

A NOTE ON HOMOGENIZATION OF THE HYPERBOLIC PROBLEMS WITH IMPERFECT INTERFACES

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Abstract: In this paper, we are concerned with a class of hyperbolic problems with non-periodic coefficients in two-component domains. By the periodic unfolding method, we derive the homogenization and corrector results, which generalize those achieved by Donato, Faella and Monsurro.

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1 Introduction

In this paper, we study the homogenization and corrector results for the following hyperbolic problem with $-1 < \gamma < 1$,

$$\left\{ \begin{array}{ll} u_{1\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} & \text{in } \Omega_{1\varepsilon} \times (0, T), \\ u_{2\varepsilon}'' - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times (0, T), \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times (0, T), \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times (0, T), \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times (0, T), \\ u_{1\varepsilon}(x, 0) = U_{1\varepsilon}^0(x), \quad u_{1\varepsilon}'(x, 0) = U_{1\varepsilon}^1(x) & \text{in } \Omega_{1\varepsilon}, \\ u_{2\varepsilon}(x, 0) = U_{2\varepsilon}^0(x), \quad u_{2\varepsilon}'(x, 0) = U_{2\varepsilon}^1(x) & \text{in } \Omega_{2\varepsilon}, \end{array} \right. \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is the union of two ε -periodic sub-domains $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$, separated by an interface Γ^ε , such that $\Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}} = \Omega$ and $\Gamma^\varepsilon = \partial\Omega_{2\varepsilon}$. Here, $\Omega_{1\varepsilon}$ is connected and the number of connected components of $\Omega_{2\varepsilon}$ is of order ε^{-n} . This problem models the wave propagation

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in a medium made up of two materials with different coefficients of propagation. For the physical model, we refer the reader to Carslaw and Jaeger [1].

Let $Y = [0, l_1) \times \cdots \times [0, l_n)$ be the reference cell with $l_i > 0$, $i = 1, \dots, n$. We suppose that Y_1 and Y_2 are two nonempty open disjoint subsets of Y such that $Y = Y_1 \cup \overline{Y_2}$, where Y_1 is connected and $\Gamma = \partial Y_2$ is Lipschitz continuous. Throughout this paper, we have the following assumptions.

- For any ε , $A^\varepsilon(x) = (a_{ij}^\varepsilon(x))_{1 \leq i, j \leq n}$ is a matrix satisfying the following:

A^ε is symmetric and there exist $\alpha, \beta \in \mathbb{R}^+$ ($0 < \alpha < \beta$) such that $(A^\varepsilon \lambda, \lambda) \geq \alpha |\lambda|^2$, $|A^\varepsilon \lambda| \leq \beta |\lambda|$ for all $\lambda \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

- For any ε , $h^\varepsilon(x) = h(x/\varepsilon)$, where h is a Y -periodic function such that $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ such that $0 < h_0 < h(y)$ a.e. on Γ .

- The initial data satisfy the assumptions:

$$U_\varepsilon^0 := (U_{1\varepsilon}^0, U_{2\varepsilon}^0) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon}), \quad U_\varepsilon^1 := (U_{1\varepsilon}^1, U_{2\varepsilon}^1) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon})$$

and

$$f_\varepsilon := (f_{1\varepsilon}, f_{2\varepsilon}) \in L^2(0, T; L^2(\Omega_{1\varepsilon})) \times L^2(0, T; L^2(\Omega_{2\varepsilon})).$$

For the classical case $A^\varepsilon(x) = A(x/\varepsilon)$ with A being periodic, symmetric, bounded and uniformly elliptic, Donato, Faella and Monsurrò gave the homogenization for $\gamma \leq 1$ in [2]. Later, they obtained the corrector results in [3] for $-1 < \gamma \leq 1$. Their proofs are based on the oscillating test functions method. In [4], the first author gave the corrector results for $\gamma < -1$ by the unfolding method. However, the above methods do not work for the case that $A^\varepsilon(x)$ is non-periodic coefficient matrix.

In this paper, we will consider problem (1.1) with $A^\varepsilon(x)$ being non-periodic for $-1 < \gamma < 1$. More precisely, suppose that there exists a matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ such that

$$\mathcal{T}_\varepsilon(A^\varepsilon) \rightarrow A \text{ strongly in } (L^1(\Omega \times Y))^{n \times n}, \quad (1.2)$$

where $\mathcal{T}_\varepsilon$ is the unfolding operator. By the unfolding method, we derive the homogenization and corrector results for $-1 < \gamma < 1$. Next, we state our main theorems, in which we will use some notations to be defined in the next section. We first state the homogenization results whose unfolded formulation will be provided for the study of correctors in Section 3.

Theorem 1.1 For $-1 < \gamma < 1$, let u_ε be the solution of problem (1.1) with (1.2). We further suppose that

$$\begin{aligned} \|U_\varepsilon^0\|_{H_\gamma^\varepsilon} &\text{ is uniformly bounded,} \\ \widetilde{U}_\varepsilon^0 &\rightharpoonup (\theta_1 U_1^0, \theta_2 U_2^0) \text{ weakly in } L^2(\Omega) \times L^2(\Omega), \text{ where } U_2^0 \in H_0^1(\Omega), \\ \widetilde{U}_\varepsilon^1 &\rightharpoonup (\theta_1 U_1^1, \theta_2 U_2^1) \text{ weakly in } L^2(\Omega) \times L^2(\Omega), \\ \widetilde{f}_\varepsilon &\rightharpoonup (\theta_1 f_1, \theta_2 f_2) \text{ weakly in } L^2(0, T, L^2(\Omega)) \times L^2(0, T, L^2(\Omega)). \end{aligned} \quad (1.3)$$

Then there exists $u_1 \in L^\infty(0, T; H_0^1(\Omega))$ such that

$$\widetilde{u}_{i\varepsilon} \rightharpoonup \theta_i u_1 \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \quad i = 1, 2.$$

Also, u_1 is the unique solution of the following problem:

$$\begin{cases} u_1'' - \operatorname{div}(A^0 \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 & \text{in } \Omega \times (0, T), \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T), \\ u_1(x, 0) = \theta_1 U_1^0 + \theta_2 U_2^0 & \text{in } \Omega, \\ u_1'(x, 0) = \theta_1 U_1^1 + \theta_2 U_2^1 & \text{in } \Omega, \end{cases} \quad (1.4)$$

where the homogenized matrix $A^0 = (a_{ij}^0(x))_{1 \leq i, j \leq n}$ is defined by

$$a_{ij}^0(x) = \theta_1 \mathcal{M}_{Y_1} \left(a_{ij} + \sum_{k=1}^n a_{ik} \frac{\partial \chi_j}{\partial y_k} \right), \quad (1.5)$$

and $\chi_j \in L^\infty(\Omega; H_{\text{per}}^1(Y_1))$ ($j = 1, \dots, n$) is the solution of the following cell problem:

$$\begin{cases} -\operatorname{div}(A \nabla(\chi_j + y_j)) = 0 & \text{in } Y_1, \\ A \nabla(\chi_j + y_j) \cdot n_1 = 0 & \text{on } \Gamma, \\ \mathcal{M}_{Y_1}(\chi_j) = 0, & \chi_j \text{ is } Y\text{-periodic.} \end{cases} \quad (1.6)$$

Further, we have the following precise convergence of flux:

$$\begin{aligned} A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} &\rightharpoonup A^0 \nabla u_1 \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} &\rightharpoonup 0 \quad \text{weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (1.7)$$

Notice that the homogenized matrix A^0 still depends on x , compared with the classical constant matrix (see for instance [2, 4]).

In order to investigate the corrector results, we need stronger assumptions on the initial data than that of the convergence results, as already evidenced in the classical works (see, for instance, [3, 5]). Here we impose some assumptions, introduced by the first author (see [4] for more details), which are slightly weaker than those in [3]. Now we list them as follows.

(i) For $f_{i\varepsilon} \in L^2(0, T; L^2(\Omega_{i\varepsilon}))$ ($i = 1, 2$), there exists f_i in $L^2(0, T; L^2(\Omega))$ such that

$$\|f_{i\varepsilon} - f_i\|_{L^2(0, T; L^2(\Omega_{i\varepsilon}))} \rightarrow 0 \quad \text{for } i = 1, 2. \quad (1.8)$$

(ii) For $U_{i\varepsilon}^1 \in L^2(\Omega_{1\varepsilon})$ ($i = 1, 2$), there exists $U^1 \in L^2(\Omega)$ such that

$$\|U_{i\varepsilon}^1 - U^1\|_{L^2(\Omega_{i\varepsilon})} \rightarrow 0. \quad (1.9)$$

(iii) For U_ε^0 , we assume that

$$\begin{cases} \text{(i)} \quad \|U_\varepsilon^0\|_{H_\gamma^\varepsilon} \text{ is uniformly bounded,} \\ \text{(ii)} \quad \widetilde{U_{1\varepsilon}^0} \rightharpoonup \theta_i U^0 \quad \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ \text{(iii)} \quad \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} dx + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} dx \\ \quad + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})^2 d\sigma_x \longrightarrow \int_{\Omega} A^0 \nabla U^0 \nabla U^0 dx, \end{cases} \quad (1.10)$$

where U^0 is given in $H_0^1(\Omega)$.

These assumptions ensure the convergence of the energy in $\mathcal{C}^0([0, T])$. Moreover, we obtain the following corrector results.

Theorem 1.2 For $-1 < \gamma < 1$, let u_ε be the solution of problem (1.1) with (1.2). Suppose that the initial data satisfy (1.8)–(1.10). Let u_1 be the solution of the homogenized problem (1.4), then we have the following corrector results:

$$\begin{aligned} \|\widetilde{u'_{1\varepsilon}} + \widetilde{u'_{2\varepsilon}} - u'_1\|_{L^2(0, T; L^2(\Omega))} &\rightarrow 0, \\ \|\nabla u_{1\varepsilon} - \nabla u_1 - \sum_{i=1}^n \mathcal{U}_1^\varepsilon\left(\frac{\partial u_1}{\partial x_i}\right) \mathcal{U}_1^\varepsilon(\nabla_y \chi_i)\|_{L^2(0, T; L^2(\Omega_{1\varepsilon}))} &\rightarrow 0, \\ \|\nabla u_{2\varepsilon}\|_{L^2(0, T; L^2(\Omega_{2\varepsilon}))} &\rightarrow 0, \end{aligned} \quad (1.11)$$

where $\chi_j \in L^\infty(\Omega; H_{\text{per}}^1(Y))$ ($j = 1, \dots, n$) is the solution of the cell problem (1.6).

For the parabolic case, Jose [6] proved the homogenization for $\gamma \leq 1$. Later, the corrector results for $-1 < \gamma \leq 1$ were given by Donato and Jose [7]. Recently, by the unfolding method, the first author obtained the homogenization and corrector results for $\gamma \leq 1$ in [8]. Our results are also related to those of hyperbolic problems in perforated domains which were studied in [9, 10].

The paper is organized as follows. In Section 2, we briefly recall the unfolding method in perforated domains. Section 3 is devoted to the homogenization result. In Section 4, we prove the corrector results.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz continuous boundary. Let ε be the general term of a sequence of positive real numbers which converges to zero.

For any $k \in \mathbb{Z}^n$, we denote

$$Y^k = k_l + Y, \quad \Gamma_k = k_l + \Gamma, \quad Y_i^k = k_l + Y_i,$$

where $k_l = (k_1 l_1, \dots, k_n l_n)$ and $i = 1, 2$. For any fixed ε , let $K_\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon Y_i^k \cap \Omega \neq \emptyset, i = 1, 2\}$. We suppose that

$$\partial\Omega \cap \left(\bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma_k) \right) = \emptyset$$

and define the two components of Ω and the interface respectively by

$$\Omega_{2\varepsilon} = \bigcup_{k \in K_\varepsilon} \varepsilon Y_2^k, \quad \Omega_{1\varepsilon} = \Omega \setminus \Omega_{2\varepsilon}, \quad \Gamma^\varepsilon = \partial\Omega_{2\varepsilon}.$$

Observe that $\partial\Omega$ and Γ^ε are disjoint, the component $\Omega_{1\varepsilon}$ is connected and the component $\Omega_{2\varepsilon}$ is union of ε^{-n} disjoint translated sets of εY_2 .

The following notations are related to the unfolding method in [11–13]:

$$\begin{aligned}\widehat{K}_\varepsilon &= \{k \in \mathbb{Z}^n \mid \varepsilon Y^k \subset \Omega\}, \quad \widehat{\Omega}_\varepsilon = \text{int} \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon(k_l + \overline{Y}), \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon, \\ \widehat{\Omega}_{i\varepsilon} &= \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_{i\varepsilon} = \Omega_{i\varepsilon} \setminus \widehat{\Omega}_{i\varepsilon}, \quad i = 1, 2, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_{2\varepsilon}.\end{aligned}$$

This paper will also use the following notations:

- $\theta_i = |Y_i|/|Y|$, $i = 1, 2$.
- $\mathcal{M}_\mathcal{O}(v) = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} v dx$.
- $\widetilde{g}$ is the zero extension to Ω (respectively $\Omega \times \mathcal{A}$) of any function g defined on $\Omega_{i\varepsilon}$ (respectively $\Omega_{i\varepsilon} \times \mathcal{A}$) for $i = 1, 2$.
- V^ε is defined by

$$V^\varepsilon := \{v \in H^1(\Omega_{1\varepsilon}) \mid v = 0 \text{ on } \partial\Omega\}$$

endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_{1\varepsilon})}$.

- For any $\gamma \in \mathbb{R}$, the product space

$$H_\gamma^\varepsilon := \{u = (u_1, u_2) \mid u_1 \in V^\varepsilon, u_2 \in H^1(\Omega_{2\varepsilon})\}$$

is equipped with the norm

$$\|u\|_{H_\gamma^\varepsilon}^2 = \|\nabla u_1\|_{L^2(\Omega_{1\varepsilon})}^2 + \|\nabla u_2\|_{L^2(\Omega_{2\varepsilon})}^2 + \varepsilon^\gamma \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2.$$

- C denotes generic constant which does not depend upon ε .
- The notation $L^p(\mathcal{O})$ will be used both for scalar and vector-valued functions defined on the set $\mathcal{O}$, since no ambiguity will arise.

In the rest of this section, we give a brief review of the unfolding operators in two-component domains. We refer the reader to [9] and [14] for further properties and related comments.

For any $x \in \mathbb{R}^n$, we use $[x]_Y$ to denote its integer part $(k_1 l_1, \dots, k_n l_n)$ such that $x - [x]_Y \in Y$, and set $\{x\}_Y = x - [x]_Y$. Then one has

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \quad \text{for any } x \in \mathbb{R}^n.$$

Definition 2.1 [2] Let $i = 1, 2$. For $p \in [1, +\infty)$ and $q \in [1, \infty]$, let $\phi \in L^q(0, T; L^p(\Omega_{i\varepsilon}))$. The unfolding operator $\mathcal{T}_i^\varepsilon : L^q(0, T; L^p(\Omega_{i\varepsilon})) \rightarrow L^q(0, T; L^p(\Omega \times Y_i))$ is defined as follows:

$$\mathcal{T}_i^\varepsilon(\phi)(x, y, t) = \begin{cases} \phi\left(\varepsilon \left[\frac{x}{\varepsilon} \right]_Y + \varepsilon y, t\right) & \text{a.e. for } (x, y, t) \in \widehat{\Omega}_\varepsilon \times Y_i \times (0, T), \\ 0 & \text{a.e. for } (x, y, t) \in \Lambda_\varepsilon \times Y_i \times (0, T). \end{cases}$$

Definition 2.2 [2] Let $i = 1, 2$. For $p \in [1, +\infty)$ and $q \in [1, +\infty]$, let ϕ be in $L^q(0, T; L^p(\Omega \times Y_i))$. The averaging operator $\mathcal{U}_i^\varepsilon : L^q(0, T; L^p(\Omega \times Y_i)) \mapsto L^q(0, T; L^p(\Omega_{i\varepsilon}))$

is defined as follows:

$$\mathcal{U}_i^\varepsilon(\phi)(x, t) = \begin{cases} \frac{1}{|Y|} \int_Y \phi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y, t\right) dz & \text{a.e. for } (x, t) \in \widehat{\Omega}_{i\varepsilon} \times (0, T), \\ 0 & \text{a.e. for } (x, t) \in \Lambda_{i\varepsilon} \times (0, T). \end{cases}$$

Proposition 2.3 For $p \in [1, +\infty)$ and $q \in [1, \infty]$, let $\phi \in L^q(0, T; L^1(\Omega_{i\varepsilon}))$. Then for a.e. $t \in (0, T)$, we have

$$\frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\phi)(x, y, t) dx dy = \int_{\widehat{\Omega}_{i\varepsilon}} \phi(x, t) dx = \int_{\Omega_{i\varepsilon}} \phi(x, t) dx - \int_{\Lambda_{i\varepsilon}} \phi(x, t) dx.$$

Proposition 2.4 (some convergence properties)

(i) Let $\omega \in L^2(0, T; L^2(\Omega))$, then $\|\mathcal{U}_i^\varepsilon(\omega) - \omega\|_{L^2(0, T; L^2(\Omega_{i\varepsilon}))} \rightarrow 0$.
(ii) Let $\omega_\varepsilon \in L^2(0, T; L^2(\Omega_{i\varepsilon}))$ and $\omega \in L^2(0, T; L^2(\Omega))$, then the following two assertions are equivalent:

- (a) $\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \rightarrow \omega$ strongly in $L^2(0, T; L^2(\Omega \times Y_i))$ and $\|\omega_\varepsilon\|_{L^2(0, T; L^2(\Lambda_{i\varepsilon}))} \rightarrow 0$,
 - (b) $\|\omega_\varepsilon - \omega\|_{L^2(0, T; L^2(\Omega_{i\varepsilon}))} \rightarrow 0$.
- (iii) Let $\omega_\varepsilon \in L^2(0, T; L^2(\Omega_{i\varepsilon}))$ and $\omega \in L^2(0, T; L^2(\Omega \times Y_i))$, then the following two assertions are equivalent:

- (a) $\mathcal{T}_i^\varepsilon(\omega_\varepsilon) \rightarrow \omega$ strongly in $L^2(0, T; L^2(\Omega \times Y_i))$ and $\|\omega_\varepsilon\|_{L^2(0, T; L^2(\Lambda_{i\varepsilon}))} \rightarrow 0$,
- (b) $\|\omega_\varepsilon - \mathcal{U}_i^\varepsilon(\omega)\|_{L^2(0, T; L^2(\Omega_{i\varepsilon}))} \rightarrow 0$.

Following the arguments in the proof of [Proposition 1.7, 14] (see also [Proposition 2.13, 9]), we can obtain the following result which will be used to get the corrector results.

Proposition 2.5 Let $p, q \in [1, \infty)$, for $i = 1, 2$, let $f \in L^q(0, T; L^p(\Omega))$ and $g \in L^\infty(\Omega; L^p(Y_i))$, then we have

$$\|\mathcal{U}_i^\varepsilon(fg) - \mathcal{U}_i^\varepsilon(f)\mathcal{U}_i^\varepsilon(g)\|_{L^q(0, T; L^p(\Omega_{i\varepsilon}))} \rightarrow 0.$$

We end this subsection with the following convergence theorem which is crucial to obtaining our homogenization result.

Theorem 2.6 Let $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ and $\{u_\varepsilon\}$ be in $L^\infty(0, T; H_\gamma^\varepsilon)$ with $-1 < \gamma < 1$. If

$$\|u_\varepsilon\|_{L^\infty(0, T; H_\gamma^\varepsilon)} + \|u'_\varepsilon\|_{L^\infty(0, T; L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \leq C,$$

then there exist $u_1 \in L^\infty(0, T; H_0^1(\Omega))$ and $\widehat{u}_1 \in L^\infty(0, T; L^2(\Omega, H_{\text{per}}^1(Y_1)))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for a.e. $x \in \Omega$, such that, up to a subsequence (still denoted by ε),

- (i) $\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \rightarrow u_1$ strongly in $L^q(0, T; L^2(\Omega, H^1(Y_1)))$ for any $q \in (1, +\infty)$,
- (ii) $\mathcal{T}_1^\varepsilon(u_{1\varepsilon}) \rightharpoonup u_1$ weakly* in $L^\infty(0, T; L^2(\Omega, H^1(Y_1)))$,
- (iii) $\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_1))$,
- (iv) $\mathcal{T}_2^\varepsilon(u_{2\varepsilon}) \rightharpoonup u_1$ weakly* in $L^\infty(0, T; L^2(\Omega, H^1(Y_2)))$,
- (v) $\mathcal{T}_2^\varepsilon(\nabla u_{2\varepsilon}) \rightharpoonup 0$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_2))$,
- (vi) $\mathcal{T}_i^\varepsilon(u'_{i\varepsilon}) \rightharpoonup u'_1$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_i))$, $i = 1, 2$.

In fact, the proof can be obtained by following the lines of the proofs of [Theorem 2.12, 14] (see also [Theorem 2.19, 9]) and [Theorem 2.20, 13].

3 Homogenization Results

In this section, we are devoted to the asymptotic behavior of the hyperbolic problem (1.1). For every fixed ε , the Galerkin method provides that problem (1.1) has a unique solution u_ε . Under assumption (1.3), following the arguments in [2], we can obtain the following uniform estimate,

$$\|u_\varepsilon\|_{L^\infty(0,T;H^\varepsilon_\gamma)} + \|u'_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}))} \leq C. \quad (3.1)$$

Now, we state the unfolded formulation of the homogenization results (see Theorem 1.1) which will be used for getting the corrector results.

Theorem 3.1 Under the assumptions of Theorem 1.1, there exist $u_1 \in L^\infty(0, T; H^1_0(\Omega))$ with $u'_1 \in L^\infty(0, T; L^2(\Omega))$ and $\widehat{u}_1 \in L^\infty(0, T; L^2(\Omega, H^1_{\text{per}}(Y_1)))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ such that

- (i) $\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) \rightarrow u_1$ strongly in $L^q(0, T; L^2(\Omega, H^1(Y_1)))$ for any $q \in (1, +\infty)$,
- (ii) $\mathcal{T}^\varepsilon_1(u_{1\varepsilon}) \rightharpoonup u_1$ weakly* in $L^\infty(0, T; L^2(\Omega, H^1(Y_1)))$,
- (iii) $\mathcal{T}^\varepsilon_1(\nabla u_{1\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_1))$,
- (iv) $\mathcal{T}^\varepsilon_2(u_{2\varepsilon}) \rightharpoonup u_1$ weakly* in $L^\infty(0, T; L^2(\Omega, H^1(Y_2)))$,
- (v) $\mathcal{T}^\varepsilon_2(\nabla u_{2\varepsilon}) \rightharpoonup 0$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_2))$,
- (vi) $\mathcal{T}^\varepsilon_i(u'_{i\varepsilon}) \rightharpoonup u'_1$ weakly* in $L^\infty(0, T; L^2(\Omega \times Y_i))$,
- (vii) $\widehat{u}_{i\varepsilon} \rightharpoonup \theta_i u_1$ weakly* in $L^\infty(0, T; L^2(\Omega))$, $i = 1, 2$.

And the pair $(u_1, \widehat{u}_1)$ is the unique solution in $L^2(0, T; H^1_0(\Omega)) \times L^2(0, T; L^2(\Omega, H^1_{\text{per}}(Y_1)))$ with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for a.e. $x \in \Omega$, of the problem

$$\left\{ \begin{array}{l} \int_0^T \int_\Omega u_1 \Psi \varphi'' dx dt + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_1} A(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \Psi + \nabla_y \Phi) \varphi dx dy dt \\ = \int_0^T \int_\Omega (\theta_1 f_1 + \theta_2 f_2) \Psi \varphi dx dt \quad \text{for all } \varphi \in \mathcal{D}(0, T), \Psi \in H^1_0(\Omega) \text{ and } \Phi \in L^2(\Omega, H^1_{\text{per}}(Y_1)), \\ u_1(x, 0) = \theta_1 U_1^0 + \theta_2 U_2^0 \quad \text{in } \Omega, \\ u'_1(x, 0) = \theta_1 U_1^1 + \theta_2 U_2^1 \quad \text{in } \Omega. \end{array} \right.$$

Moreover, we have

$$\widehat{u}_1 = \sum_{j=1}^n \frac{\partial u_1}{\partial x_j} \chi_j, \quad (3.3)$$

where $\chi_j \in L^\infty(\Omega; H^1_{\text{per}}(Y))$ ($j = 1, \dots, n$) is the solution of the cell problem (1.6).

The proofs of Theorem 3.1 and Theorem 1.1 mainly rely on the periodic unfolding method. Indeed, following the lines of proof of Theorem 3.1 [4], we can use Theorem 2.6 to obtain the proofs of these two theorems.

Remark 3.2 Following the framework in the proof of Theorem 3.2 [8], we derive

$$\begin{aligned} \int_{\Omega} A^0 \nabla u_1 \nabla u_1 dx &= \frac{1}{|Y|} \int_{\Omega \times Y_1} A(\nabla u_1 + \nabla_y \hat{u}_1)(\nabla u_1 + \nabla_y \hat{u}_1) dx dy \\ &+ \frac{1}{|Y|} \int_{\Omega \times Y_2} A(\nabla_y \hat{u}_2)(\nabla_y \hat{u}_2) dx dy, \end{aligned} \quad (3.4)$$

which will be used in the proof of Corollary 4.2.

Remark 3.3 In Theorem 1.1, we exclude the case $\gamma = 1$. For this case, the homogenized problem is a coupled system of a PDE and an ODE. As a result, the corrector results are more complicated.

4 Proof of Theorem 1.2

In this section, we are devoted to the proof of corrector results. To do that, we need some stronger assumptions than those of the homogenization results. Here, we impose the assumptions (1.8)–(1.10), as presented in [4], which are slightly weaker than those in [3]. Under these assumptions, the energy of problem (1.1) converges in $C^0([0, T])$ to that of the homogenized one. Moreover, we obtain that some convergences in (3.2) are strong ones.

For each ε , the energy $E^\varepsilon(t)$, associated to the problem (1.1), is defined by

$$\begin{aligned} E^\varepsilon(t) &:= \frac{1}{2} \left[\int_{\Omega_{1\varepsilon}} |u'_{1\varepsilon}(t)|^2 dx + \int_{\Omega_{2\varepsilon}} |u'_{2\varepsilon}(t)|^2 dx + \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} dx \right. \\ &\quad \left. + \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} dx + \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2 d\sigma_x \right]. \end{aligned}$$

The energy associated to the homogenized problem (1.4) is defined by

$$E(t) := \frac{1}{2} \left[\int_{\Omega} |u'_1|^2 dx + \int_{\Omega} A^0 \nabla u_1 \nabla u_1 dx \right].$$

Following the classical arguments (see for instance [3]), we have the following result.

Theorem 4.1 Let $\gamma \in (-1, 1)$. Suppose that u_ε is the solution of problem (1.1) with the initial data satisfying (1.8)–(1.10). Let u_1 be the solution of the homogenized problem (1.4), then we have

$$E^\varepsilon(t) \rightarrow E(t) \text{ strongly in } C^0([0, T]).$$

Corollary 4.2 Under the assumptions of Theorem 4.1, we have

$$\begin{aligned} \text{(i)} \quad &\|u'_{i\varepsilon}\|_{L^2(0, T; L^2(\Lambda_{i\varepsilon}))} \rightarrow 0, \quad \|\nabla u_{1\varepsilon}\|_{L^2(0, T; L^2(\Lambda_{1\varepsilon}))} \rightarrow 0 \quad \text{and} \\ &\mathcal{T}_i^\varepsilon(u'_{i\varepsilon}) \rightarrow u'_1 \quad \text{strongly in } L^2(0, T; L^2(\Omega \times Y_i)) \quad \text{for } i = 1, 2, \\ \text{(ii)} \quad &\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \rightarrow \nabla u_1 + \nabla_y \hat{u}_1 \quad \text{strongly in } L^2(0, T; L^2(\Omega \times Y_1)), \\ \text{(iii)} \quad &\|\nabla u_{2\varepsilon}\|_{L^2(0, T; L^2(\Omega_{2\varepsilon}))} \rightarrow 0, \end{aligned} \quad (4.1)$$

where $\hat{u}_1$ is given by Theorem 3.1.

To prove this corollary, we need the following classical result.

Proposition 4.3 (see [14]) Let $\{D_\varepsilon\}$ be a sequence of $n \times n$ matrices in $M(\alpha, \beta, \mathcal{O})$ for some open set $\mathcal{O}$, such that $D_\varepsilon \rightarrow D$ a.e. on $\mathcal{O}$ (or more generally, in measure in $\mathcal{O}$). If $\zeta_\varepsilon \rightharpoonup \zeta$ weakly in $L^2(\mathcal{O})$, then

$$\int_{\mathcal{O}} D\zeta\zeta \, dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\mathcal{O}} D_\varepsilon \zeta_\varepsilon \zeta_\varepsilon \, dx.$$

Proof of Corollary 4.2 From (3.4), we have

$$\begin{aligned} 2 \int_0^T E(t) \, dt &= \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_1} |u'_1|^2 \, dx \, dy \, dt + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_2} |u'_1|^2 \, dx \, dy \, dt \\ &\quad + \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_1} A(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla u_1 + \nabla_y \widehat{u}_1) \, dx \, dy \, dt. \end{aligned}$$

By Proposition 4.3 and the weak lower-semicontinuity, we deduce

$$\begin{aligned} 2 \int_0^T E(t) \, dt &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_1} [\mathcal{T}_1^\varepsilon(u'_{1\varepsilon})]^2 \, dx \, dy \, dt \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_2} [\mathcal{T}_2^\varepsilon(u'_{2\varepsilon})]^2 \, dx \, dy \, dt \\ &\quad + \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y_1} A(y) \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \, dx \, dy \, dt. \end{aligned}$$

Thus, Proposition 2.3 allows us to get that

$$\int_0^T E(t) \, dt \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \widehat{E}^\varepsilon(t) \, dt \leq \limsup_{\varepsilon \rightarrow 0} \int_0^T \widehat{E}^\varepsilon(t) \, dt \leq \lim_{\varepsilon \rightarrow 0} \int_0^T E^\varepsilon(t) \, dt = \int_0^T E(t) \, dt,$$

where

$$\widehat{E}^\varepsilon(t) := \frac{1}{2} \left[\int_{\widehat{\Omega}_{1\varepsilon}} |u'_{1\varepsilon}|^2 \, dx + \int_{\widehat{\Omega}_{2\varepsilon}} |u'_{2\varepsilon}|^2 \, dx + \int_{\widehat{\Omega}_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} \, dx \right].$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T E^\varepsilon(t) \, dt = \lim_{\varepsilon \rightarrow 0} \int_0^T \widehat{E}^\varepsilon(t) \, dt = \int_0^T E(t) \, dt. \quad (4.2)$$

The former equality implies that

$$\begin{aligned} \int_0^T \int_{\Lambda_{1\varepsilon}} |u'_{1\varepsilon}|^2 \, dx \, dt + \int_0^T \int_{\Lambda_{2\varepsilon}} |u'_{2\varepsilon}|^2 \, dx \, dt &\rightarrow 0, \\ \int_0^T \int_{\Lambda_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} \, dx \, dt &\rightarrow 0, \\ \int_0^T \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} \, dx \, dt &\rightarrow 0. \end{aligned}$$

These give the first line and (iii) in (4.1) due to the ellipticity of A^ε .

By the latter equality in (4.2) and Proposition 2.3, we know

$$\begin{aligned} & \int_0^T \int_{\Omega \times Y_1} [\mathcal{T}_1^\varepsilon(u'_{1\varepsilon})]^2 dx dy dt + \int_0^T \int_{\Omega \times Y_2} [\mathcal{T}_2^\varepsilon(u'_{2\varepsilon})]^2 dx dy dt \\ & + \int_0^T \int_{\Omega \times Y_1} A(y) \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) \mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) dx dy dt \longrightarrow 2|Y| \int_0^T E(t) dt. \end{aligned}$$

Combining this with (3.2), we obtain

$$\begin{aligned} & \int_0^T \int_{\Omega \times Y_1} [\mathcal{T}_1^\varepsilon(u'_{1\varepsilon}) - u'_1]^2 dx dy dt + \int_0^T \int_{\Omega \times Y_2} [\mathcal{T}_2^\varepsilon(u'_{2\varepsilon}) - u'_1]^2 dx dy dt \\ & + \int_0^T \int_{\Omega \times Y_1} A(y) \left[\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) - (\nabla u_1 + \nabla_y \widehat{u}_1) \right] \left[\mathcal{T}_1^\varepsilon(\nabla u_{1\varepsilon}) - (\nabla u_1 + \nabla_y \widehat{u}_1) \right] dx dy dt \\ & \longrightarrow 2|Y| \int_0^T E(t) dt - |2Y| \int_0^T E(t) dt + 2|Y| \int_0^T E(t) dt - 2|Y| \int_0^T E(t) dt = 0. \end{aligned}$$

This together with the ellipticity of A , allows us to obtain the rest convergences in (4.1).

Proof of Theorem 1.2 Observe that u_1 is independent of y . By (ii) of Proposition 2.4, the first convergence in (1.11) follows from (i) in Corollary 4.2. By (i) and (ii) in Corollary 4.2, we use (iii) of Proposition 2.4 to get

$$\|\nabla u_{1\varepsilon} - \mathcal{U}_1^\varepsilon(\nabla u_1 + \nabla_y \widehat{u}_1)\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} \rightarrow 0.$$

By the fact that ∇u_1 is independent of y , (i) of Proposition 2.4 gives

$$\|\nabla u_1 - \mathcal{U}_1^\varepsilon(\nabla u_1)\|_{L^2(0,T;L^2(\Omega_{1\varepsilon}))} \rightarrow 0.$$

Together with (3.3) and Proposition 2.5, we complete the proof of Theorem 1.2.

References

- [1] Carslaw H S, Jaeger J C. Conduction of heat in solids[M]. Oxford: Clarendon Press, 1947.
- [2] Donato P, Faella L, Monsurrò S. Homogenization of the wave equation in composites with imperfect interface: A memory effect[J]. J. Math. Pures Appl., 2007, 87(2): 119–143.
- [3] Donato P, Faella L, Monsurrò S. Correctors for the homogenization of a class of hyperbolic equations with imperfect interfaces[J]. SIAM J. Math. Anal., 2009, 40(5): 1952–1978.
- [4] Yang Zhanying. Homogenization and correctors for the hyperbolic problems with imperfect interfaces via the periodic unfolding method[J]. Commu. Pure Appl. Anal., 2014, 13(1): 249–272.
- [5] Cioranescu D, Donato P. An Introduction to Homogenization[M]. Oxford: Oxford Univ. Press, 1999.
- [6] Jose E C. Homogenization of a parabolic problem with an imperfect interface[J]. Rev. Rouma. Math. Pures Appl., 2009, 54(3): 189–222.
- [7] Donato P, Jose E C. Corrector results for a parabolic problem with a memory effect[J]. ESAIM: Math. Model. Num. Anal., 2010, 44(3): 421–454.
- [8] Yang Zhanying. The periodic unfolding method for a class of parabolic problems with imperfect interfaces[J]. ESAIM: Math. Model. Num. Anal., 2014, 48(5): 1279–1302.

- [9] Donato P, Yang Zhanying. The periodic unfolding method for the wave equation in domains with holes[J]. Adv. Math. Sci. Appl., 2012, 22(2): 521–551.
- [10] Nabil A. A corrector result for the wave equations in perforated domains[J]. GAKUTO Internat. Ser. Math. Sci. Appl., 1997, 9: 309–321.
- [11] Cioranescu D, Damlamian A, Griso G. Periodic unfolding and homogenization[J]. C. R. Math. Acad. Sci. Paris, 2002, 335(1): 99–104.
- [12] Cioranescu D, Damlamian A, Griso G. The periodic unfolding method in homogenization[J]. SIAM J. Math. Anal., 2008, 40(4): 1585–1620.
- [13] Donato P, Le Nguyen K H, Tardieu R. The periodic unfolding method for a class of imperfect transmission problems[J]. J. Math. Sci., 2011, 176(6): 891–927.
- [14] Cioranescu D, Damlamian A, Donato P, Griso G, Zaki R. The periodic unfolding method in domains with holes[J]. SIAM J. Math. Anal., 2012, 44(2): 718–760.
- [15] Chen Jiajia, Mu Chunlai. The upper and lower bound on the blow-up phenomena for some nonlinear parabolic systems[J]. J. Math., 2012, 32(5): 897–903.

带不完美界面的双曲问题均匀化的一个注记

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摘要: 本文研究了一类二分区域上的具有非周期系数的双曲问题. 利用周期Unfolding方法, 得到了均匀化及其矫正结果, 推广了Donato, Faella 和Monsurrò 的工作.

关键词: 双曲问题; 周期Unfolding 方法; 均匀化; 矫正

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