

A MASCHKE TYPE THEOREM FOR PARTIAL π -COMODULES

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Abstract: In this paper, we study the Maschke type theorems of partial group comodules. By the methods of weak Hopf group coalgebras, we obtain the classical Maschke type theorems of Hopf algebras, which generalized those of Hopf algebras and results of [8].

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1 Introduction

Partial actions of groups as powerful tools were introduced during the study of operator algebras by Exel [2]. With the further development, many positive results were proposed [3–6]. Caenepeel and the other authors developed a theory of partial actions of Hopf algebras [1] and introduced the notion of a partial entwining structure as a generalization of entwining structure (see [9]).

On other hand, the notion of a Hopf π -coalgebra which generalized that of a Hopf algebra was introduced and played an important role, consequently group entwining structures and group weak entwining structure were carefully studied. Motivated by this fact, we introduce the notion of a partial group comodule and give a Maschke type theorem for them. Because the “coassociativity” of a partial structure is destroyed, the generalization is not trivial and easy.

In this paper, we first recall basic definitions of partial group comodules and give some examples. Then we state a Maschke-type theorem of partial group Hopf modules which generalizes the relevant results of Hopf modules (see [7, 8]), entwined modules, group Hopf modules, etc..

The organization of the paper is as follows: First we introduce the notion of partial group comodules and then give our main result-Maschke type theorem.

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Conventions We work over a commutative ring k . We denote by i the unit of the group π and use the standard (co)algebra notation, i.e., Δ is a coproduct, ε is a counit, m is a product and 1 is a unit. If 1 appears more than once in the same expression, then we use different $1'$. The identity map from any k -space V to itself is denoted by id_V . Write a_α for any element in A_α and $[a]$ for an element in $\bar{A} = A/\ker f$, where f is a k -linear map. For a right π - C -comodule M , we write $\rho_{\alpha,\beta}(m) = \sum m_{([0],\alpha)} \otimes m_{([1],\beta)}$ for any $\alpha, \beta \in \pi$ and $m \in M_{\alpha\beta}$.

2 The Main Results

Definition 2.1 A π -coalgebra over k is a family of $C = \{C_\alpha\}_{\alpha \in \pi}$ of k -spaces endowed with a family k -linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \rightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta \in \pi}$ and a k -linear map $\varepsilon : C_i \rightarrow k$ such that for any $\alpha, \beta, \gamma \in \pi$,

- (1) $(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}$.
- (2) $(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,i} = (\varepsilon \otimes id_{C_\alpha})\Delta_{i,\alpha} = id_{C_\alpha}$.

Here we extend the Sweedler notation for comultiplication, we write

$$\Delta_{\alpha,\beta}(c_{\alpha\beta}) = \sum c_{\alpha\beta 1\alpha} \otimes c_{\alpha\beta 2\beta}, \quad \alpha, \beta \in \pi, \quad c \in C_{\alpha\beta}.$$

Remark $(C_i, \Delta_{i,i}, \varepsilon)$ is a coalgebra in the usual sense.

Definition 2.2 A Hopf π -coalgebra (Hopf group coalgebra) is a family of algebras $H = \{H_\alpha\}_{\alpha \in \pi}$ and also a π -coalgebra $\{H_\alpha, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon\}_{\alpha,\beta \in \pi}$ endowed with a family $S = \{S_{\alpha^{-1}} : H_\alpha \rightarrow H_{\alpha^{-1}}\}_{\alpha \in \pi}$ of k -linear maps called an antipode such that for any $\alpha \in \pi$.

- (3) $\sum S_{\alpha^{-1}}(h_{1\alpha})h_{2\alpha^{-1}} = \varepsilon(h)1_{\alpha^{-1}}, \quad \sum h_{1\alpha}S_\alpha(h_{2\alpha^{-1}}) = \varepsilon(h)1_\alpha$.

Definition 2.3 Let H be a Hopf group coalgebra and $A = \{A_\alpha\}_{\alpha \in \pi}$ be a family of algebras endowed with a family of k -linear maps $\{\rho_{\alpha,\beta} : A_{\alpha\beta} \rightarrow A_\alpha \otimes A_\beta\}_{\alpha,\beta \in \pi}$. A is called a right partial group comodule-algebra if the following conditions are satisfied:

- (4) $\rho_{\alpha,\beta}(ab) = \rho_{\alpha,\beta}(a)\rho_{\alpha,\beta}(b), \quad a, b \in A_{\alpha\beta}$.
- (5) $(\rho_{\alpha,\beta} \otimes id_{H_\gamma})\rho_{\alpha\beta,\gamma}(c) = \sum c_{([0],\alpha)} 1_{([0],\alpha)} \otimes c_{([1],\beta\gamma)} 1_{([1],\beta)} \otimes c_{([1],\beta\gamma)} 2_\gamma, \quad c \in A_{\alpha\beta\gamma}$.
- (6) $\sum \varepsilon(d_{([1],i)})d_{([0],\alpha)} = d, \quad d \in A_\alpha$.

Example 1 Let H be a Hopf group coalgebra and $e = \{e_\alpha\}_{\alpha \in \pi}$ be a central idempotent such that $\Delta_{\alpha,\beta}(e_{\alpha\beta})(e_\alpha \otimes 1_\beta) = e_\alpha \otimes e_\beta$ and $\varepsilon(e_i) = 1$, then H is a right partial group comodule-algebra.

Definition 2.4 Let H be a Hopf group coalgebra and A be a right partial group comodule-algebra. An A -module $M = \{M_\alpha\}_{\alpha \in \pi}$ with a family of k -linear maps

$$\{\rho_{\alpha,\beta} : M_{\alpha\beta} \rightarrow M_\alpha \otimes H_\beta\}_{\alpha,\beta \in \pi}$$

is called a partial (H, A) -Hopf module if the following conditions are verified for any $m \in M_\alpha$, $m' \in M_{\alpha\beta\gamma}$, $m'' \in M_{\alpha\beta}$, $a \in A_{\alpha\beta}$:

- (7) $\sum \varepsilon(m_{([1],i)})m_{([0],\alpha)} = m$.
- (8) $(\rho_{\alpha,\beta} \otimes id_{H_\gamma})\rho_{\alpha\beta,\gamma}(m') = \sum m'_{([0],\alpha)} \cdot 1_{([0],\alpha)} \otimes m'_{([1],\beta\gamma)} 1_{([1],\beta)} \otimes m'_{([1],\beta\gamma)} 2_\gamma$.

$$(9) \quad \rho_{\alpha,\beta}(m'' \cdot a) = \sum m''_{([0],\alpha)} \cdot a_{([0],\alpha)} \otimes m''_{([1],\beta)} a_{([1],\beta)}.$$

We define the coinvariants of M as

$$M^{COH} = \{m = \{m_\alpha\}_{\alpha \in \pi} \mid \rho_{\alpha,\beta}(m_{\alpha\beta}) = \sum m_\alpha \cdot 1_{([0],\alpha)} \otimes 1_{([1],\beta)}\}$$

and denote $M_A^{\pi-H}$ the category of partial (H, A) -Hopf modules.

Example 2 Let H be a Hopf group coalgebra and A be a right partial group comodule-algebra. It is easy to prove that A is a partial (H, A) -Hopf module with the multiplications as A -actions.

Definition 2.5 Let H be a Hopf group coalgebra and A be a right partial group comodule-algebra. A right partial π - H -comodule map $\theta = \{\theta_\alpha : H_\alpha \rightarrow A_\alpha\}_{\alpha \in \pi}$ such that $\sum 1_{([0],\alpha)} \theta_\alpha(S_\alpha(1_{([1],\alpha^{-1})})) = 1_\alpha$ is called a right total integral of A .

Definition 2.6 Let $M \in M_A^{\pi-H}$ and $\text{tr} = \{\text{tr}_\alpha : M_i \rightarrow M_\alpha\}_{\alpha \in \pi}$ be a family of k -linear maps such that $\text{tr}_\alpha(m) = \sum m_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}))$. Then tr is called a trace map of M .

We define $\hat{\text{tr}} : \prod M_\alpha \rightarrow \prod M_\alpha$, $\hat{\text{tr}}((m_\alpha)_{\alpha \in \pi}) = (\text{tr}_\alpha(m_i))_{\alpha \in \pi}$.

In the following parts we always suppose that

$$\sum 1_{([0],\alpha)} a \otimes 1_{([1],\beta)} b = \sum a 1_{([0],\alpha)} \otimes b 1_{([1],\beta)}$$

for any $a \in A_\alpha$ and $b \in H_\beta$.

Proposition 2.7 $m = (m_\alpha)_{\alpha \in \pi}$ is a coinvariant of M if and only if $\hat{\text{tr}}(m) = m$.

Proof If $m = (m_\alpha)_{\alpha \in \pi} \in M^{COH}$, then

$$\hat{\text{tr}}(m) = \sum m_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})) = \sum m_\alpha \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(1_{([1],\alpha^{-1})})) = m_\alpha.$$

Conversely, if $\hat{\text{tr}}(m) = m$, we have

$$\begin{aligned} & \rho_{\alpha,\beta}(\text{tr}_{\alpha\beta}(m)) \\ &= \sum m_{([0],\alpha\beta)} \cdot \theta_{\alpha\beta}(S_{\alpha\beta}(m_{([1],(\alpha\beta)^{-1})})) \otimes m_{([0],\alpha\beta)} S_\beta(m_{([1],(\alpha\beta)^{-1})}) \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})) \otimes m_{([1],\alpha^{-1})} S_\beta(m_{([1],\alpha^{-1})}) \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})) \otimes 1_{([1],\beta)} = \sum m_\alpha \cdot 1_{([0],\alpha)} \otimes 1_{([1],\beta)}. \end{aligned}$$

Given $M \in M_A^{\pi-H}$, we define

$$\omega_\alpha : M_i \otimes H_\alpha \rightarrow M_i \otimes H_\alpha, \quad \omega_\alpha(m \otimes h) = \sum m \cdot 1_{([0],i)} \otimes h 1_{([1],\alpha)}, \quad m \in M_i, \quad h \in H_\alpha.$$

It is trivial to prove that $\omega_\alpha^2 = \omega_\alpha$, so we can define

$$\overline{M_i \otimes H_\alpha} = (M_i \otimes H_\alpha) / \ker \omega_\alpha$$

as a k -space.

We claim that $\overline{M_i \otimes H} = \{\overline{M_i \otimes H_\alpha}\}_{\alpha \in \pi} \in M_A^{\pi-H}$ with the actions given by

$$[m \otimes h_\alpha] \cdot a_\alpha = \sum [m \cdot a_{([0],i)} \otimes h_\alpha a_{([1],\alpha)}]$$

and the partial coactions given by

$$\rho_{\alpha,\beta}([m \otimes h_{\alpha\beta}]) = \sum [m \cdot 1_{([0],i)} \otimes h_{\alpha\beta 1\alpha} 1_{([1],\alpha\beta)1\alpha}] \otimes h_{\alpha\beta 2\beta} 1_{([1],\alpha\beta)2\beta}.$$

It is easy to prove that the actions are well-defined. We only have to show the coactions are well-defined. In fact, we have

$$\begin{aligned} & \rho_{\alpha,\beta}(m \otimes h_{\alpha\beta} - m \cdot 1_{([0],i)} \otimes h_{\alpha\beta} a_{([1],\alpha\beta)}) \\ = & \sum m \cdot 1_{([0],i)} \otimes h_{\alpha\beta 1\alpha} 1_{([1],\alpha\beta)1\alpha} \otimes h_{\alpha\beta 2\beta} 1_{([1],\alpha\beta)2\beta} \\ & - m \cdot 1_{([0],i)} 1'_{([0],i)} \otimes h_{\alpha\beta 1\alpha} 1_{([1],\alpha\beta)1\alpha} 1'_{([1],\alpha\beta)1\alpha} \otimes h_{\alpha\beta 2\beta} 1_{([1],\alpha\beta)2\beta} 1'_{([1],\alpha\beta)2\beta} = 0. \\ & \sum [m \otimes h_{\alpha\beta\gamma}]_{([0],\alpha)} \cdot 1_{([0],\alpha)} \otimes [m \otimes h_{\alpha\beta\gamma}]_{([1],\beta\gamma)1\beta} 1_{([1],\beta)} \otimes [m \otimes h_{\alpha\beta\gamma}]_{([1],\beta\gamma)2\gamma} \\ = & \sum [m \cdot 1'_{([0],i)} 1_{([0],i)} 1''_{([0],i)} \otimes h_{\alpha\beta\gamma 1\alpha} 1'_{([1],\alpha\beta\gamma)1\alpha} 1_{([1],\alpha\beta)1\alpha} 1''_{([1],\alpha)1\alpha}] \\ & \otimes h_{\alpha\beta\gamma 2\beta\gamma 1\beta} 1'_{([1],\alpha\beta\gamma)2\beta\gamma 1\beta} 1_{([1],\alpha\beta)2\beta} \otimes h_{\alpha\beta\gamma 2\beta\gamma 2\gamma} 1'_{([1],\alpha\beta\gamma)2\beta\gamma 2\gamma} \\ = & \sum [m \cdot 1'_{([0],i)} 1_{([0],i)} \otimes h_{\alpha\beta\gamma 1\alpha\beta 1\alpha} 1'_{([1],\alpha\beta\gamma)1\alpha\beta 1\alpha} 1_{([1],\alpha\beta)1\alpha} \\ & \otimes h_{\alpha\beta\gamma 1\alpha\beta 2\beta} 1'_{([1],\alpha\beta\gamma)1\alpha\beta 2\beta} 1_{([1],\alpha\beta)2\beta} \otimes h_{\alpha\beta\gamma 2\gamma} 1'_{([1],\alpha\beta\gamma)2\gamma} \\ = & (\rho_{\alpha,\beta} \otimes id_{H_\gamma}) \rho_{\alpha\beta,\gamma}([m \otimes h_{\alpha\beta\gamma}]), \\ & \sum \varepsilon([m \otimes h_\alpha]_{([1],i)}) [m \otimes h_\alpha]_{([0],\alpha)} = \sum \varepsilon(h_{\alpha 2i} 1_{([1],\alpha)2i}) [m \cdot 1_{([0],i)} \otimes h_{\alpha 1\alpha} 1_{([1],\alpha)1\alpha}] \\ = & [m \otimes h_\alpha]. \\ & \rho_{\alpha,\beta}([m \otimes h_{\alpha\beta}] \cdot a_{\alpha\beta}) = \sum \rho_{\alpha,\beta}([m \cdot a_{\alpha\beta}([0],i)} \otimes h_{\alpha\beta} a_{\alpha\beta}([1],\alpha\beta)]) \\ = & \sum [m \cdot a_{\alpha\beta}([0],i)} 1_{([0],i)} \otimes h_{\alpha\beta 1\alpha} a_{\alpha\beta}([1],\alpha\beta) 1_{([1],\alpha\beta)1\alpha}] \otimes h_{\alpha\beta 2\beta} a_{\alpha\beta}([1],\alpha\beta) 2\beta 1_{([1],\alpha\beta)2\beta} \\ = & \sum [m \otimes h_{\alpha\beta}]_{([0],\alpha)} \cdot a_{\alpha\beta}([0],\alpha) \otimes [m \otimes h_{\alpha\beta}]_{([1],\beta)} a_{\alpha\beta}([1],\beta). \end{aligned}$$

We define $\xi = \{\xi_\alpha : \overline{M_i \otimes H_\alpha} \rightarrow M_\alpha\}_{\alpha \in \pi}$, where

$$\xi_\alpha([m \otimes h]) = \sum m_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h), \quad m \in M_i, \quad h \in H_\alpha.$$

Then we claim that ξ_α is well-defined for any $\alpha \in \pi$. Indeed, for any $m \in M_i$, $h \in H_\alpha$,

$$\begin{aligned} & \xi_\alpha(m \otimes h - m \cdot 1_{([0],i)} \otimes h 1_{([1],\alpha)}) \\ = & \sum m_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h) - (m \cdot 1_{([0],i)})_{([0],\alpha)} \theta_\alpha(S_\alpha((m \cdot 1_{([0],i)})_{([1],\alpha^{-1})})h 1_{([1],\alpha)}) \\ = & \sum m_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h) - m_{([0],\alpha)} \cdot 1_{([0],i)} 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})} 1_{([0],i)} 1_{([1],\alpha^{-1})})h 1_{([1],\alpha)}) \\ = & \sum m_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h) - m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})} 1_{([0],i)} 1_{\alpha^{-1}})h 1_{([1],i)2\alpha}) = 0. \end{aligned}$$

Lemma 2.8 For any $\alpha \in \pi$, $\xi_\alpha \circ p_\alpha \circ \rho_{i,\alpha}^M = id_{M_\alpha}$, where $p_\alpha : M_i \otimes H_\alpha \rightarrow \overline{M_i \otimes H_\alpha}$ is a canonical map.

Proof In fact, for any $m \in M_\alpha$, we have

$$\begin{aligned} \xi_\alpha \circ p_\alpha \circ \rho_{i,\alpha}^M(m) &= \sum m_{([0],i)([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([0],i)([1],\alpha^{-1})})m_{([1],\alpha)}) \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([0],i)1\alpha^{-1}}1_{([1],\alpha^{-1})})m_{([1],i)2\alpha}) = m. \end{aligned}$$

Lemma 2.9 If for any $\alpha \in \pi$, $h \in H_\alpha$, $a \in A_\alpha$,

$$\sum a_{([0],\alpha)} \theta_\alpha(S_\alpha(a_{([0],i)1\alpha^{-1}})ha_{([1],i)2\alpha}) = \theta_\alpha(h)a,$$

then ξ_α is a right A_α -linear map.

Proof Indeed for any $\alpha \in \pi$, $h \in H_\alpha$, $a \in A_\alpha$, we have

$$\begin{aligned} \xi_\alpha([m \otimes h] \cdot a) &= (m \cdot a_{([0],i)})_{([0],\alpha)} \theta_\alpha(S_\alpha((m \cdot a_{([0],i)})_{([1],\alpha^{-1})})ha_{([1],\alpha)}) \\ &= m_{([0],\alpha)} \cdot a_{([0],i)([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}a_{([0],i)([1],\alpha^{-1})})ha_{([1],\alpha)}) \\ &= m_{([0],\alpha)} \cdot a_{([0],\alpha)} 1'_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}a_{([0],i)1\alpha^{-1}}1'_{([1],\alpha^{-1})})ha_{([1],i)2\alpha}) \\ &= m_{([0],\alpha)} \cdot a_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}a_{([0],i)1\alpha^{-1}})ha_{([1],i)2\alpha}) \\ &= m_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h)a = \xi_\alpha([m \otimes h]) \cdot a. \end{aligned}$$

Lemma 2.10 Let H be a Hopf group coalgebra and A be a right partial group comodule-algebra. If the condition in Lemma 2.9 is satisfied, then $\xi = \{\xi_\alpha\} \in M_A^{\pi-H}$.

Proof It is sufficed to prove that ξ is right partial π - H -colinear. In fact, for any $\alpha, \beta \in \pi$, $h \in H_{\alpha\beta}$, $m \in M_i$, on one hand,

$$\begin{aligned} &\rho_{\alpha,\beta}^M \xi_{\alpha\beta}([m \otimes h]) \\ &= \sum m_{([0],\alpha\beta)([0],\alpha)} \cdot \theta_{\alpha\beta}(S_{\alpha\beta}(m_{([1],(\alpha\beta)^{-1})})h)_{([0],\alpha)} \\ &\quad \otimes m_{([0],\alpha\beta)([1],\beta)} \theta_{\alpha\beta}(S_{\alpha\beta}(m_{([1],(\alpha\beta)^{-1})})h)_{([1],\beta)}) \\ &= \sum m_{([0],\alpha\beta)([0],\alpha)} \cdot \theta_\alpha(S_\alpha(m_{([1],(\alpha\beta)^{-1})2\alpha^{-1}})h_{1\alpha}) \otimes m_{([0],\alpha\beta)([1],\beta)} S_\beta(m_{([1],(\alpha\beta)^{-1})1\beta^{-1}})h_{2\beta} \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})2(\alpha\beta)^{-1}2\alpha^{-1}})h_{1\alpha}) \otimes m_{([1],\alpha^{-1})1\beta} 1_{([1],\beta)} \\ &\quad S_\beta(m_{([1],\alpha^{-1})2(\alpha\beta)^{-1}1\beta^{-1}})h_{2\beta} = \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h_{1\alpha}) \otimes 1_{([1],\beta)} h_{2\beta}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\xi_\alpha \otimes id_{H_\beta}) \overline{\rho_{\alpha,\beta}^{M_i \otimes H_{\alpha\beta}}}([m \otimes h]) \\ &= \sum (m \cdot 1_{([0],i)})_{([0],\alpha)} \cdot \theta_\alpha(S_\alpha((m \cdot 1_{([0],i)})_{([1],\alpha^{-1})})h_{1\alpha}1_{([1],\alpha\beta)1\alpha}) \otimes h_{2\beta}1_{([1],\alpha\beta)2\beta} \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],i)([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}1_{([0],i)([1],\alpha^{-1})})h_{1\alpha}1_{([1],\alpha\beta)1\alpha}) \otimes h_{2\beta}1_{([1],\alpha\beta)2\beta} \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} 1'_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}1_{([1],\beta)1\alpha^{-1}}1'_{([1],\alpha^{-1})})h_{1\alpha}1_{([1],\beta)2\alpha\beta1\alpha}) \\ &\quad \otimes h_{2\beta}1_{([1],\beta)2\alpha\beta2\beta} \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})}1_{([1],i)1\alpha^{-1}})h_{1\alpha}1_{([1],i)2\alpha}) \otimes h_{2\beta}1_{([1],\beta)} \\ &= \sum m_{([0],\alpha)} \cdot 1_{([0],\alpha)} \theta_\alpha(S_\alpha(m_{([1],\alpha^{-1})})h_{1\alpha}) \otimes 1_{([1],\beta)} h_{2\beta}. \end{aligned}$$

Therefore we complete the proof. Now we can give our main result.

Theorem 2.11 Let H be a Hopf group coalgebra and A be a right partial group comodule-algebra with a total integral θ , and $M, N \in M_A^{\pi-H}$. Supposing the condition in Lemma 2.9 is satisfied, if $f_i : M_i \rightarrow N_i$ splits as A_i -module map, then $f = \{f_\alpha : M_\alpha \rightarrow N_\alpha\}_{\alpha \in \pi}$ splits as partial π - H -comodule map.

Proof Assume that there exists an A_i -module map $g_i : N_i \rightarrow M_i$ such that $g_i f_i = id_{M_i}$. We define

$$\bar{g} = \{\bar{g}_\alpha : N_\alpha \rightarrow M_\alpha, \bar{g}_\alpha(n) = \sum \xi_\alpha([g_i(n_{([0],i)}) \otimes n_{([1],\alpha)}]), n \in N_\alpha\}_{\alpha \in \pi}.$$

First we claim that \bar{g}_α is right A_α -linear. In fact, for any $\alpha \in \pi$, $n \in N_\alpha$, $a \in A_\alpha$,

$$\begin{aligned} \bar{g}_\alpha(n \cdot a) &= \sum \xi_\alpha([g_i(n_{([0],i)}) \cdot a_{([0],i)}) \otimes n_{([1],\alpha)} a_{([1],\alpha)}]) \\ &= \sum \xi_\alpha([g_i(n_{([0],i)}) \cdot a_{([0],i)}) \otimes n_{([1],\alpha)} a_{([1],\alpha)}]) = \sum \xi_\alpha([g_i(n_{([0],i)}) \otimes n_{([1],\alpha)}]) \cdot a = \bar{g}_\alpha(n) \cdot a. \end{aligned}$$

Second we prove that \bar{g} is right partial π - H -colinear. Indeed, for any $n \in N_{\alpha\beta}$,

$$\begin{aligned} (\bar{g}_\alpha \otimes id_{H_\beta}) \rho_{\alpha,\beta}^N(n) &= \sum \xi_\alpha([g_i(n_{([0],\alpha)} n_{([0],i)}) \otimes n_{([0],\alpha)} n_{([1],\alpha)}]) \otimes n_{([1],\beta)} \\ &= \sum \xi_\alpha([g_i(n_{([0],i)}) \cdot 1_{([0],i)} \otimes n_{([1],\alpha\beta)} 1_{([1],\alpha)}]) \otimes n_{([1],\alpha\beta)} 2_\beta \\ &= \sum \rho_{\alpha,\beta}^M \xi_\alpha([g_i(n_{([0],i)}) \otimes n_{([1],\alpha\beta)}]) = \rho_{\alpha,\beta}^M \bar{g}_{\alpha\beta}(n). \end{aligned}$$

Finally we have to show $\bar{g}f = id_M$. In fact, for any $\alpha \in \pi$, $m \in M_\alpha$, we have

$$\begin{aligned} \bar{g}_\alpha f_\alpha(m) &= \sum \xi_\alpha([g_i(f_\alpha(m)_{([0],i)}) \otimes f_\alpha(m)_{([1],\alpha)}]) \\ &= \sum \xi_\alpha([g_i(f_i(m_{([0],i)})) \otimes m_{([1],\alpha)}]) = \sum \xi_\alpha([m_{([0],i)} \otimes m_{([1],\alpha)}]) \\ &= \xi_\alpha \circ p_\alpha \circ \xi_{i,\alpha}(m) = m. \end{aligned}$$

Hence we complete the proof.

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偏 π -余模的Maschke型定理

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摘要: 本文研究了偏群余模的Maschke型定理. 利用弱Hopf群余代数推广Hopf代数的方法, 获得了偏群余模的Maschke型定理. 推广了Hopf代数理论中的Maschke型定理和[8] 的相关结论.

关键词: 偏群余模; 迹映射; Maschke型定理

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