Vol. 36 (2016)
No. 5

数 学 杂 志
J. of Math. (PRC)

NUMERICAL STABILITY ANALYSIS FOR EQUATION $x'(t) = ax(t) + bx(3[(t+1)/3])$

WANG Qi, WANG Xiao-ming, CHEN Xue-song

(*School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510006, China*)

Abstract: In this paper, we investigate the numerical stability of Euler-Maclaurin method for differential equation with piecewise constant arguments $x'(t) = ax(t) + bx(3[(t+1)/3])$. By the method of characteristic analysis, the sufficient conditions of stability for the numerical solution are obtained. Moreover, we show that the Euler-Maclaurin method preserves the stability of the exact solution. Finally, some numerical examples are given.

Keywords: Euler-Maclaurin method; piecewise constant arguments; stability; numerical solution

2010 MR Subject Classification: 65L07; 65L20

Document code: A **Article ID:** 0255-7797(2016)05-0955-08

1 Introduction

We are interested in the numerical stability of the Euler-Maclaurin method for the following differential equation with piecewise constant arguments (EPCA):

$$\begin{cases} x'(t) = ax(t) + bx\left(3\left[\dfrac{t+1}{3}\right]\right), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $t > 0$, $a \neq 0$, b and x_0 are real constants and $[\cdot]$ denotes the greatest integer function.

EPCA belongs to one special kind of delay differential equations [1–3]. They described hybrid dynamical systems and combine properties of both differential and difference equations. So EPCA had many applications in science and engineering. In the past twenty years, many researchers investigated the properties of the exact solution of EPCA (see [4–6] and the references therein). In particularly, stability of solutions of EPCA received much attention (see [7–9] and the extensive bibliography therein). For more information on this type of equations, the interested readers can refer Wiener' s book [10]. Recently, special interest was shown to the properties of numerical solution of EPCA, such as stability [11, 12], dissipativity [13] and oscillation [14]. In this paper, we will study the stability of the numerical

* **Received date:** 2014-08-30 **Accepted date:** 2015-03-16

Foundation item: Supported by National Natural Science Foundation of China (11201084); China Postdoctoral Science Foundation (2013M531842) and Science and Technology Program of Guangzhou (2014KP000039).

Biography: Wang Qi (1978–), male, born at Yichun, Heilongjiang, associate professor, major in numerical computation of differential equation.

solution in the Euler-Maclaurin method for (1.1). Whether the numerical method preserves stability of the exact solution is considered. Two numerical examples for demonstrating the theoretical results are also provided.

The following results give the definition and stability of exact solution for (1.1).

Definition 1.1 (see [10]) A solution of (1.1) on $[0,\infty)$ is a function $x(t)$ which satisfies the conditions

(i) $x(t)$ is continuous on $[0,\infty)$;

(ii) the derivative $x'(t)$ exists at each point $t\in[0,\infty)$, with the possible exception of the points $t=3n-1$ for $n\in N$, where one-sided derivatives exist;

(iii) (1.1) is satisfied on each interval $[3n-1,3n+2)$ for $n\in N$.

Theorem 1.2 (see [10]) Assume that a, b and $x_0\in R$, then (1.1) has on $[0,\infty)$ a unique solution $x(t)$ given by

$$x(t)=\lambda(\Omega(t))\left(\frac{\lambda_1}{\lambda_{-1}}\right)^{\left[\frac{t+1}{3}\right]}x_0,$$

where

$$\lambda(t)=e^{at}+\frac{b}{a}(e^{at}-1),\quad \Omega(t)=t-3\left[\frac{t+1}{3}\right],\quad \lambda_1=\lambda(2),\quad \lambda_{-1}=\lambda(-1).$$

Theorem 1.3 (see [10]) The solution $x(t)=0$ of (1.1) is asymptotically stable ($x(t)\to0$ as $t\to\infty$) if and only if any one of the following conditions is satisfied

$$\begin{aligned}&-\frac{a(e^{3a}+1)}{\phi(a)}<b<-a,\quad a>\bar a,\\ &b>-\frac{a(e^{3a}+1)}{\phi(a)}\quad\text{or}\quad b<-a,\quad a<\bar a,\\ &b<-a,\quad a=\bar a,\end{aligned}\tag{1.2}$$

where $\bar a$ is the nonzero solution of equation $\phi(x)=e^{3x}-2e^x+1=0$.

2 Stability of Numerical Solution

2.1 The Euler-Maclaurin Method

Let h be a given stepsize, $m\geq1$ be a given integer and satisfies $h=1/m$. The gridpoints t_i be defined by $t_i=ih$ $(i=0,1,2,\cdots)$. Applying the Euler-Maclaurin formula to (1.1), we have

$$x_{i+1}=x_i+\frac{ha}{2}(x_{i+1}+x_i)-\sum_{j=1}^{n}\frac{B_{2j}(ha)^{2j}}{(2j)!}(x_{i+1}-x_i)+hbx_i^{(n)},\tag{2.1}$$

where B_{2j} denotes the $2j$-th Bernoulli number, x_i and x_{i+1} are approximations to $x(t)$ at t_n and t_{n+1}, respectively, $x_i^{(n)}$ is an approximation to $x(3[(t+1)/3])$ at t_n. Let us denote $i=3km+l$, $l=-m,-m+1,\cdots,2m-1$ for $k\geq1$ and $l=0,1,\cdots,2m-1$ for $k=0$. Then

$x_i^{(n)}$ can be defined as x_{3km} according to Definition 1.1. So we have

$$\left[1 - \frac{ha}{2} + \sum_{j=1}^{n} \frac{B_{2j}(ha)^{2j}}{(2j)!}\right] x_{i+1} = \left[1 + \frac{ha}{2} + \sum_{j=1}^{n} \frac{B_{2j}(ha)^{2j}}{(2j)!}\right] x_i + hbx_{3km}, \tag{2.2}$$

which is equivalent to

$$x_{3km+l+1} = R(z)x_{3km+l} + \frac{b}{a}(R(z) - 1)x_{3km}, \tag{2.3}$$

where

$$z = ha, \quad R(z) = 1 + \frac{z}{\Phi(z)}, \quad \Phi(z) = 1 - \frac{z}{2} + \sum_{j=1}^{n} \frac{B_{2j}z^{2j}}{(2j)!}.$$

Thus

$$\begin{aligned} x_{3km+l} &= \left[R(z)^l + \frac{b}{a}(R(z)^l - 1)\right] x_{3km}, \\ x_{3(k+1)m} &= \frac{R(z)^{2m} + \frac{b}{a}(R(z)^{2m} - 1)}{R(z)^{-m} + \frac{b}{a}(R(z)^{-m} - 1)} x_{3km}. \end{aligned} \tag{2.4}$$

Similar to Theorem 2.2 in [14], we have the following result for convergence.

Theorem 2.1 For any given $n \in N$, the Euler-Maclaurin method is of order $2n + 2$.

2.2 Stability Analysis

Definition 2.2 The Euler-Maclaurin method is called asymptotically stable at (a, b) if there exists a constant M_0 such that x_n defined by (2.3) tends to zero as $n \to \infty$ for all $h = 1/m$ and any given x_0.

Lemma 2.3 (see [15]) If $|z| < 1$, then $\Phi(z) \geq 1/2$ for $z > 0$ and $\Phi(z) \geq 1$ for $z < 0$.

Lemma 2.4 (see [15]) If $|z| < 1$, then

$$\Phi(z) \leq \frac{z}{e^z - 1}$$

for n is even and

$$\Phi(z) \geq \frac{z}{e^z - 1}$$

for n is odd.

Theorem 2.5 The Euler-Maclaurin method is asymptotically stable if any one of the following conditions is satisfied

$$\begin{aligned} &-\frac{a(R(z)^{3m} + 1)}{\bar{\phi}(z)} < b < -a, \quad a > a^0, \\ &b > -\frac{a(R(z)^{3m} + 1)}{\bar{\phi}(z)} \quad \text{or} \quad b < -a, \quad a < a^0, \\ &b < -a, \quad a = a^0, \end{aligned} \tag{2.5}$$

where $z = ha$, $\bar{\phi}(z) = R(z)^{3m} - 2R(z)^m + 1$, a^0 is the nonzero solution of equation $\bar{\phi}(z) = 0$.

Proof Let
$$\overline{\lambda_1} = R(z)^{2m} + \frac{b}{a}(R(z)^{2m} - 1)$$
and
$$\overline{\lambda_{-1}} = R(z)^{-m} + \frac{b}{a}(R(z)^{-m} - 1),$$
so we need to verify
$$\left|\frac{\overline{\lambda_1}}{\overline{\lambda_{-1}}}\right| < 1. \tag{2.6}$$
Assume $\overline{\lambda_{-1}} > 0$, i.e.,
$$b < \frac{a}{R(z)^m - 1}.$$
Then (2.6) is equivalent to
$$\begin{aligned} &-\frac{a(R(z)^{3m}+1)}{\bar{\phi}(z)} < b < -a, \quad a > a^0, \\ &b < -a, \quad a < a^0, \\ &b < -a, \quad a = a^0. \end{aligned}$$
Assume $\overline{\lambda_{-1}} < 0$, i.e.,
$$b > \frac{a}{R(z)^m - 1}.$$
Then (2.6) is equivalent to
$$b > -\frac{a(R(z)^{3m}+1)}{\bar{\phi}(z)}, \quad a < a^0.$$
The proof is completed.

The following two lemmas are given naturally.

Lemma 2.6 Let $f(r) = r^3 - 2r + 1$, $r > 0$, then

(a) the function $f(r)$ has a minimum at $r_1 = \sqrt{2}/\sqrt{3}$, and $f(r)$ is decreasing in $[0, r_1)$ and increasing in $[r_1, +\infty)$;

(b) the function $f(r)$ has a unique solution $1 > r_0 \neq 1$;

(c) $f(r) < 0$ if $r \in [r_0, 1)$ and $f(r) > 0$ if $r \in [0, r_0)$ or $r \in [1, +\infty)$.

Lemma 2.7 Let
$$g(\omega) = \frac{\omega^3 + 1}{\omega^3 - 2\omega + 1},$$
then

(a) the function $g(\omega)$ has extremum at $\omega_1 = 1/\sqrt[3]{2}$;

(b) $g(\omega)$ is increasing in $(0, r_0)$ and (r_0, ω_1);

(c) $g(\omega)$ is decreasing in $(\omega_1, 1)$ and $(1, +\infty)$.

By Lemmas 2.6 and 2.7, we obtain

Corollary 2.8 Assume that $r_0 \neq 1$ is a unique solution of the function $f(r) = r^3 - 2r + 1$, then $r_0 < \omega_1 < r_1 < 1$.

So we have the following result.

Theorem 2.9 Assume that (1.1) is asymptotically stable, then the Euler-Maclaurin method is asymptotically stable if one of the following conditions is satisfied

(a) $R(z)^m \le e^a \quad (a \le \ln \omega_1)$;

(b) $R(z)^m \ge e^a \quad (\ln \omega_1 < a < 0)$;

(c) $R(z)^m \le e^a \quad (a \ge 0)$.

Proof In view of Theorems 1.2 and 2.5, we will prove that condition (2.5) is satisfied under condition (1.2).

If (a) holds, then we know from Lemmas 2.3 and 2.4 that $f(r)$ is decreasing and $g(\omega)$ is increasing. Hence $\bar{a} < a^0$ and

$$-\frac{a(R(z)^{3m}+1)}{\bar{\phi}(z)} \le -\frac{a(e^{3a}+1)}{\phi(a)}. \tag{2.7}$$

If $a > \bar{a}$, then the first inequality of (1.2) holds. Then by (2.7) we get the first inequality of (2.5). If $a < \bar{a}$, then the second inequality of (1.2) holds. Then by (2.7) we obtain the second inequality of (2.5). If $a = \bar{a}$, then the third inequality of (1.2) holds which implies the first inequality of (2.5). The other cases can be proved in the same way. The proof is completed.

From Lemmas 2.3, 2.4 and Theorem 2.9, we have the following main result in this paper.

Theorem 2.10 The Euler-Maclaurin method preserves the stability of (1.1) if one of the following conditions is satisfied

(a) n is odd if $e^a > \omega_1$,

(b) n is even if $e^a \le \omega_1$.

3 Numerical Experiments

Consider the following two problems

$$\begin{cases} x'(t) = x(t) - 1.2x\left(3\left[\dfrac{t+1}{3}\right]\right), \\ x(0) = 1 \end{cases} \tag{3.1}$$

and

$$\begin{cases} x'(t) = -x(t) + 4x\left(3\left[\dfrac{t+1}{3}\right]\right), \\ x(0) = 1. \end{cases} \tag{3.2}$$

In Figures 1 and 2, we plot the exact solution and the numerical solution for (3.1), respectively. Moreover, for (3.2), we also plot the exact solution and the numerical solution in Figures 3 and 4, respectively. We can see from these figures that the Euler-Maclaurin method preserves the stability of (3.1) and (3.2), which is coincide with Theorem 2.10.

Acknowledgements The authors would like to thank the anonymous reviewers for their careful reading. Many thanks to Professors Mingzhu Liu, Minghui Song and Zhanwen Yang for their great help and valuable suggestions.

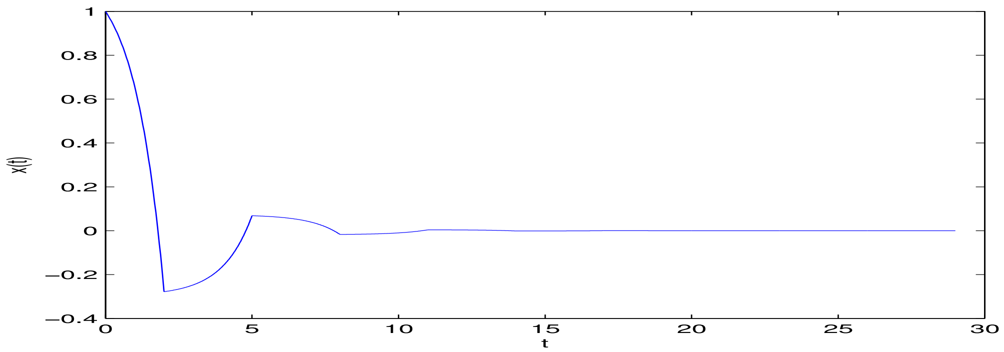

Figure 1: the exact solution of (3.1)

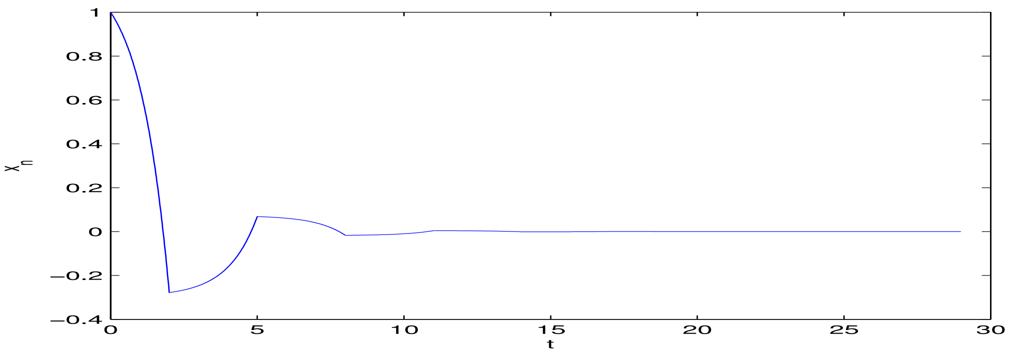

Figure 2: the numerical solution of (3.1) with $n = 3$ and $m = 50$

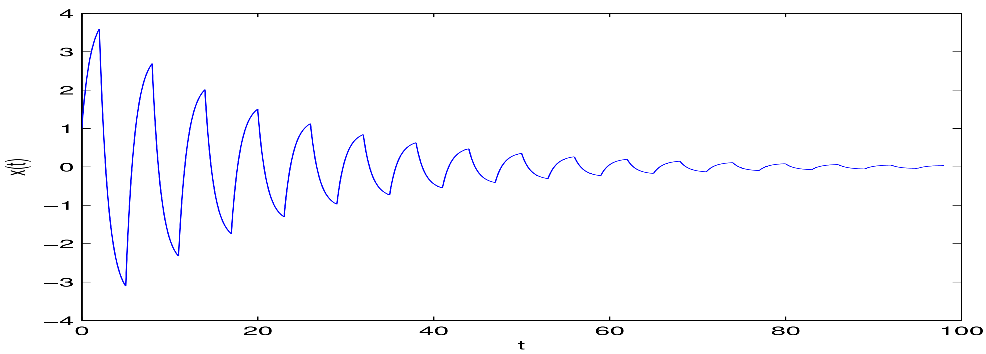

Figure 3: the exact solution of (3.2)

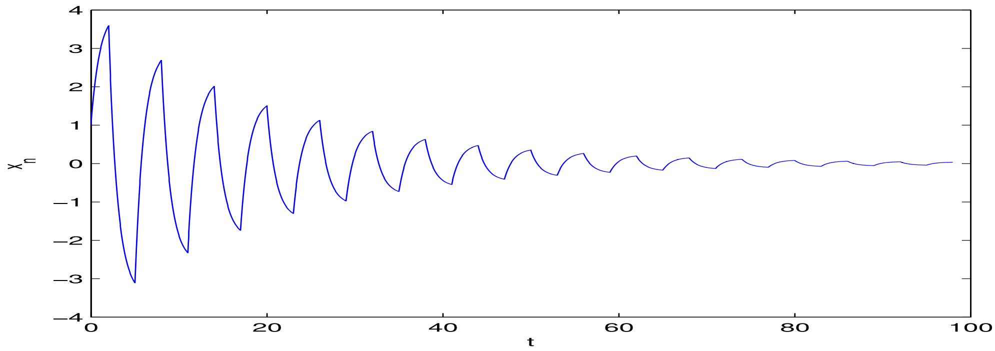

Figure 4: the numerical solution of (3.2) with $n = 2$ and $m = 40$

References

[1] Cheng Shengmin, Zhou Shaobo. Convergence and stability of numerical methods for stochastic differential delay equation[J]. J. Math., 2014, 34(6): 1073–1084.

[2] Wang Xiao, Cui Cheng, Xiao Li, et al. Existence and uniqueness of solutions for differential equations with time delay and impulsive differential equations with time delay[J]. J. Math., 2013, 33(4): 683–688.

[3] Ashyralyev A, Agirseven D. On convergence of difference schemes for delay parabolic equations[J]. Comput. Math. Appl., 2013, 66(7): 1232–1244.

[4] Cooke K L, Wiener J. Retarded differential equations with piecewise constant delays[J]. J. Math. Anal. Appl., 1984, 99(1): 265–297.

[5] Wiener J, Aftabizadeh A R. Differential equations alternately of retarded and advanced type[J]. J. Math. Anal. Appl., 1988, 129(1): 243–255.

[6] Liang Haihua, Wang Genqiang. Oscillation criteria of certain third-order differential equation with piecewise constant argument[J]. J. Appl. Math., 2012, 2012: 1–18.

[7] Alwan M S, Liu Xinzhi, Xie Weichau. Comparison principle and stability of differential equations with piecewise constant arguments[J]. J. Franklin I., 2013, 350(2): 211–230.

[8] Akhmet M U. Stability of differential equations with piecewise constant arguments of generalized type[J]. Nonl. Anal., 2008, 68(4): 794–803.

[9] Li Huaixing, Muroya Y, Nakata Y, et al. Global stability of nonautonomous logistic equations with a piecewise constant delay[J]. Nonl. Anal., RWA, 2010, 11(3): 2115–2126.

[10] Wiener J. Generalized solutions of functional differential equations[M]. Singapore: World Scientific, 1993.

[11] Song Minghui, Yang Zhanwen, Liu Mingzhu. Stability of θ-methods for advanced differential equations with piecewise continuous arguments[J]. Comput. Math. Appl., 2005, 49(9-10): 1295–1301.

[12] Liang Hui, Liu Mingzhu, Yang Zhanwen. Stability analysis of Runge-Kutta methods for systems $u'(t) = Lu(t) + Mu([t])$[J]. Appl. Math. Comput., 2014, 228(1): 463–476.

[13] Wang Wansheng, Li Shoufu. Dissipativity of Runge-Kutta methods for neutral delay differential equations with piecewise constant delay[J]. Appl. Math. Lett., 2008, 21(9): 983–991.

[14] Liu Mingzhu, Gao Jianfang, Yang Zhanwen. Preservation of oscillations of the Runge-Kutta method for equation $x'(t) + ax([t]) + a_1x([t-1]) = 0$[J]. Comput. Math. Appl., 2009, 58(6): 1113–1125.

[15] Lü Wanjin, Yang Zhanwen, Liu Mingzhu. Stability of the Euler-Maclaurin methods for neutral differential equations with piecewise continuous arguments[J]. Appl. Math. Comput., 2007, 186(2): 1480–1487.

方程$x'(t)=ax(t)+bx(3[(t+1)/3])$的数值稳定性分析

王 琦, 汪小明, 陈学松

(广东工业大学应用数学学院, 广东 广州 510006)

摘要: 本文研究了分段连续型微分方程$x'(t)=ax(t)+bx(3[(t+1)/3])$ Euler-Maclaurin方法的数值稳定性问题. 利用特征分析的方法, 获得了数值解稳定的充分条件, 进而证明了Euler-Maclaurin方法保持了精确解的稳定性. 最后给出了一些数值例子.

关键词: Euler-Maclaurin方法; 分段连续项; 稳定性; 数值解

MR(2010)**主题分类号**: 65L07; 65L20 **中图分类号**: O241.81