{A_\alpha}). \tag{3.3}$$

A monoidal Hom-Hopf G-algebra is a G-algebra $H = (\{H_\alpha, \xi_{H_\alpha}\}, m, \eta)$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in G}$ such that each $(H_\alpha, \xi_{H_\alpha})$ is a monoidal Hom-coalgebra with a comultiplication Δ_α and a counit ε_α; the map $\eta : k \to A_1$ and the maps $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_{\alpha\beta}$ (for all $\alpha, \beta \in G$) are coalgebra homomorphisms; and for any $\alpha \in G$, one has that

$$m_{\alpha^{-1},\alpha}(S_\alpha \otimes id_{H_\alpha})\Delta_\alpha = \varepsilon_\alpha 1_1 = m_{\alpha,\alpha^{-1}}(id_{H_\alpha} \otimes S_\alpha)\Delta_\alpha. \tag{3.4}$$

A monoidal Hom-Hopf G-algebra H is said to be of finite type if, for all $\alpha \in G$, H_α is finite dimensional as k-space.

Furthermore, a monoidal Hom-Hopf T-algebra is a monoidal Hom-Hopf G-algebra H with a set of coalgebra isomorphisms $\psi = \{\psi_\beta : H_\alpha \to H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta \in G}$ called a conjugation, satisfying the following conditions:

- ψ is multiplicative, i.e., $\psi_\beta \circ \psi_\gamma = \psi_{\beta\gamma}$ for any $\beta, \gamma \in G$. It follows that, for any $\alpha \in G$, $\psi_1 | H_\alpha = id_{H_\alpha}$.

- ψ is compatible with m, i.e., for any $\alpha, \beta, \gamma \in G$, we have $m_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}} \circ (\psi_\gamma \otimes \psi_\gamma) = \psi_\gamma \circ m_{\alpha,\beta}$.

- ψ is compatible with η, i.e., $\eta \circ \psi_\gamma = \eta$ for any $\gamma \in G$.

Let H be a monoidal Hom-Hopf T-algebra. Similar to that of [9] we have the construction H_{pk} (called a packed form of H) which can form a Hom-Hopf algebra.

Remark 3.3 Let H be a finite type monoidal Hom-Hopf T-algebra. The dual of H is the monoidal Hom-Hopf T-algebra defined as follows. For any $\alpha \in G$, the αth component of H^* is the dual coalgebra $(H_\alpha^*, \xi_\alpha^{*-1})$ of the algebra (H_α, ξ_α). The multiplication of H^* is given by

$$\langle m_{\alpha,\beta}(f \otimes g), h \rangle = \langle f \otimes g, \Delta_{\alpha,\beta} \rangle \tag{3.5}$$

for any $f \in (H_\alpha^*, \xi_\alpha^{*-1}), g \in (H_\beta^*, \xi_\beta^{*-1})$ and $h \in (H_{\alpha\beta}, \xi_{\alpha\beta})$, with $\alpha, \beta \in G$. The unit of H^* is given by $\varepsilon \in H_1^* \subset H^*$. The antipode $\mathscr{A}^*$ of H^* is given by $\mathscr{A}_\alpha^* = S_{\alpha^{-1}}^*$ for any $\alpha \in G$. For any $\beta \in G$, the conjugation isomorphism $\psi_\beta^* = \varphi_{\beta^{-1}}^*$.

Remark 3.4 Given any crossed monoidal Hom-Hopf T-coalgebra, then $((H^*)_{pk})^{cop}$ is the monoidal Hom-Hopf algebra obtained from $(H^*)_{pk}$ by replacing its comultiplication with the new one $\Delta^* = \Delta^{*t,cop}$ given by

$$\langle \Delta^*(f), h \otimes k \rangle = \langle f, kh \rangle \tag{3.6}$$

for any $f \in H^*_\alpha \subset \oplus_{\beta \in G} H^*_\beta$ and $h, k \in H_\alpha$, with $\alpha \in G$. We also replace the antipode with the new one obtained by $\mathscr{A}_* = \mathscr{A}^{*t} = (S^*)^{-1}$. In particular, we have $< \mathscr{A}_*(f), h >=< f, S^{-1}_\alpha(h) >$, for any $f \in H^*_\alpha$ and $h \in H_{\alpha^{-1}}$, with $\alpha \in G$. We can obtain the crossed monoidal Hom-Hopf T-coalgebra denoted by $H^{*t,cop}$ based on $((H^*)_{pk})^{cop}$. Note that $\varphi_{H^{*t,cop},\alpha} = \varphi_{H^{*t},\alpha} = \sum\limits_{\beta \in G} \varphi^*_{\beta^{-1}}$ for any $\alpha \in G$.

Let H be a finite type monoidal Hom-Hopf T-coalgebra. We define the Drinfeld quantum double $D(T)$ of H as follows. Consider the following vector spaces

$$H_{\alpha^{-1}} \otimes H^{*t,cop}_\alpha = H_{\alpha^{-1}} \otimes H^{*t,cop}_1 = \bar{H}_\alpha \otimes \bigoplus_{\beta \in G} H^*_\beta$$

for any $\alpha \in G$. A multiplication is obtained by setting, for any $h, k \in H_{\alpha^{-1}}, f \in H^*_\gamma$, and $g \in H^*_\delta$ with $\gamma, \delta \in G$,

$$\begin{aligned}
& (h \circledast f)(k \circledast g) \\
= {} & \xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})k \circledast f\langle g, S^{-1}_\delta(h_{(2,\delta^{-1})})((\cdot)\varphi_\alpha(h_{(11,\alpha^{-1}\delta\alpha)}))\rangle \\
= {} & \xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})k \circledast f\langle g_{11}, \varphi_\alpha(h_{(11,\alpha^{-1}\delta\alpha)})\rangle\langle g_2, S^{-1}_\delta(h_{(2,\delta^{-1})})\rangle\xi^{*-2}_\delta(g_{12}).
\end{aligned} \tag{3.7}$$

For any $h, k \in H_{\alpha^{-1}}$ and $f \in H^*_\gamma$

$$(h \circledast f)(k \circledast \varepsilon) = hk \circledast \xi^{*-1}_\gamma(f).$$

We now have the following main result of this section.

Theorem 3.5 Let H be a finite-type monoidal Hom-Hopf T-coalgebra. Then $D(H)$ is a crossed monoidal Hom-Hopf T-coalgebra with the following structures:

- For any $\alpha \in G$, αth component $D_\alpha(H)$ is an associative algebra with the multiplication given in eq. (3.7) and with unit $1_{\alpha^{-1}} \circledast \varepsilon$;

- The comultiplication is given by

$$\Delta_{\alpha,\beta}(h \circledast F) = [\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast F_1] \otimes [h_{(2,\beta^{-1})} \circledast F_2] \tag{3.8}$$

  for any $\alpha, \beta \in G, h \in \bar{H}_{\alpha\beta}$ and $F \in H^{*t,cop}$, where we have that $\Delta^*(F) = F_1 \otimes F_2$ defined by eq. (3.6);

- The counit is obtained by setting

$$\varepsilon(h \circledast f) = \langle \varepsilon, h \circledast f \rangle = \langle \varepsilon, h \rangle \langle f, 1_\gamma \rangle \tag{3.9}$$

  for any $h \in H_1$ and $f \in H^*_\gamma$ with $\gamma \in G$;

- For any $\alpha \in G$, the αth component of the antipode of $D(H)$ is given by

$$\begin{aligned} S_\alpha(h \circledast F) &= [\bar{S}_\alpha \xi^{-1}_{\alpha^{-1}}(h) \circledast \varepsilon][1_\alpha \circledast \mathscr{A}_*(\xi^*_\alpha(F))] \\ &= [\varphi_\alpha S_{\alpha^{-1}}(\xi^{-1}_{\alpha^{-1}}(h)) \circledast \varepsilon][1_\alpha \circledast \mathscr{A}_*(\xi^*_\alpha(F))] \end{aligned} \tag{3.10}$$

  for any $h \in \bar{H}_\alpha$ and $F \in H^{*t,cop}$, where $\mathscr{A}_*$ is the antipode of $H^{*t,cop}$ and $\bar{S}_\alpha = \varphi_\alpha \circ S_{\alpha^{-1}}$ is the antipode of $\bar{H}$;

- For any $\alpha \in G$, the conjugation isomorphism is given by

$$\varphi_\beta(h \circledast f) = [\varphi_\beta(h) \circledast \varphi_{H^{*t,cop},\beta}(f)] = [\varphi_\beta(h) \circledast \varphi^*_{\beta^{-1}}(f)] \tag{3.11}$$

  for any $h \in \bar{H}_\alpha$ and $f \in H^{*t,cop}_\gamma$ with $\gamma \in G$.

Proof First, for any $\alpha \in G$, we will show that $D_\alpha(H)$ is an Hom-associative algebra with unit. Then we will show that Δ, defined as above, is multipilcative, i.e., that any $\Delta_{\alpha,\beta}$ is an algebra map. After that, we show that ε is an algebra map. Finally, we will check axioms for the antipode and the conjugation isomorphisms are compatible with the multiplication.

Hom-associativity Let α be in G. The multiplication definition eq.(3.7) is associative if and only if, for any $h, k, l \in (H_{\alpha^{-1}}, \xi_{\alpha^{-1}})$, $f \in (H^*_\beta, \xi^{*-1}_\beta)$, $q \in (H^*_\delta, \xi^{*-1}_\delta)$, and $p \in (H^*_\gamma, \xi^{*-1}_\gamma)$ with $\beta, \delta, \gamma \in G$,

$$((h \circledast f)(k \circledast q))\xi_{D_\alpha(H)}(l \circledast p) = \xi_{D_\alpha(H)}(h \circledast f)((k \circledast q)(l \circledast p)). \tag{3.12}$$

By computing the left-hand side of (3.12), we obtain

$$\begin{aligned} & ((h \circledast f)(k \circledast q))\xi_{D_\alpha(H)}(l \circledast p) \\ = \; & \xi^2_{\alpha^{-1}}((\xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})k)_{(12,\alpha^{-1})})\xi_{\alpha^{-1}}(l) \circledast (f\langle q_{11}, \varphi_\alpha(h_{(11,\alpha^{-1}\delta\alpha)})\rangle \\ & \langle q_2, S^{-1}_\delta(h_{(2,\delta^{-1})})\rangle \xi^{*-2}_\delta(q_{12}))\langle \xi^{*-1}_\gamma(p_{11}), \varphi_\alpha((\xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})k)_{(11,\alpha^{-1}\gamma\alpha)})\rangle \\ & \langle \xi^{*-1}_\gamma(p_2), S^{-1}_\gamma((\xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})k)_{(2,\gamma^{-1})})\rangle \xi^{*-2}_\gamma(\xi^{*-1}_\gamma(p_{12})), \end{aligned}$$

by the antimultiplicativity of S and the multiplicativity of φ,

$$\begin{aligned} = \; & \xi^5_{\alpha^{-1}}(h_{(1212,\alpha^{-1})})(\xi^2_{\alpha^{-1}}(k_{(12,\alpha^{-1})})l) \circledast \langle q_{11}, \varphi_\alpha(h_{(11,\alpha^{-1}\delta\alpha)})\rangle \langle q_2, S^{-1}_\delta(h_{(2,\delta^{-1})})\rangle \\ & \langle \xi^{*-1}_\gamma(p_{111}), \varphi_\alpha(k_{(11,\alpha^{-1}\gamma\alpha)})\rangle \langle \xi^{*-1}_\gamma(p_{112}), \varphi_\alpha(\xi^2_{\alpha^{-1}\gamma\alpha}(h_{(1211,\alpha^{-1}\gamma\alpha)}))\rangle \\ & \langle \xi^{*-1}_\gamma(p_{21}), S^{-1}_\gamma(\xi^2_{\gamma^{-1}}(h_{(122,\gamma^{-1})}))\rangle \langle \xi^{*-1}_\gamma(p_{22}), S^{-1}_\gamma(k_{(2,\gamma^{-1})})\rangle \xi^{*-1}_\beta(f)(\xi^{*-2}_\delta(q_{12})\xi^{*-2}_\gamma(p_{12})) \\ = \; & \xi^3_{\alpha^{-1}}(h_{(12,\alpha^{-1})})(\xi^2_{\alpha^{-1}}(k_{(12,\alpha^{-1})})l) \circledast \langle q_{11}, \varphi_\alpha(\xi_{\alpha^{-1}\delta\alpha}(h_{(111,\alpha^{-1}\delta\alpha)}))\rangle \langle q_2, S^{-1}_\delta(\xi_{\delta^{-1}}(h_{(22,\delta^{-1})}))\rangle \\ & \langle \xi^{*-1}_\gamma(p_{111}), \varphi_\alpha(k_{(11,\alpha^{-1}\gamma\alpha)})\rangle \langle \xi^{*-1}_\gamma(p_{112}), \varphi_\alpha(\xi_{\alpha^{-1}\gamma\alpha}(h_{(112,\alpha^{-1}\gamma\alpha)}))\rangle \\ & \langle \xi^{*-1}_\gamma(p_{21}), S^{-1}_\gamma(\xi_{\gamma^{-1}}(h_{(21,\gamma^{-1})}))\rangle \langle \xi^{*-1}_\gamma(p_{22}), S^{-1}_\gamma(k_{(2,\gamma^{-1})})\rangle \xi^{*-1}_\beta(f)(\xi^{*-2}_\delta(q_{12})\xi^{*-2}_\gamma(p_{12})) \end{aligned}$$

while, by computing the right-hand side, we obtain

$$
\begin{aligned}
& \xi_{D_\alpha(H)}(h \circledast f)((k \circledast q)(l \circledast p)) \\
= \ & \xi^3_{\alpha^{-1}}(h_{(12,\alpha^{-1})})(\xi^2_{\alpha^{-1}}(k_{(12,\alpha^{-1})})l) \circledast \langle p_{11}, \varphi_\alpha(k_{(11,\alpha^{-1}\gamma\alpha)})\rangle \langle p_2, S^{-1}_\gamma(k_{(2,\gamma^{-1})})\rangle \\
& \langle (q\xi^{*-2}_\gamma(p_{12}))_{11}, \varphi_\alpha(\xi_{\alpha^{-1}}(h)_{(11,\alpha^{-1}\delta\gamma\alpha)}))\rangle \langle (q\xi^{*-2}_\gamma(p_{12}))_2, S^{-1}_{\gamma\delta}(\xi_{\alpha^{-1}}(h)_{(2,\delta^{-1}\gamma^{-1})}))\rangle \\
& \xi^{*-1}_\beta(f)(\xi^{*-2}_\delta(q_{12})\xi^{*-4}_\gamma(p_{1212}))
\end{aligned}
$$

by the antimultiplicativity of S and the comultiplicativity of φ,

$$
\begin{aligned}
= \ & \xi^3_{\alpha^{-1}}(h_{(12,\alpha^{-1})})(\xi^2_{\alpha^{-1}}(k_{(12,\alpha^{-1})})l) \circledast \langle p_{11}, \varphi_\alpha(k_{(11,\alpha^{-1}\gamma\alpha)})\rangle \langle p_2, S^{-1}_\gamma(k_{(2,\gamma^{-1})})\rangle \\
& \langle q_{11}, \varphi_\alpha(\xi_{\alpha^{-1}\delta\alpha}(h_{(111,\alpha^{-1}\delta\alpha)}))\rangle \langle \xi^{*-2}_\gamma(p_{1211})), \varphi_\alpha(\xi_{\alpha^{-1}\gamma\alpha}(h_{(112,\alpha^{-1}\gamma\alpha)}))\rangle \\
& \langle q_2, S^{-1}_\delta(\xi_{\delta^{-1}}(h_{(22,\delta^{-1})}))\rangle \langle \xi^{*-2}_\gamma(p_{122})), S^{-1}_\gamma(\xi_{\gamma^{-1}}(h_{(21,\gamma^{-1})}))\rangle \\
& \xi^{*-1}_\beta(f)(\xi^{*-2}_\delta(q_{12})\xi^{*-4}_\gamma(p_{1212})) \\
= \ & \xi^3_{\alpha^{-1}}(h_{(12,\alpha^{-1})})(\xi^2_{\alpha^{-1}}(k_{(12,\alpha^{-1})})l) \circledast \langle q_{11}, \varphi_\alpha(\xi_{\alpha^{-1}\delta\alpha}(h_{(111,\alpha^{-1}\delta\alpha)}))\rangle \langle q_2, S^{-1}_\delta(\xi_{\delta^{-1}}(h_{(22,\delta^{-1})}))\rangle \\
& \langle \xi^{*-1}_\gamma(p_{111}), \varphi_\alpha(k_{(11,\alpha^{-1}\gamma\alpha)})\rangle \langle \xi^{*-1}_\gamma(p_{112})), \varphi_\alpha(\xi_{\alpha^{-1}\gamma\alpha}(h_{(112,\alpha^{-1}\gamma\alpha)}))\rangle \\
& \langle \xi^{*-1}_\gamma(p_{21})), S^{-1}_\gamma(\xi_{\gamma^{-1}}(h_{(21,\gamma^{-1})}))\rangle \langle \xi^{*-1}_\gamma(p_{22}), S^{-1}_\gamma(k_{(2,\gamma^{-1})})\rangle \xi^{*-1}_\beta(f)(\xi^{*-2}_\delta(q_{12})\xi^{*-2}_\gamma(p_{12})).
\end{aligned}
$$

Unit Let α be in G. For any $h \in H_{\alpha^{-1}}$ and $f \in H^*_\gamma$ with $\gamma \in G$, we have

$$
\begin{aligned}
(1_{\alpha^{-1}} \circledast \varepsilon)(h \circledast f) &= \xi_{\alpha^{-1}}(h) \circledast \xi^{*-1}_\delta(f) \\
&= \xi^2_{\alpha^{-1}}(h_{(12,\alpha^{-1})})1_{\alpha^{-1}} \circledast f\varepsilon_2(S^{-1}_1(h_{(2,1)}))\varepsilon_{11}(\varphi_\alpha(h_{(11,1)}))\xi^{*-2}_\delta(\varepsilon_{12}) \\
&= (h \circledast f)(1_{\alpha^{-1}} \circledast \varepsilon).
\end{aligned}
$$

Remark 3.6 Where we use the fact that both S_1 and φ_α commute with ε.

Multiplicativity of Δ Let us prove that $\Delta_{\alpha,\beta}$ is an algebra map for any $\alpha, \beta \in G$. For any $h, k \in H_{\beta^{-1}\alpha^{-1}}$, $f \in H^*_\gamma$ and $g \in H^*_\delta$ with $\gamma, \delta \in G$, we have

$$
\Delta_{\alpha,\beta}((h \circledast f)(k \circledast g)) = \Delta_{\alpha,\beta}(h \circledast f)\Delta_{\alpha,\beta}(k \circledast g). \tag{3.13}
$$

This is proved by evaluating both terms in the above equation (3.13) against the general term $p \otimes x \otimes q \otimes y$ ($p \in H^*_{\alpha^{-1}}, q \in H^*_{\beta^{-1}}$, and $x, y \in H_{\gamma\delta}$).

Multiplicativity of ε Let us prove that ε is an algebra map for any $h, k \in H_1$, $f \in H^*_\gamma$, and $g \in H^*_\delta$ with $\gamma, \delta \in G$,

$$
\langle \varepsilon, h \circledast f\rangle \langle \varepsilon, k \circledast g\rangle = \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle \langle \varepsilon, k\rangle \langle g, 1_\delta\rangle
$$

and

$$
\begin{aligned}
\langle \varepsilon, (h \circledast f)(k \circledast g)\rangle &= \langle \varepsilon, \xi^2_1(h_{(12,1)})k \circledast f\langle g_{11}, h_{(11,\delta)})\rangle \langle g_2, S^{-1}_\delta(h_{2,\delta^{-1}})\rangle \xi^{*-2}_\delta(g_{12})\rangle \\
&= \langle \varepsilon, \xi^2_1(h_{12})\rangle \langle \varepsilon, k\rangle \langle f, 1_\gamma\rangle \langle g_{11}, h_{(11,\delta)})\rangle \langle g_2, S^{-1}_\delta(h_{(2,\delta^{-1})})\rangle \langle \xi^{*-2}_\delta(g_{12}), 1_\delta\rangle \\
&= \langle \varepsilon, k\rangle \langle f, 1_\gamma\rangle \langle g, S^{-1}_\delta(h_{(2,\delta^{-1})})h_{(1,\delta)})\rangle = \langle \varepsilon, h\rangle \langle f, 1_\gamma\rangle \langle \varepsilon, k\rangle \langle g, 1_\delta\rangle.
\end{aligned}
$$

This proves that ε is multiplicative. Moreover, since ε is obviously unitary, it is an algebra homomorphism.

Antipode Let $h \in H_1$ and let $f \in H_\gamma^*$ with $\gamma \in G$,

$$\begin{aligned}
&(h \circledast f)_{(1,\alpha)} S_{\alpha^{-1}}((h \circledast f)_{(2,\alpha^{-1})})\\
= \ & (\varphi_{\alpha^{-1}}(h_{(1,\alpha^{-1})}) \circledast f_1)[((\varphi_{\alpha^{-1}} \circ S_\alpha)\xi_\alpha^{-1}(h_{(2,\alpha)}) \circledast \varepsilon)(1_{\alpha^{-1}} \circledast \mathscr{A}_*(\xi_\gamma^*(f_2)))]\\
= \ & (\varphi_{\alpha^{-1}}(\xi_{\alpha^{-1}}^{-1}(h_{(1,\alpha^{-1})}) S_\alpha \xi_\alpha^{-1}(h_{(2,\alpha)})) \circledast \xi_\gamma^*(f_1)\varepsilon)(1_{\alpha^{-1}} \circledast \mathscr{A}_*(f_2))\\
= \ & \langle \varepsilon, h \circledast f\rangle 1_{\alpha^{-1}} \circledast \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
& S_{\alpha^{-1}}((h \circledast f)_{(1,\alpha^{-1})})(h \circledast f)_{(2,\alpha)}\\
= \ & [(S_\alpha \xi_\alpha^{-1}(h_{(1,\alpha)}) \circledast \varepsilon)(1_{\alpha^{-1}} \circledast \mathscr{A}_*(\xi_\gamma^*(f_1)))](h_{(2,\alpha^{-1})} \circledast f_2)\\
= \ & (S_\alpha(h_{(1,\alpha)}) \circledast \varepsilon)(h_{(2,\alpha^{-1})} \circledast \mathscr{A}_*(\xi_\gamma^*(f_1)) \langle \xi_\gamma^*(f_2)_{11}, 1_\gamma\rangle \langle \xi_\gamma^*(f_2)_2, 1_\gamma\rangle \xi_\gamma^{*-2}(\xi_\gamma^*(f_2)_{12})\\
= \ & \langle f, 1_\gamma\rangle S_\alpha(h_{(1,\alpha)}) h_{(2,\alpha^{-1})} \circledast \varepsilon\\
= \ & \langle \varepsilon, h \circledast f\rangle 1_{\alpha^{-1}} \circledast \varepsilon.
\end{aligned}$$

Conjugation Let us check that φ_β is an algebra isomorphism for any $\alpha, \beta \in G$. For all $h, k \in H_\alpha^{-1}$, $f \in H_\gamma^*$, and $g \in H_\delta^*$ with $\gamma, \delta \in G$,

$$\begin{aligned}
& (\varphi_\beta(h \circledast f)\varphi_\beta(k \circledast g))(p \otimes x)\\
= \ & (\xi^2_{\beta\alpha^{-1}\beta^{-1}}(\varphi_\beta(h)_{12})\varphi_\beta(k) \circledast \varphi^*_{\beta^{-1}}(f)\langle \varphi^*_{\beta^{-1}}(g)_{11}, \varphi_\alpha(\varphi_\beta(h)_{(11,\alpha^{-1}\beta^{-1}\delta\beta\alpha)})\rangle\\
& \langle \varphi^*_{\beta^{-1}}(g)_2, S^{-1}_{\beta\delta\beta^{-1}}(\varphi_\beta(h)_{(2,\beta\delta^{-1}\beta^{-1})})\rangle)\xi^{*-2}_{\beta^{-1}\delta\beta}(\varphi^*_{\beta^{-1}}(g)_{12}))(p \otimes x)\\
= \ & \varphi_\beta(\xi^2_{\alpha^{-1}}(h_{12})k)p \circledast \langle g_{11}, \varphi_\alpha(h_{(11,\alpha^{-1}\delta\alpha)})\rangle \langle g_2, S_\delta^{-1}(h_{(2,\delta^{-1})})\rangle\\
& \varphi^*_{\beta^{-1}}(f\xi_\delta^{*-2}(g_{12}))(x)\\
= \ & \varphi_\beta((h \circledast f)(k \circledast g))(p \otimes x)
\end{aligned}$$

for all $x \in H_{\beta\gamma\delta\beta^{-1}}, p \in H^*_{\beta\alpha^{-1}\beta^{-1}}$.

For any $\alpha \in G$, we set $n_\alpha = \dim H_\alpha$. Let $(\kappa_{(\alpha,i)})_{i=1,\cdots,n_\alpha}$ and $(\kappa^{(\alpha,i)})_{i=1,\cdots,n_\alpha}$ be dual bases in H_α and H_α^*. Then we have the following proposition.

Definition 3.7 A quasitriangular Hom-Hopf T-coalgebra is a Hom-Hopf T-coalgebra endowed with a family $R = \{R_{\alpha,\beta} = \kappa_{\alpha,i} \otimes \kappa_{\beta,i} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta \in G}$, called a universal R-matrix, such that $R_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in G$ and the following conditions are satisfied:

$$R_{\alpha,\beta}\Delta_{\alpha,\beta}(h) = (\tau \circ (\varphi_{\alpha^{-1}} \otimes id_\alpha) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})(h)R_{\alpha,\beta}, \tag{3.14}$$

$$(\xi_\alpha \otimes \xi_\beta)R_{\alpha,\beta} = R_{\alpha,\beta} \tag{3.15}$$

for all $h \in H_{\alpha\beta}$ and $\alpha, \beta \in G$,

$$\kappa_{\alpha,i} \otimes \kappa_{(1,\beta),i} \otimes \kappa_{(2,\gamma),i} = \kappa_{\alpha,i}\kappa_{\alpha,j} \otimes \kappa_{\beta,j} \otimes \kappa_{\gamma,i}, \tag{3.16}$$

$$\kappa_{(1,\alpha),i} \otimes \kappa_{(2,\beta),i} \otimes \kappa_{\gamma,i} = \varphi_\beta(\kappa_{\beta^{-1}\alpha\beta,i}) \otimes \kappa_{\beta,j} \otimes \kappa_{\gamma,i}\kappa_{\gamma,j}, \tag{3.17}$$

$$(\varphi_\beta \otimes \varphi_\beta)(R_{\alpha,\gamma}) = R_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}} \tag{3.18}$$

for all $\alpha, \beta, \gamma \in G$.

Remark 3.8 (1) We introduce the notation $\tilde{\kappa}_{\alpha,i} \otimes \tilde{\kappa}_{\beta,i} = \tilde{R}_{\alpha,\beta} = (R^{-1})_{\alpha,\beta}$.

(2) $R_{1,1}$ is an R-matrix for the Hom-Hopf algebra H_1 (see [8]).

Proposition 3.9 The Drinfeld double $D(H) = \{D_\alpha(H)\}_{\alpha \in G}$ has a quasitriangular structure given by

$$R_{\alpha,\beta} = (\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \kappa^{(\alpha^{-1},i)}) \in D_\alpha(H) \otimes D_\beta(H) \tag{3.19}$$

and

$$\bar{R}_{\alpha,\beta} = (S_\alpha(\kappa_{(\alpha,i)}) \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \kappa^{(\alpha,i)}) \in D_\alpha(H) \otimes D_\beta(H). \tag{3.20}$$

Proof Relation (3.14):

$$R_{\alpha,\beta}\Delta_{\alpha,\beta}(h) = (\tau \circ (\varphi_{\alpha^{-1}} \otimes id_\alpha) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})(h) R_{\alpha,\beta}.$$

Let $\alpha, \beta, \gamma \in G$. Given $h \in H_{\beta^{-1}\alpha^{-1}}$ and $f \in H^*_\gamma$, we have

$$\begin{aligned}
& R_{\alpha,\beta}\Delta_{\alpha,\beta}(h \circledast f) \\
= \ & ((\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \otimes \kappa^{(\alpha^{-1},i)}))((\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast f_1) \otimes (h_{(2,\beta^{-1})} \circledast f_2)) \\
= \ & (\xi^2_{\alpha^{-1}}((\kappa_{(\alpha^{-1},i)})_{12})\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast \varepsilon\langle f_{111}, \varphi_\alpha((\kappa_{(\alpha^{-1},i)})_{(11,\alpha^{-1}\gamma\alpha)})\rangle \\
& \langle f_{12}, S^{-1}_\gamma((\kappa_{(\alpha^{-1},i)})_{(2,\gamma^{-1})})\rangle \xi^{*-2}_\gamma(f_{112})) \otimes (\xi^2_{\beta^{-1}}(1_{\beta^{-1}})h_{(2,\beta^{-1})} \circledast \kappa^{(\alpha^{-1},i)}) \\
& \langle f_{211}, 1_\gamma\rangle\langle f_{22}, 1_\gamma\rangle \xi^{*-2}_\gamma(f_{212}) \\
= \ & (\xi^2_{\alpha^{-1}}((\kappa_{(\alpha^{-1},i)})_{12})\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast \langle f_{111}, \varphi_\alpha((\kappa_{(\alpha^{-1},i)})_{(11,\alpha^{-1}\gamma\alpha)})\rangle \\
& \langle f_{12}, S^{-1}_\gamma((\kappa_{(\alpha^{-1},i)})_{(2,\gamma^{-1})})\rangle \xi^{*-3}_\gamma(f_{112})) \otimes (\xi_{\beta^{-1}}(h_{2,\beta^{-1}}) \circledast \kappa^{(\alpha^{-1},i)} f_2)
\end{aligned}$$

and

$$\begin{aligned}
& ((\tau \circ (\varphi_{\alpha^{-1}} \otimes id_{D_\alpha(H)}) \circ \Delta_{\alpha\beta\alpha^{-1},\alpha})(h \circledast f))R_{\alpha,\beta} \\
= \ & (h_{(2,\alpha^{-1})} \circledast f_2)(\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes (h_{(1,\beta^{-1})} \circledast \varphi^*_\alpha(f_1))(1_{\beta^{-1}} \otimes \kappa^{(\alpha^{-1},i)}) \\
= \ & (\xi^2_{\alpha^{-1}}(h_{(212,\alpha^{-1})})\kappa_{\alpha^{-1},i} \circledast f_2\langle \varepsilon, S^{-1}_1(h_{(22,1)})\rangle\langle \varepsilon, \varphi_\alpha(h_{(211,1)})\rangle \xi^{*-2}_1(\varepsilon)) \\
& \otimes(\xi^2_{\beta^{-1}}(h_{(112,\beta^{-1})})1_{\beta^{-1}} \circledast \varphi^*_\alpha(f_1)\langle \kappa^{(\alpha^{-1},i)}_{11}, \varphi_\beta(h_{(111,\beta^{-1}\alpha^{-1}\beta)})\rangle \\
& \langle \kappa^{(\alpha^{-1},i)}_2, S^{-1}_{\alpha^{-1}}(h_{(12,\alpha)})\rangle \xi^{*-2}_{\alpha^{-1}}(\kappa^{(\alpha^{-1},i)}_{12})) \\
= \ & (h_{(2,\alpha^{-1})}\kappa_{(\alpha^{-1},i)} \circledast \xi^{*-1}_\gamma(f_2)) \otimes (\xi^3_{\beta^{-1}}(h_{(112,\beta^{-1})}) \circledast \langle \kappa^{(\alpha^{-1},i)}_{11}, \varphi_\beta(h_{(111,\beta^{-1}\alpha^{-1}\beta)})\rangle \\
& \langle \kappa^{(\alpha^{-1},i)}_2, S^{-1}_{\alpha^{-1}}(h_{(12,\alpha)})\rangle \varphi^*_\alpha(f_1)\xi^{*-2}_{\alpha^{-1}}(\kappa^{(\alpha^{-1},i)}_{12})).
\end{aligned}$$

Relation (3.14) is proved by observing that evaluating the two expressions above against the tensor $id_{\alpha^{-1}} \otimes id^*_\gamma \otimes id_{\beta^{-1}} \otimes \langle \cdot, x\rangle$ (for $x \in H_{\alpha^{-1}\gamma}$), we get the same result

$$\begin{aligned}
& (\xi^2_{\alpha^{-1}}((\kappa_{(\alpha^{-1},i)})_{12})\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast \langle f_{111}, \varphi_\alpha((\kappa_{(\alpha^{-1},i)})_{(11,\alpha^{-1}\gamma\alpha)})\rangle \\
& \langle f_{12}, S^{-1}_\gamma((\kappa_{(\alpha^{-1},i)})_{(2,\gamma^{-1})})\rangle \xi^{*-3}_\gamma(f_{112})) \otimes (\xi_{\beta^{-1}}(h_{(2,\beta^{-1})}) \circledast \langle \kappa^{(\alpha^{-1},i)} f_2, x\rangle) \\
= \ & (x_{(2,\alpha^{-1})}\varphi_\beta(h_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast \langle f_1, \varphi_\alpha(x_{(1,\alpha^{-1}\gamma\alpha)})\rangle \xi^{*-1}_\gamma(f_2)) \otimes \xi_{\beta^{-1}}(h_{(2,\beta^{-1})}) \\
= \ & (h_{(2,\alpha^{-1})}\kappa_{(\alpha^{-1},i)} \circledast \xi^{*-1}_\gamma(f_2)) \otimes (\xi^3_{\beta^{-1}}(h_{(112,\beta^{-1})}) \circledast \langle \kappa^{(\alpha^{-1},i)}_{11}, \varphi_\beta(h_{(111,\beta^{-1}\alpha^{-1}\beta)})\rangle \\
& \langle \kappa^{(\alpha^{-1},i)}_2, S^{-1}_{\alpha^{-1}}(h_{(12,\alpha)})\rangle\langle \varphi^*_\alpha(f_1)\xi^{*-2}_{\alpha^{-1}}(\kappa^{(\alpha^{-1},i)}_{12}), x\rangle),
\end{aligned}$$

where we used

$$\sum_i \langle f, \kappa_{(\alpha^{-1},i)}\rangle \kappa^{(\alpha^{-1},i)} = f \quad \text{and} \quad \sum_i \langle \kappa^{(\alpha^{-1},i)}, h\rangle \kappa_{(\alpha^{-1},i)} = h$$

for all $f \in H^*_{\alpha^{-1}}, h \in H_{\alpha^{-1}}$ and $\alpha \in G$.

Then we check Relation (3.16) and (3.17). The identities

$$\begin{aligned} &(\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \kappa_1^{(\alpha^{-1},i)}) \otimes (1_{\gamma^{-1}} \circledast \kappa_2^{(\alpha^{-1},i)}) \\ = \;& (\kappa_{(\alpha^{-1},i)}\kappa_{(\alpha^{-1},j)} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \kappa^{(\alpha^{-1},j)}) \otimes (1_{\gamma^{-1}} \circledast \kappa^{(\alpha^{-1},i)}) \end{aligned}$$

and

$$\begin{aligned} &(\varphi_\beta((\kappa_{((\alpha\beta)^{-1},i)})_{(1,\beta^{-1}\alpha^{-1}\beta)}) \circledast \varepsilon) \otimes ((\kappa_{((\alpha\beta)^{-1},i)})_{(2,\beta^{-1})} \circledast \varepsilon) \otimes (1_{\gamma^{-1}} \circledast \kappa^{((\alpha\beta)^{-1},i)}) \\ = \;& (\varphi_\beta(\kappa_{(\beta^{-1}\alpha^{-1}\beta,i)}) \circledast \varepsilon) \otimes (\kappa_{(\beta^{-1},j)} \circledast \varepsilon) \otimes (1_{\gamma^{-1}} \circledast \kappa^{(\beta^{-1}\alpha^{-1}\beta,i)}\kappa^{(\beta^{-1},j)}) \end{aligned}$$

can be written as (identifying $\bar{H}_\alpha \otimes \varepsilon$ with $\bar{H}_\alpha$ and $1_{\beta^{-1}} \otimes H^*$ with H^*)

$$\kappa_{(\alpha^{-1},i)} \otimes \kappa_1^{(\alpha^{-1},i)} \otimes \kappa_2^{(\alpha^{-1},i)} = \kappa_{(\alpha^{-1},i)}\kappa_{(\alpha^{-1},j)} \otimes \kappa^{(\alpha^{-1},j)} \otimes \kappa^{(\alpha^{-1},i)} \tag{3.21}$$

$$\begin{aligned} &(\kappa_{((\alpha\beta)^{-1},i)})_{(1,\beta^{-1}\alpha^{-1}\beta)} \otimes (\kappa_{((\alpha\beta)^{-1},i)})_{(2,\beta^{-1})} \otimes \kappa^{((\alpha\beta)^{-1},i)} \\ = \;& \kappa_{(\beta^{-1}\alpha^{-1}\beta,i)} \otimes \kappa_{(\beta^{-1},j)} \otimes \kappa^{(\beta^{-1}\alpha^{-1}\beta,i)}\kappa^{(\beta^{-1},j)}. \end{aligned} \tag{3.22}$$

The above equalities can be verified by evaluating both sides on element $f \in H^*_{\alpha^{-1}}$ in the first factor (respectively, on $h \in \overline{H}_{\alpha\beta}$ in the third factor) (see Zunino [9], Theorem 11).

Finally, let us check that

$$\begin{aligned} (\xi_{D_\alpha(H)} \otimes \xi_{D_\beta(H)})R_{\alpha,\beta} &= R_{\alpha,\beta}, \\ (\xi_{D_\alpha(H)} \otimes \xi_{D_\beta(H)})R_{\alpha,\beta} &= \xi_{D_\alpha(H)}(\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes \xi_{D_\beta(H)}(1_{\beta^{-1}} \circledast \kappa^{(\alpha^{-1},i)}) \\ &= (\xi_{\alpha^{-1}}(\kappa_{(\alpha^{-1},i)}) \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \xi^{*-1}_{\alpha^{-1}}(\kappa^{(\alpha^{-1},i)})). \end{aligned}$$

Now, $\xi_{\alpha^{-1}}$ is a linear isomorphism, so $(\xi_{\alpha^{-1}}(\kappa_{(\alpha^{-1},i)}))_{i=1,\cdots,n_\alpha}$ is a basis of $H_{\alpha^{-1}}$, and

$$(\xi^{*-1}_{\alpha^{-1}}(\kappa_{(\alpha^{-1},i)}))_{i=1,\cdots,n_\alpha}$$

is its dual basis. So $(\xi_{D_\alpha(H)} \otimes \xi_{D_\beta(H)})R_{\alpha,\beta} = R_{\alpha,\beta}$,

$$\begin{aligned} &(\xi_{\alpha^{-1}}(\kappa_{(\alpha^{-1},i)}) \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \langle \xi^{*-1}_{\alpha^{-1}}(\kappa^{(\alpha^{-1},i)}), x\rangle) \\ = \;& (\xi_{\alpha^{-1}}(\kappa_{(\alpha^{-1},i)}\langle \kappa^{(\alpha^{-1},i)}, \xi^{-1}_{\alpha^{-1}}(x)\rangle) \circledast \varepsilon) \otimes 1_{\beta^{-1}} \\ = \;& (x \circledast \varepsilon) \otimes 1_{\beta^{-1}} \\ = \;& (\kappa_{(\alpha^{-1},i)} \circledast \varepsilon) \otimes (1_{\beta^{-1}} \circledast \langle \kappa^{(\alpha^{-1},i)}, x\rangle) \end{aligned}$$

for $x \in H_{\alpha^{-1}}$.

This completes the proof.

References

[1] Turaev V, Homotopy field theory in dimension 3 and crossed group-categories[GT]. http://arxiv.org/abs/math/0005291.

[2] Freyd P, Yetter D, Braided compact closed categories with applications to low-dimensional topology[J]. Adv. Math., 1989, 77(2): 156–182.

[3] Zhou Xuan, Yang Tao, Kegel's theorem over weak Hopf group coalgebras[J]. J. Math., 2013, 33(2): 228–236.

[4] Yang Tao, Another construction of the braided T-category[RA]. http://arxiv.org/abs/1409.6936.

[5] Caenepeel S, Goyvaerts I, Monoidal Hom-Hopf algebras[J]. Commun. Alg., 2011, 39: 2216–2240.

[6] Virelizier A. Hopf group-coalgebra[J]. J. Pure Appl. Alg., 2002, 171: 75–122.

[7] Drinfeld V. Quantum groups[A]. Proceedings of the international congress of mathematicians[C]. (Berkeley, CA), Vol. 1, Providence, RI: Amer. Math. Soc., 1987: 789–820.

[8] Chen Yuanyuan, Wang Zhongwei, Zhang Liangyun. Quasitriangular Hom-Hopf algebra[J]. Colloq. Math., 2014, 137(1): 67–88.

[9] Zunino M, Double construction for crossed Hopf coalgebra[J]. J. Algebra., 2004, 278: 43–75.

Monoidal Hom-Hopf 群-余代数上的Drinfeld量子偶

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摘要: 本文研究了monoidal Hom-Hopf 群-余代数上的Drinfeld量子偶的问题. 利用交叉monoidal Hom-Hopf T-余代数的定义及拟三角monoidal Hom-Hopf 群-余代数的定义, 获得了此Drinfeld量子偶是拟三角monoidal Hom-Hopf 群-余代数的结果.

关键词: 拟三角; Monoidal Hom-Hopf 群-余代数; Drinfeld量子偶

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