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GEOMETRIC TRANSIENCE FOR NON-LINEAR AUTOREGRESSIVE MODELS

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Abstract: In the paper, we study the stochastic stability for non-linear autoregressive models. By establishing an appropriate Foster-Lyapunov criterion, a sufficient condition for geometric transience is presented.

Keywords: geometric transience; non-linear autoregressive model; Foster-Lyapunov criterion

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1 Introduction

Consider a non-linear autoregressive Markov chain $\{\Phi_n : n \in \mathbb{Z}_+\}$ on $\mathbb{R}$ defined by

$$\Phi_{n+1} = F(\Phi_n) + U_{n+1}, \quad n \in \mathbb{Z}_+,$$

where F: $\mathbb{R} \to \mathbb{R}$ is a continuous function, $\{U_n : n \in \mathbb{Z}_+\}$ is a sequence of i.i.d. random variables with distribution

$$\Gamma(-\infty, x] = \mathbb{P}\{U_n \le x\}, \quad x \in \mathbb{R},$$

and Φ_0 is independent of $\{U_n : n \in \mathbb{Z}_+\}$. Assume that the distribution Γ is absolutely continuous with respect to the Lebesgue measure λ, and has a density which is positive everywhere. The non-linear autoregressive model attracted a large amount of attention in the literature. Most of the studies focued on conditions implying ergodicity, sub-geometric ergodicity and geometric ergodicity, see e.g. [1–4] and references therein. In the paper, we aim to present a sufficient condition for geometric transience for the non-linear autoregressive model.

First, let us recall some notations and definitions, see [3, 5, 6] for details. Denote by $\mathscr{B}(\mathbb{R})$ the Borel σ-field on $\mathbb{R}$, and write $\mathscr{B}^+(\mathbb{R}) = \{A \in \mathscr{B}(\mathbb{R}) : \lambda(A) > 0\}$. The n-step

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transition kernel of the chain Φ_n is defined as

$$P^n(x, A) = \mathbb{P}_x\{\Phi_n \in A\}, \quad n \in \mathbb{Z}_+,\ x \in \mathbb{R},\ A \in \mathscr{B}(\mathbb{R}),$$

where $\mathbb{P}_x$ is the conditional distribution of the chain given $\Phi_0 = x$. The corresponding expectation operator will be denoted by $\mathbb{E}_x$. The operator P acts on non-negative measurable functions f via

$$Pf(x) = \int_{\mathbb{R}} f(y)P(x, dy), \quad x \in \mathbb{R}.$$

The chain Φ_n is Lebesgue-irreducible, if for every $A \in \mathscr{B}^+(\mathbb{R})$,

$$\sum_{n=0}^{\infty} P^n(x, A) > 0, \quad x \in \mathbb{R}.$$

A set $A \in \mathscr{B}(\mathbb{R})$ is called petite, if there exist a probability distribution $a = \{a_n : n \in \mathbb{Z}_+\}$ and a non-trivial measure ν_a satisfying for all $x \in A$ and $B \in \mathscr{B}(\mathbb{R})$,

$$\sum_{n=0}^{\infty} a_n P^n(x, B) \geq \nu_a(B).$$

Obviously, the subset of a petite set is still petite. By [7, Lemma 2.1] or [8, Theorem 1], we know that the non-linear autoregressive model is Lebesgue-irreducible, and every compact set in $\mathscr{B}^+(\mathbb{R})$ is petite.

For $A \in \mathscr{B}(\mathbb{R})$, let

$$\tau_A = \inf\{n \geq 1 : \Phi_n \in A\} \quad \text{and} \quad \sigma_A = \inf\{n \geq 0 : \Phi_n \in A\}$$

be the first return and first hitting times, respectively, on A. It is obvious that $\tau_A = \sigma_A$ if $\Phi_0 \in A^c$. Denote by $L(x, A) = \mathbb{P}_x\{\tau_A < \infty\}$ the probability of the chain Φ_n ever returning to A.

Recall that a set $A \in \mathscr{B}^+(\mathbb{R})$ is called a uniformly geometrically transient set of the chain Φ_n, if there exists a constant $\kappa > 1$ such that

$$\sup_{x \in A} \sum_{n=1}^{\infty} \kappa^n P^n(x, A) < \infty.$$

The chain Φ_n is called geometrically transient, if it is ψ-irreducible for some non-trivial measure ψ, and $\mathbb{R}$ can be covered ψ-a.e. by a countable number of uniformly geometrically transient sets. That is, there exist sets D and A_i, $i = 1, 2, \cdots$ such that $\mathbb{R} = D \cup \left(\bigcup_{i=1}^{\infty} A_i\right)$, where $\psi(D) = 0$ and each A_i is a uniformly geometrically transient set of the chain Φ_n.

To state the main result of this paper, we need the following assumptions:

(A1) $\int \mathrm{e}^{s|x|}\Gamma(dx) < \infty$ for some constant $s > 0$;

(A2) $\liminf\limits_{|x| \to \infty} \dfrac{|F(x)|}{|x|} > 1$.

Theorem 1.1 Assume (A1) and (A2). Then the non-linear autoregressive model Φ_n is geometrically transient.

Remark 1.2 It is easy to see that (A2) is equivalent to the condition in [9, Theorem 3.1], where transience for the the non-linear autoregressive model Φ_n was confirmed. Here, we get a stronger result (i.e. geometric transience) in Theorem 1.1.

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1 by using the Foster-Lyapunov (or drift) condition for geometric transience.

It is well known that Foster-Lyapunov conditions were widely used to study the stochastic stability for Markov chains. For examples, Down, Meyn and Tweedie [10–13] studied the drift conditions for recurrence, ergodicity, geometric ergodicity and uniform ergodicity. The drift conditions for sub-geometric ergodicity were discussed in [1, 4, 14–17] and so on. In [18, 19], the drift conditions for transience were obtained.

Recently, we investigated the drift condition for geometric transience in [6]. One of the main results shows that the chain Φ_n is geometrically transient, if there exist some set $A \in \mathscr{B}^+(\mathbb{R})$, constants $\lambda, b \in (0,1)$, and a function $W \geq 1_A$ (with $W(x_0) < \infty$ for some $x_0 \in \mathbb{R}$) satisfying the drift condition

$$PW(x) \leq \lambda W(x) 1_{A^c}(x) + b 1_A(x), \quad x \in \mathbb{R}.$$

As far as we know, however, this drift condition can not be applied directly for the non-linear autoregressive model considered in this paper. Alternatively, we will establish a more practical drift condition for geometric transience. First, we need the following two lemmas, which are taken from [6].

Lemma 2.1 The chain Φ_n is geometrically transient if and only if there exist some set $A \in \mathscr{B}^+(\mathbb{R})$ and a constant $\kappa > 1$ such that

$$\sup_{x\in A} L(x, A) < 1, \quad \sup_{x\in A} \mathbb{E}_x \left[\kappa^{\tau_A} 1_{\{\tau_A<\infty\}}\right] < \infty.$$

Lemma 2.2 (1) For $A \in \mathscr{B}^+(\mathbb{R})$ and $\kappa \geq 1$,

$$\mathbb{E}_x \left[\kappa^{\tau_A} 1_{\{\tau_A<\infty\}}\right] = \kappa \int_{A^c} \mathbb{E}_y \left[\kappa^{\sigma_A} 1_{\{\sigma_A<\infty\}}\right] P(x, dy) + \kappa P(x, A), \quad x \in \mathbb{R}.$$

(2) $\left\{\mathbb{E}_x \left[\kappa^{\sigma_A} 1_{\{\sigma_A<\infty\}}\right], x \in \mathbb{R}\right\}$ is the minimal non-negative solution to the equations

$$\begin{cases} g(x) = \kappa \displaystyle\int_{A^c} g(y) P(x, dy) + \kappa P(x, A), & x \in A^c, \\ g(x) = 1, & x \in A. \end{cases}$$

Proposition 2.3 The chain Φ_n is geometrically transient, if there exist a petite set $A \in \mathscr{B}^+(\mathbb{R})$, constants $\lambda \in (0,1)$, $b \in (0,\infty)$, and a non-negative measurable function W bounded on A satisfying

$$PW(x) \leq \lambda W(x) + b 1_A(x), \quad x \in \mathbb{R} \tag{2.1}$$

and

$$D := \left\{ x : W(x) < \inf_{y \in A} W(y) \right\} \in \mathscr{B}^+(\mathbb{R}). \tag{2.2}$$

Proof Since W is non-negative and $D \in \mathscr{B}^+(\mathbb{R})$, we have $\inf\limits_{y \in A} W(y) > 0$. Set

$$\overline{W}(x) = \frac{W(x)}{\inf\limits_{y \in A} W(y)}, \quad x \in \mathbb{R}.$$

Then $\overline{W}(x) \geq 1$ for $x \in D^c$, $\overline{W}(x) < 1$ for $x \in D$, and (2.1) yields that

$$\begin{cases} \overline{W}(x) \geq \lambda^{-1} P\overline{W}(x) \geq \lambda^{-1} \displaystyle\int_{A^c} \overline{W}(y) P(x, dy) + \lambda^{-1} P(x, A), & x \in A^c, \\ \overline{W}(x) \geq 1, & x \in A. \end{cases} \tag{2.3}$$

According to Lemma 2.2 (2), $\left\{ \mathbb{E}_x \left[\lambda^{-\sigma_A} 1_{\{\sigma_A < \infty\}} \right], x \in \mathbb{R} \right\}$ is the minimal non-negative solution to the equations

$$\begin{cases} g(x) = \lambda^{-1} \displaystyle\int_{A^c} g(y) P(x, dy) + \lambda^{-1} P(x, A), & x \in A^c, \\ g(x) = 1, & x \in A. \end{cases} \tag{2.4}$$

Hence by the comparison theorem of the minimal non-negative solution (see [20, Theorem 2.6]), we know from (2.3) and (2.4) that

$$\mathbb{E}_x \left[\lambda^{-\sigma_A} 1_{\{\sigma_A < \infty\}} \right] \leq \overline{W}(x), \quad x \in A^c. \tag{2.5}$$

By (2.5) and noting that $D \subset A^c$, we have for all $x \in \mathbb{R}$,

$$\begin{aligned} L(x, A) &= \int_D L(y, A) P(x, dy) + \int_{D^c} L(y, A) P(x, dy) \\ &\leq \int_D L(y, A) P(x, dy) + P(x, D^c) \leq \int_D \mathbb{E}_y \left[\lambda^{-\sigma_A} 1_{\{\sigma_A < \infty\}} \right] P(x, dy) + P(x, D^c) \\ &\leq \int_D \overline{W}(y) P(x, dy) + P(x, D^c) < P(x, D) + P(x, D^c) = 1. \end{aligned}$$

Thus there exists some set $C \subset A$ with $C \in \mathscr{B}^+(\mathbb{R})$ such that

$$\sup_{x \in C} L(x, C) < 1. \tag{2.6}$$

According to Lemma 2.1, in the following, it is enough to prove that for some $\kappa > 1$, $\sup\limits_{x \in C} \mathbb{E}_x \left[\kappa^{\tau_C} 1_{\{\tau_C < \infty\}} \right] < \infty$.

Combining Lemma 2.2 (1) with (2.5) and (2.1), we get for all $x \in A$,

$$\begin{aligned} \mathbb{E}_x \left[\lambda^{-\tau_A} 1_{\{\tau_A < \infty\}} \right] &= \lambda^{-1} \int_{A^c} \mathbb{E}_y \left[\lambda^{-\sigma_A} 1_{\{\sigma_A < \infty\}} \right] P(x, dy) + \lambda^{-1} P(x, A) \\ &\leq \lambda^{-1} \int_{A^c} \overline{W}(y) P(x, dy) + \lambda^{-1} P(x, A) \\ &\leq \lambda^{-1} P\overline{W}(x) \leq \overline{W}(x) + \frac{b}{\lambda \inf\limits_{y \in A} W(y)}. \end{aligned}$$

Since W is bounded on A,

$$\sup_{x\in A}\mathbb{E}_x\left[\lambda^{-\tau_A}1_{\{\tau_A<\infty\}}\right]<\infty. \tag{2.7}$$

Noting that A is petite and $C\subset A$, according to (2.7) and the proof of [3, Theorem 15.2.1], we obtain that for all $1<\kappa\le\lambda^{-1/2}$, $\sup_{x\in C}\mathbb{E}_x\left[\kappa^{\tau_C}1_{\{\tau_C<\infty\}}\right]<\infty$. This together with (2.6) yields the desired assertion.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By (A2), there exist constants $\theta>0$ and $c>0$ satisfying

$$|F(x)|\ge(1+\theta)|x|,\quad |x|\ge c. \tag{2.8}$$

Choose

$$A=\left\{x:|x|<c\vee\frac{1+\log\int \mathrm{e}^{s|x|}\Gamma(dx)}{s\theta}\right\},\quad W(x)=\mathrm{e}^{-s|x|}.$$

Then $A\in\mathscr{B}^+(\mathbb{R})$ is petite and $D\in\mathscr{B}^+(\mathbb{R})$, where D is defined in (2.2). From (A1) and (2.8), we have for $x\in A^c$,

$$\begin{aligned}\frac{PW(x)-W(x)}{W(x)}&=\int\left(\frac{W\left(F(x)+y\right)}{W(x)}-1\right)\Gamma(dy)=\int\left(\mathrm{e}^{-s|F(x)+y|+s|x|}-1\right)\Gamma(dy)\\&\le\int\left(\mathrm{e}^{-s|F(x)|+s|y|+s|x|}-1\right)\Gamma(dy)\le\int\left(\mathrm{e}^{-s(1+\theta)|x|+s|y|+s|x|}-1\right)\Gamma(dy)\\&=\mathrm{e}^{-s\theta|x|}\int\mathrm{e}^{s|y|}\Gamma(dy)-1\le\mathrm{e}^{-1}-1.\end{aligned}$$

That is,

$$PW(x)\le\mathrm{e}^{-1}W(x),\quad x\in A^c. \tag{2.9}$$

Noting that W is bounded, it is obvious that for some $b\in(0,\infty)$,

$$PW(x)\le\mathrm{e}^{-1}W(x)+b,\quad x\in A.$$

Combining this with (2.9), the drift condition (2.1) holds. Thus the non-negative autoregressive model Φ_n is geometrically transient by Proposition 2.3.

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非线性自回归模型的几何非常返性

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摘要: 本文研究了非线性自回归模型的随机稳定性. 通过建立恰当的 Foster-Lyapunov 条件, 得到了非线性自回归模型几何非常返的充分条件.

关键词: 几何非常返; 非线性自回归模型; Foster-Lyapunov 条件

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