Vol. 37 (2017)
No. 1

数 学 杂 志
J. of Math. (PRC)

MAJORIZATION OF THE GENERALIZED MARTIN FUNCTIONS FOR THE STATIONARY SCHRÖDINGER OPERATOR AT INFINITY IN A CONE

LONG Pin-hong, HAN Hui-li

(*School of Mathematics and Computer Science, Ningxia University, Yinchuan 750021, China*)

Abstract: In the paper, we mainly study Dirichlet problem for the stationary Schrödinger operator and the boundary behavior of Martin function. Depended on the generalized Martin representation and the fundamental system of solutions of an ordinary differential equation corresponding to stationary Schrödinger operator, we obtain some characterizations for the majorization of the generalized Martin functions associated with the stationary Schrödinger operator in a cone with smooth boundary, and generalize some classical results in Laplace setting.

Keywords: stationary Schrödinger operator; Martin function; harmonic majorization; minimally thin; cone

2010 MR Subject Classification: 31B05; 31B25; 31C35

Document code: A **Article ID:** 0255-7797(2017)01-0051-12

1 Introduction

Let $\mathbf{R}^n (n \geq 2)$ be the n-dimensional Euclidean space and $\mathbf{S}$ its an open set. The boundary and the closure of $\mathbf{S}$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively. In cartesian coordinate a point P is denoted by (X, x_n), where $X = (x_1, x_2, \cdots, x_{n-1})$. For P and Q in $\mathbf{R}^n$, let $|P|$ be the Euclidean norm of P and $|P-Q|$ the Euclidean distance. The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ be the open ball of radius r centered at P in $\mathbf{R}^n$, then $S_r = \partial B(O, r)$. Furthermore, denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r.

In the paper we are mainly concerned with some properties for the generalized Martin function associated with the stationary Schrödinger operator in a cone. Our aim is to give precise characterization for majorization of the generalized Martin functions in a cone. Deng et al. (see [17] and [23]) ever considered the growth for the potential functions in the half space. However, Miyamoto et al. (see [10, 11] and [12]) focused on the potential theories

* **Received date:** 2014-11-22 **Accepted date:** 2015-02-11

Foundation item: Supported by National Natural Science Foundation of China (11271045; 11261041); Natural Science Foundation of Ningxia University (NDZR1301); Startup Foundation for Doctor Scientific Research of Ningxia University.

Biography: Long Pinhong (1978–), male, born at Chenzhou, Hunan, lecturer, major in complex analysis and potential theory.

in a cone. Levin and Kheyfits (see [9]) paid attention to the problems associated with the stationary Schrödinger operator in a cone. In addition, Long and Qiao et al. (see [7, 8, 13–15] and [16]) considered some related problems about Dirichlet problem for the stationary Schrödinger operator at ∞ with respect to a cone as well as Levin and Kheyfits (see [9]). Based on the above statement, we will mainly generalize some results from Miyamoto and Yoshida (see [10]) to the stationary Schrödinger operator's setting. Unfortunately we don't have Riesz-Herglotz type theorem as the classical results which needed in the proof. To get over this difficulty, here we will depend on the generalized Martin representation theorem (see [8]). For the better statements about our results, we will introduce some notations and background materials below.

Relative to system of spherical coordinates, the Laplace operator Δ may be written by

$$\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Delta^*}{r^2},$$

where the explicit form of the Beltrami operator Δ^* is given by Azarin (see [1]).

For an arbitrary domain D in $\mathbf{R}^n$, $\mathcal{A}_D$ denotes the class of nonnegative radial potentials $a(P)$, i.e., $0 \le a(P) = a(r)$, $P = (r, \Theta) \in D$, such that $a \in L^b_{\mathrm{loc}}(D)$ with some $b > n/2$ if $n \ge 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

If $a \in \mathcal{A}_D$, then the stationary Schrödinger operator with a potential $a(\cdot)$

$$\mathcal{L}_a = -\Delta + a(\cdot)I$$

can be extended in the usual way from the space $C_0^\infty(D)$ to an essentially self-adjoint operator on $L^2(D)$, where Δ is the Laplace operator and I the identical operator(see [18, Chap.13]). Then $\mathcal{L}_a$ has a Green a-function $G_D^a(\cdot,\cdot)$. Here $G_D^a(\cdot,\cdot)$ is positive on D and its inner normal derivative $\partial G_D^a(\cdot,\cdot)/\partial n_Q$ is not negative, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into D. We write this derivative by $PI_D^a(\cdot,\cdot)$, which is called the Poisson a-kernel with respect to D. Denote by $G_D^0(\cdot,\cdot)$ the Green function of Laplacian.

For simplicity, a point $(1,\Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1,\Theta) \in \Omega\}$ for a set Ω ($\Omega \subset \mathbf{S}^{n-1}$) are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r,\Theta) \in \mathbf{R}^n; r \in \Xi, (1,\Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$. By $C_n(\Omega)$ we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$ and call it a cone. We mean the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}_+$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$, and $C_n(\Omega) \cap S_r$ by $C_n(\Omega; r)$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) \setminus \{O\}$. From now on, we always assume $D = C_n(\Omega)$ and write $G_\Omega^a(\cdot,\cdot)$ instead of $G_{C_n(\Omega)}^a(\cdot,\cdot)$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary and λ the least positive eigenvalue for $-\Delta^*$ on Ω (see [19, p. 41]),

$$\begin{aligned}&(\Delta^* + \lambda)\varphi(\Theta) = 0 \quad \text{on } \Omega,\\ &\varphi(\Theta) = 0 \quad \text{on } \partial\Omega.\end{aligned}$$

The corresponding eigenfunction is denoted by $\varphi(\Theta)$ satisfying $\int_\Omega \varphi^2(\Theta)dS_1 = 1$. In order to ensure the existence of λ and $\varphi(\Theta)$, we put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g., see [6, p. 88–89] for the definition of $C^{2,\alpha}$-domain).

Solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda}{r^2} + a(r)\right)Q(r) = 0 \text{ for } 0 < r < \infty \tag{1.1}$$

are known (see [22] for more references) that if the potential $a \in \mathcal{A}_D$. We know the equation (1.3) has a fundamental system of positive solutions $\{V, W\}$ such that V is nondecreasing with

$$0 \leq V(0+) \leq V(r) \text{ as } r \to +\infty \tag{1.2}$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \text{ as } r \to +\infty. \tag{1.3}$$

We remark that both $V(r)\varphi(\Theta)$ and $W(r)\varphi(\Theta)$ are a-harmonic on $C_n(\Omega)$ and vanish continuously on $S_n(\Omega)$.

We will also consider the class $\mathcal{B}_D$, consisting of the potentials $a \in \mathcal{A}_D$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = \kappa \in [0, \infty)$, moreover, $r^{-1}|r^2 a(r) - \kappa| \in L(1, \infty)$. If $a \in \mathcal{B}_D$, then the (super)subfunctions are continuous (e.g. see [20]). For simplicity, in the rest of paper we assume that $a \in \mathcal{B}_D$.

Denote

$$\iota_\kappa^\pm = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(\kappa + \lambda)}}{2},$$

then the solutions $V(r)$ and $W(r)$ to equation (1.1) normalized by $V(1) = W(1) = 1$ have the asymptotic (see [6])

$$V(r) \approx r^{\iota_\kappa^+}, \quad W(r) \approx r^{\iota_\kappa^-} \quad \text{as} \quad r \to \infty \tag{1.4}$$

and

$$\chi = \iota_\kappa^+ - \iota_\kappa^- = \sqrt{(n-2)^2 + 4(\kappa + \lambda)}, \quad \chi' = \omega(V(r), W(r))\,|_{r=1},$$

where χ' is their Wronskian at $r = 1$.

Remark 1 If $a = 0$ and $\Omega = \mathbf{S}_+^{n-1}$, then $\iota_0^+ = 1$, $\iota_0^- = 1-n$ and $\varphi(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

The function M_Ω^a defined on $C_n(\Omega) \times C_n(\Omega) \setminus \{(P_0, P_0)\}$ by

$$M_\Omega^a(P, Q) = \frac{G_\Omega^a(P, Q)}{G_\Omega^a(P_0, Q)}$$

is called the generalized Martin kernel of $C_n(\Omega)$ (relative to P_0). If $Q = P_0$, the above quotient is interpreted as 0 (for a=0, refer to Armitage and Gardiner [3]).

The rest of the paper is organized as follows: in Section 2, we shall give our main theorems; in Section 3, some necessary lemmas are given; in Section 4, we shall prove the main results.

2 Statement of Main Results

It is known that the Martin boundary $\triangle$ of $C_n(\Omega)$ is the set $\partial C_n(\Omega)\cup\{\infty\}$. When we denote the Martin kernel associated with the stationary Schrödinger operator by $M^a_\Omega(P,Q)$ $(P\in C_n(\Omega), Q\in\partial C_n(\Omega)\cup\{\infty\})$ with respect to a reference point chosen suitably, for any $P\in C_n(\Omega)$, we see

$$M^a_\Omega(P,\infty)=V(r)\varphi(\Theta),\ \ M^a_\Omega(P,O)=\kappa W(r)\varphi(\Theta), \tag{2.1}$$

where O denotes the origin of $\mathbf{R^n}$ and κ is a positive constant.

For a set $E\subset D$ and $\ell\in(0,1)$, put

$$E_\ell=\bigcup_{P\in E}B(P,\ell d(P)),$$

where $d(P)=\inf\limits_{Q\in D^c}\mid P-Q\mid$. Next we start to sate our main theorems.

Theorem 1 Let E be a set in $C_n(\Omega)$ satisfying $\overline{E}\cap\partial C_n(\Omega)=\phi$. If E_ℓ with a positive number $\ell(0<\ell<1)$ is a-minimally thin at ∞, then there exists a positive generalized harmonic function $u(P)$ on $C_n(\Omega)$ such that

$$\inf_{P\in C_n(\Omega)}\frac{u(P)}{M^a_\Omega(P,\infty)}<\inf_{P\in E}\frac{u(P)}{M^a_\Omega(P,\infty)}.$$

For $E\subseteq C_n(\Omega)$ and a fixed point $Q\in\partial C_n(\Omega)$, E is a-minimally thin at Q if and only if $\widehat{R}^E_{M^a_\Omega(\cdot,Q)}\neq M^a_\Omega(\cdot,Q)$, where $\widehat{R}^E_{M^a_\Omega(\cdot,Q)}$ is the regularized reduced function of $M^a_\Omega(\cdot,Q)$ relative to E and a superfunction on $C_n(\Omega)$ (refer to [8]).

Following the Armitage and Kuran (see [4]) as well as Miyamoto et al. (see [10]), we call that set $E\subset D$ characterizes the positive generalized harmonic majorization of $M^a_\Omega(\cdot,Q)$, if every positive generalized harmonic function υ in D which majorizes $M^a_\Omega(\cdot,Q)$ on E can majorize $M^a_\Omega(\cdot,Q)$ on D, that is to say

$$\inf_{P\in D}\frac{\upsilon(P)}{M^a_\Omega(P,Q)}=\inf_{P\in E}\frac{\upsilon(P)}{M^a_\Omega(P,Q)}.$$

Theorem 2 Let E be a subset $C_n(\Omega)$. The following conditions on E are equivalent:

(a) E characterizes the positive generalized harmonic majorization of $M^a_\Omega(P,\infty)$;

(b) for any $\ell\in(0,1)$, E_ℓ is not a-minimally thin at ∞;

(c) for some $\ell\in(0,1)$, E_ℓ is not a-minimally thin at ∞.

Theorem 3 Let E be a subset $C_n(\Omega)$. The following conditions on E are equivalent:

(a) E characterizes the positive generalized harmonic majorization of $M^a_\Omega(P,\infty)$;

(b) for any $\ell \in (0,1)$,

$$\int_{E_\ell} \frac{V(1+r)W(1+r)dP}{(1+r)^2} = +\infty;$$

(c) for some $\ell \in (0,1)$,

$$\int_{E_\ell} \frac{V(1+r)W(1+r)dP}{(1+r)^2} = +\infty.$$

A sequence $P_m \subset D$ is called to be separated if there exists a positive constant C such that

$$\mid P_i - P_j \mid \geq Cd(P_i) \quad (i,j = 1,2,\cdots, i \neq j)$$

(see [2]). With Theorem 3, we have the corollary as follows.

Corollary 1 Let $\{P_m\} \subset C_n(\Omega)$ be a separated sequence such that

$$\inf_m \mid P_m \mid > 0.$$

The sequence $\{P_m\}$ characterizes the positive generalized harmonic majorization of $M^a_\Omega(P,\infty)$ if and only if

$$\sum_{m=1}^{\infty} \frac{d(P_m)^n V(\mid P_m \mid)W(\mid P_m \mid)}{\mid P_m \mid^2} = +\infty.$$

Remark 2 When $a = 0$, the theorems and corollary above are due to Miyamoto et al. (see [10]). If $a = 0$ and $\Omega = \mathbf{S}^{n-1}_+$, Theorem 1, Theorem 2 and Theorem 3 are from the Dahlberg's results in upper-half space or Liapunov-Dini domain in $\mathbf{R}^n$ (see [5]), and Corollary 1 results from Armitage and Kuran (see [4]).

3 Some Lemmas

For our arguments we collect the following results.

Lemma 1 (see [13])

$$PI^a_\Omega(P,Q) \approx t^{-1}V(t)W(r)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \tag{3.1}$$

$$(\text{resp. } PI^a_\Omega(P,Q) \approx V(r)t^{-1}W(t)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}) \tag{3.2}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < \frac{t}{r} \leq \frac{1}{2}$ (resp. $0 < \frac{r}{t} \leq \frac{1}{2}$),

$$PI^0_\Omega(P,Q) \lesssim \frac{\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} + \frac{r\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} \tag{3.3}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega;(\frac{1}{2}r,2r))$.

Lemma 2 (see [13])

$$G^a_\Omega(P,Q) \approx V(t)W(r)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \tag{3.4}$$

$$(\text{resp. } G^a_\Omega(P,Q) \approx V(r)W(t)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}) \tag{3.5}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega)$ satisfying $0<\frac{t}{r}\leq\frac{1}{2}$ (resp. $0<\frac{r}{t}\leq\frac{1}{2}$);

$$G^0_\Omega(P,Q)\lesssim\frac{\varphi(\Theta)}{t^{n-2}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}+\frac{rt\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} \tag{3.6}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega;(\frac{1}{2}r,2r))$.

Lemma 3 (The generalized Martin representation, see [7]) If u is a positive a-harmonic function on $C_n(\Omega)$, then there exists a measure μ_u on $\triangle$, uniquely determined by u, such that

$$u(P)=\int_\triangle M^a_\Omega(P,Q)d\mu_u(Q)\quad(P\in C_n(\Omega)),$$

where $\triangle$ is the Martin boundary of $C_n(\Omega)$.

It is well-known that a cube is of the form

$$[\ell_1 2^{-k},(\ell_1+1)2^{-k}]\times\cdots\times[\ell_n 2^{-k},(\ell_n+1)2^{-k}],$$

where $k,\ell_1,\cdots,\ell_n$ are integers. Now we introduce a family of so-called Whitney cubes of $C_n(\Omega)$ having the following properties:

(a) $\cup_j W_j=C_n(\Omega)$;

(b) $\mathrm{int}W_j\cap\mathrm{int}W_k=\emptyset(j\neq k)$;

(c) $\mathrm{diam}W_j\leq\mathrm{dist}(W_j,\mathbf{R}^n\setminus C_n(\Omega))\leq 4\mathrm{diam}W_j$,

where $\mathrm{int}S$, $\mathrm{diam}S$ and $\mathrm{dist}(S_1,S_2)$ stand for the interior of S, the diameter of S and the distance between S_1 and S_2, respectively (see [21], P.167, Theorem 1).

Lemma 4 (see [10]) Let $\{W_i\}_{i\geq1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with ℓ. Let E be a subset of $C_n(\Omega)$. Then there exists a subsequence $\{W_{i_m}\}_{i\geq1}$ of $\{W_i\}_{i\geq1}$ such that

(a) $\cup_m W_{i_m}\subset E_\ell$;

(b) $W_{i_m}\cap E_{\ell/4}\neq\emptyset\ (m=1,2,,\cdots)$, $E_{\ell/4}\subset\cup_m W_{i_m}$.

Lemma 5 (see [8]) Let a Borel subset E of $C_n(\Omega)$ be a-minimally thin at ∞ with respect to $C_n(\Omega)$. Then we see that

$$\int_E V(1+\mid P\mid)W(1+\mid P\mid)(1+\mid P\mid)^{-2}dP<\infty. \tag{3.7}$$

If E is a union of cubes from the Whitney cubes of $C_n(\Omega)$, then (3.7) is also sufficient for E to be a-minimally thin at ∞ with respect to $C_n(\Omega)$.

4 Proofs of Main Theorems

Proof of Theorem 1 When E is a bounded subset of $C_n(\Omega)$, we may assume that $u(P)$ is a constant function. Otherwise we will follow the same method as Dahlberg to make the required function.

Set $\ell\in(0,1)$. We assume that $\{P_m\}$ is a sequence of points P_m which are central points of cubes W_{i_m} in Lemma 4. From the assumption on E, it follows that $\{P_m\}$ can not

converge to any boundary point of $C_n(\Omega)$. Since $\{P_m\} \in E_\ell$ due to Lemma 4, we see that $\mid P_m \mid \to +\infty (m \to +\infty)$. Because E_ℓ is a-minimally thin at ∞ and

$$\int_{W_{im}} \frac{V(1+r)W(1+r)dP}{(1+r)^2} \approx \frac{d(P_m)^n V(\mid P_m \mid)W(\mid P_m \mid)}{\mid P_m \mid^2} \quad (m = 1, 2, \cdots), \tag{4.1}$$

we get by Lemma 4 and Lemma 5 that

$$\sum_{m=1}^{\infty} \frac{d(P_m)^n V(\mid P_m \mid)W(\mid P_m \mid)}{\mid P_m \mid^2} < \infty.$$

Hence from (1.2)–(1.4) we can take a positive integer N such that $d(P_m) \leq \frac{1}{N} \mid P_m \mid$ for each $m \geq N$.

Choose a point $Q_m = (t_m, \Phi_m) \in \partial C_n(\Omega) \setminus \{O\}$ such that

$$\mid P_m - Q_m \mid = d(P_m) \quad (m = N, N+1, \cdots).$$

Then we see that $\mid Q_m \mid \geq \frac{N-1}{N} \mid P_m \mid$ and hence $\mid Q_m \mid \to +\infty$ $(m \to +\infty)$. Define $h_1(P)$ as follow:

$$h_1(P) = \sum_{m=N}^{\infty} PI_\Omega^a(P, Q_m) \frac{d(P_m)^n V(\mid P_m \mid)}{\mid P_m \mid} \quad (P \in C_n(\Omega)),$$

then h_1 is well defined, and hence is a positive generalized harmonic function on $C_n(\Omega)$ which is due to Lemma 4.

First we will prove that

$$\inf_{P \in E} \frac{h_1(P)}{M_\Omega^a(P, \infty)} > 0. \tag{4.2}$$

Denote the Possion Kernel of the ball $B_m = B(P_m, d(P_m)))$ by $PI_{B_m}(P, Q)$ for $P \in B_m$ and $Q \in \partial B_m$. Since $PI_\Omega^a(P, Q_m) \approx PI_\Omega^0(P, Q_m)$ (see [13]), we have

$$PI_\Omega^a(P, Q_m) \gtrsim PI_{B_m}(P, Q_m) \quad (P \in B_m; m = N, N+1, \cdots)$$

and hence

$$PI_\Omega^a(P_m, Q_m) \gtrsim PI_{B_m}(P_m, Q_m) = s_n^{-1} d(P_m)^{1-n} \quad (m = N, N+1, \cdots).$$

Because

$$\varphi(\Phi) \approx d(P') \quad (P' = (1, \Phi), \Phi \in \Omega),$$

we get that

$$h_1(P_m) \geq PI_\Omega^a(P_m, Q_m) \frac{d(P_m)^n V(\mid P_m \mid)}{\mid P_m \mid} \gtrsim M_\Omega^a(P_m, \infty) \quad (m = N, N+1, \cdots). \tag{4.3}$$

For any $P \in E$,then exists a point P_m such that

$$\mid P - P_m \mid < \frac{\mathrm{diam}(W_{i_m})}{2} \lesssim \delta d(P) \leqslant \frac{\delta}{2} \mid P_m \mid .$$

When $2r \le t$ or $r \ge 2t$ ($2 \mid P_m \mid \le t$ or $\mid P_m \mid \ge 2t$), by Lemma 2 and (1.2)–(1.4) we obtain that

$$\frac{G^a_\Omega(P,Q)}{G^a_\Omega(P_m,Q)} \ge C.$$

Since

$$\ell^{n-2}G^0_\Omega(\ell P,\ell Q) = G^0_\Omega(P,Q) \quad (P,Q \in C_n(\Omega))$$

and

$$G^a_\Omega(P,Q) \approx G^0_\Omega(P,Q) \quad (P,Q \in C_n(\Omega))$$

(refer to [10] and [13]), we know that

$$\frac{G^a_\Omega(P,Q)}{G^a_\Omega(P_m,Q)} \succsim 1$$

for $\frac{t}{2} \le r$ and $\mid P_m \mid \le 2t$. From Lemma 3 (the generalized Martin representation), we may obtain that

$$h_1(P) \gtrsim h_1(P_m). \tag{4.4}$$

By (2.1) and (1.2)–(1.4) we also see that

$$M^a_\Omega(P,\infty) \lesssim M^a_\Omega(P_m,\infty). \tag{4.5}$$

With (4.4)–(4.5) and (4.3) we see that (4.2) holds.

Now, for a fixed ray L which is in $C_n(\Omega)$ and starts from O, we will show

$$\lim_{|P|\to\infty, P\in L} \frac{h_1(P)}{M^a_\Omega(P,\infty)} = 0. \tag{4.6}$$

Set

$$H_m(P) = \frac{PI^a_\Omega(P,Q_m) \mid P_m \mid}{M^a_\Omega(P,\infty)W(\mid P_m \mid)} \quad (P \in C_n(\Omega); m = N, N+1, \cdots).$$

Then we have that

$$\frac{h_1(P)}{M^a_\Omega(P,\infty)} = \sum_{m=N}^{\infty} H_m(P)\frac{d(P_m)^n V(\mid P_m \mid)W(\mid P_m \mid)}{\mid P_m \mid^2}.$$

By Lemma 1 we see that

$$\lim_{|P|\to\infty, P\in L} H_m(P) = 0$$

for any fixed $m \ge N$. Hence, if we can show that

$$\mid H_m(P) \mid \le C \quad (P \in L; m = N, N+1, \cdots) \tag{4.7}$$

for some constant C independent of m, then we will get (4.6) from (4.1) and Lebesgue's dominated convergence theorem.

To prove (4.7), we divide the proof into three cases. When $2r \leq t_m$ or $r \geq 2t_m$, by Lemma 1 we see that

$$| H_m(P) | \leq C \quad (P = (r, \Theta) \in C_n(\Omega); m = N, N+1, \cdots).$$

Finally, when $\frac{t_m}{2} \leq r \leq 2t_m$, we have

$$t_m^{n-1} PI_\Omega^0(P, Q_m) \leq C' \quad (P = (r, \Theta) \in L; m = N, N+1, \cdots)$$

for some constant C' (refer to [10, p.1051]). Since $PI_\Omega^a(P, Q_m) \approx PI_\Omega^0(P, Q_m)$, by (1.2)–(1.4) we have

$$t_m PI_\Omega^a(P, Q_m) \leq C'' V(t_m) W(t_m) \quad (P = (r, \Theta) \in L; m = N, N+1, \cdots).$$

So

$$| H_m(P) | \leq C \quad (P = (r, \Theta) \in L; m = N, N+1, \cdots).$$

At last, we put $\Upsilon = \max\limits_{1 \leq m \leq N} M_\Omega^a(P_m, \infty)$ and $h(P) = H_1(P) + \Upsilon$ for any $P \in C_n(\Omega)$. Then we easily get from (4.2) and (4.6) that $h(P)$ is a positive generalized harmonic function on $C_n(\Omega)$ which is required in Theorem 1.

Proof of Theorem 2 (a)⇒(b). Let C be a positive constant and set $E_C = \{P \in E : M_\Omega^a(P, \infty) \geq C\}$. Then E_C satisfies that $\overline{E_C} \cap \partial C_n(\Omega) = \emptyset$. Since E characterizes the positive generalized harmonic majorization of $M_\Omega^a(P, \infty)$, E_C also characterizes the positive generalized harmonic majorization of $M_\Omega^a(P, \infty)$. Otherwise, there would exists a positive generalized harmonic function $\upsilon(P)$ on $C_n(\Omega)$ satisfying

$$A = \inf_{P \in C_n(\Omega)} \frac{\upsilon(P)}{M_\Omega^a(P, \infty)} < \inf_{P \in E_C} \frac{\upsilon(P)}{M_\Omega^a(P, \infty)} = B.$$

Let $u(P) = \upsilon(P) + BC$ for any $P \in C_n(\Omega)$. Then $u(P) \geq BM_\Omega^a(P, \infty)$ for $P \in E$, and so

$$\inf_{P \in C_n(\Omega)} \frac{u(P)}{M_\Omega^a(P, \infty)} = A < B \leq \inf_{P \in E_C} \frac{u(P)}{M_\Omega^a(P, \infty)},$$

which contradicts (a).

If we can show that $(E_C)_\ell$ is not a-minimally thin at infinity when $\ell \in (0, 1)$, then for all $\ell \in (0, 1)$ the set E_ℓ also is not a-minimally thin at infinity, and hence (b) holds.

Suppose that for some $\ell \in (0, 1)$ the set $(E_C)_\ell$ is a-minimally thin at infinity. Then from Theorem 1 there exists a positive generalized harmonic function $\upsilon(P)$ on $C_n(\Omega)$ satisfying

$$\inf_{P \in C_n(\Omega)} \frac{\upsilon(P)}{M_\Omega^a(P, \infty)} < \inf_{P \in E_C} \frac{\upsilon(P)}{M_\Omega^a(P, \infty)}.$$

We see that E_C characterizes the positive generalized harmonic majorization of $M_\Omega^a(P, \infty)$, so for all $\ell \in (0, 1)$ the set $(E_C)_\ell$ is not a-minimally thin at infinity.

(c)⇒(a). Suppose that E does not characterize the positive generalized harmonic majorization of $M_\Omega^a(P,\infty)$. Then there exists a positive generalized harmonic function $\upsilon(P)$ on $C_n(\Omega)$ such that

$$A = \inf_{P\in C_n(\Omega)} \frac{\upsilon(P)}{M_\Omega^a(P,\infty)} < \inf_{P\in E} \frac{\upsilon(P)}{M_\Omega^a(P,\infty)} = B.$$

Put $h(P) = \upsilon(P) - AM_\Omega^a(P,\infty)$ for any $P \in C_n(\Omega)$. Then $h(P)$ is a positive generalized harmonic function on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)} \frac{h(P)}{M_\Omega^a(P,\infty)} = 0. \tag{4.8}$$

For any $P \in E_\ell(\ell \in (0,1)$ there exists a point P' such that $\mid P - P' \mid < \ell d(P')$, and by the generalized Martin representation and the same proof as Theorem 1 we see that

$$\inf_{P\in E_\ell} \frac{h(P)}{M_\Omega^a(P,\infty)} \gtrsim \inf_{P\in E} \frac{h(P)}{M_\Omega^a(P,\infty)} > 0. \tag{4.9}$$

From (4.8) and (4.9) we obtain that

$$\inf_{P\in C_n(\Omega)} \frac{h(P)}{M_\Omega^a(P,\infty)} \leq \inf_{P\in E_\ell} \frac{h(P)}{M_\Omega^a(P,\infty)}$$

for the positive supfunction $h(P)$ on $C_n(\Omega)$. It follows that E_ℓ is a-minimally thin at infinity. This contradicts (c).

Proof of Theorem 3 (a)⇒(b). Assume that

$$\int_{E_\ell} \frac{V(1+r)W(1+r)dP}{(1+r)^2} < \infty$$

for some $\ell \in (0,1)$. Let $\{W_{i_m}\}_{m\geq 1}$ be a subsequence of $\{W_i\}_{i\geq 1}$ from Lemma 4. With (a) of Lemma 4 we obtain

$$\int_{\cup_m W_{im}} \frac{V(1+r)W(1+r)dP}{(1+r)^2} < \infty.$$

Since $\cup_m W_{i_m}$ is a union of cubes from the Whitney cubes of $C_n(\Omega)$ with ℓ, by Lemma 5 we see that $\cup_m W_{i_m}$ is a-minimally thin at infinity. Further, from Lemma 4 we know that $E_{\frac{\ell}{4}}$ is a-minimally thin at infinity.

On the other hand, since E characterizes the positive generalized harmonic majorization of $M_\Omega^a(P,\infty)$, we see from the Theorem 2 that $E_{\frac{\ell}{4}}$ is not a-minimally thin at infinity, which contradicts the conclusion above.

(c)⇒(a). Suppose that E does not characterize the positive generalized harmonic majorization of $M_\Omega^a(P,\infty)$. Then it follows from Theorem 2 that for any $\ell \in (0,1)E_\ell$ is a-minimally thin at infinity. So we see from Lemma 5 that for any $\ell \in (0,1)$

$$\int_{E_\ell} \frac{V(1+r)W(1+r)dP}{(1+r)^2} < \infty,$$

which contradicts (c).

Proof of Corollary 1 If $\{P_m\}$ is a separated sequence,then

$$B(P_i, \ell d(P_i)) \cap B(P_j, \ell d(P_j)) = \emptyset \ \ (i, j = 1, 2, \cdots; i \neq j)$$

for a sufficiently small $\ell \in (0, 1)$, and hence

$$\int_{E_\ell} \frac{V(1+r)W(1+r)dP}{(1+r)^2} \approx \sum_{m=1}^{\infty} \frac{d(P_m)^n V(\mid P_m \mid)W(\mid P_m \mid)}{\mid P_m \mid^2}.$$

Following (c) of Theorem 3, Corollary 1 immediately holds.

References

[1] Azarin V S. Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone[J]. Trans. Amer. Math. Soc., 1969, 80(2): 119–138.

[2] Aikawa H, Essén M. Potential theory–selected topics[M]. Lect. Notes Math., 1633, Berlin: Springer-Verlag, 1996.

[3] Armitage D H, Gardiner S J. Classical potential theory[M]. London: Springer-verlag, 2001.

[4] Armitage D H, Kuran Ü. On positive harmonic majorization of y in $\mathbf{R}^n \times (0, \infty)$[J]. J. London Math. Soc. Ser. II, 1971, 3: 733–741.

[5] Dalberg B E J. A minimum principle for positive harmonic functions[J]. Proc. London Math. Soc., 1976, 33(2): 238–250.

[6] Gilbarg D, Trudinger N S. Elliptic partial differential equations of second order[M]. Berlin: Springer Verlag, 1977.

[7] Long Pinhong, Gao Zhiqiang, Deng Guantie. Criteria of Wiener type for minimally thin sets and rarefied sets associated with the stationary Schrödinger operator in a cone[J]. Abstr. Appl. Anal., DOI:10.1155/2012/453891.

[8] Long Pinhong. Characterizations for exceptional sets and growth problems in classical or nonlinear potential theory[D]. Beijng: Beijing Normal University, 2013.

[9] Levin B, Kheyfits A. Asymptotic behavior of subfunctions of the stationary Schrödinger operator[J]. http://arxiv.org/abs/math/0211328v1, 2002, 96 pp.

[10] Miyamoto I, Yanagishita M, Yoshida H. On harmonic majorization of the Martin function at infnity in a cone[J]. Czech. Math. J., 2005, 55(130): 1041–1054.

[11] Miyamoto I, Yoshida H. Two criterions of Wiener type for minimally thin sets and rarefied sets in a cone[J]. J. Math. Soc. Japan., 2002, 54(3): 487–512.

[12] Miyamoto I, Yoshida H. On a-minimally thin sets at infinity in a cone[J]. Hiroshima Math. J., 2007, 37(1): 61–80.

[13] Qiao Lei. Some researches on(generalized) harmonic and superharmonic functions[D]. Beijng: Beijing Normal University, 2010.

[14] Qiao Lei, Deng Guantie. A theorem of Phragmén Lindelöf type for subfunctions in a cone[J]. Glasg. Math. J., 2011, 53(3): 599–610.

[15] Qiao Lei, Deng Guantie. The Riesz decomposition theorem for superharmonic functions in a cone and its application (in Chinese)[J]. Scientia Sinica (Math.), 2012, 42(8): 763–774.

[16] Qiao Lei, Deng Guantie. Integral representation for the solution of the stationary Schrodinger equation in a cone[J]. Math. Nachr., 2012, 285(16): 2029–2038.

[17] Qiao Lei, Deng GuanTie, Pan GuoShuang. Exceptional sets of modified Poisson integral and Green potential in the upper-half space[J]. Scientia Sinica (Math.), 2010, 40(8): 787–800.

[18] Reed M, Simon B. Methods of modern mathematical physics[M]. Vol. 3, London, New York, San Francisco: Acad. Press, 1970.

[19] Rosenblum G V, Solomyak M Z, Shubin M A. Spectral theory of differential operators[M]. Moscow: VINITI, 1989.

[20] Simon B, Schrödinger semigroups[J]. Bull. Amer. Math. Soc., 1982, 7: 447–526.

[21] Stein E M. Singular integrals and differentiability properties of functions[M]. New Jersey: Princeton University Press, 1970.

[22] Verzhbinskii G M, Maz'ya V G. Asymptotic behavior of solutions of elliptic equations of the second order close to a boundary. I[J]. Sibirsk. Matem. Zh., 1971, 12(6): 1217–1249.

[23] Zhang Yanhui, Deng Guantie. Growth properties of generalized Poisson integral in the half space[J]. J. Math., 2013, 33(3): 473–478.

锥中与稳态的薛定谔算子相关的广义Martin函数无穷远处的控制

龙品红, 韩惠丽

(宁夏大学数学计算机学院, 宁夏 银川 750021)

摘要: 本文研究了稳态的薛定谔算子的Dirichlet 问题和Martin 函数的边界行为. 利用广义Martin 表示和稳态的薛定谔算子对应的常微分方程基本解, 在具有光滑边界的锥形区域中获得了与稳态的薛定谔算子相关的广义Martin 函数无穷远处广义调和控制的一些刻画, 推广了拉普拉斯算子情形的结果.

关键词: 稳态的薛定谔算子; Martin函数; 调和控制; 极细; 锥

MR(2010)**主题分类号**: 31B05; 31B25; 31C35 **中图分类号**: O174.3