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THE MINIMAL SOLUTION OF A SPECIAL ANTICIPATED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

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Abstract: In this paper, we study the problem of a minimal solution to a special class of anticipated backward stochastic differential equation. When the generator is continuous and satisfying a similar linear growth condition, we prove the existence of minimal solutions. Here, our hypotheses are weaker than the before papers, however, we obtain a better lemma and the same result.

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1 Introduction

The notions of non-linear backward stochastic differential equations (BSDEs) were introduced by Pardoux and Peng [11]. A solution of this equation, associated with a terminal value ξ and a generator or coefficient $f(t, \omega, y, z)$, is a couple of adapted stochastic processes $(Y(t), Z(t))_{\{t\in[0,T]\}}$ such that

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s),$$

where W is a d-dimensional standard Brownian motion. This type of nonlinear backward stochastic differential equations were first studied by Pardoux and Peng in [11], and they established the existenceness and uniqueness of adapted solution under the global Lipschitz condition. Since then, many people try to weaken the conditions of generators to get the same results and study some different forms of BSDEs. For examples, Aman and Nz'i [1] studied BSDEs with oblique reflection and local Lipschitz. Bahlali [2] studied backward

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stochastic differential equations with locally Lipschitz coefficients. Situ [9] and Royer [10] studied BSDEs with jumps. It is now well-known that BSDEs provide a useful framework for formulating a lot of mathematical problems such as used in financial mathematics, optimal control, stochastic games and partial differential equations (see [12–14]). Based on the above applications, specially in the field of finance, and optimal control, recently, a new type of BSDEs, called anticipated BSDEs (ABSDEs), were introduced by Peng and Yang [4] as the following

$$\begin{cases} Y(t)=\xi(T)+\int_t^T f(s,Y(s),Z(s),Y(s+\theta(s)),Z(s+\vartheta(s)))\mathrm{d}s-\int_t^T Z(s)\mathrm{d}W(s), t\in[0,T],\\ Y(t)=\xi(t), t\in[T,T+K],\\ Z(t)=\eta(t), t\in[T,T+K], \end{cases} \tag{1.1}$$

where $\theta(\cdot):[0,T]\to\mathbb{R}^+$, $\vartheta(\cdot):[0,T]\to\mathbb{R}^+$ are continuous functions and satisfy that

(i) there exists a constant $K\geqslant 0$ such that for each $t\in[0,T]$,

$$t+\theta(t)\leqslant T+K,\ t+\vartheta(t)\leqslant T+K;$$

(ii) there exists a constant $L\geqslant 0$ such that for each $t\in[0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s+\theta(s))\mathrm{d}s\leqslant L\int_t^{T+K} g(s)\mathrm{d}s,\quad \int_t^T g(s+\vartheta(s))\mathrm{d}s\leqslant L\int_t^{T+K} g(s)\mathrm{d}s.$$

Under global Lipschitz conditions, Peng and Yang proved the existencenee and uniqueness of solution (see Theorem 4.2 in [4]).

For anticipated BSDEs, we mention that the generator includes not only the values of solutions of presents but also the future. So ABSDEs may be used in finance. From Theorem 2.1 in [4], we know that there is a duality between stochastic differential equations with delay and anticipated BSDEs which can be used in optimal control. We also mention that, following Peng and Pardoux [11], many papers were devoted to BSDEs with continuous coefficients. Especially, many scholars studied the minimal solution of BSDEs, it is refered to [3, 5–8].

Motivated by the above papers, in this paper, we study a special class of 1-dimension ABSDEs as the following

$$\begin{cases} Y(t)=\xi(T)+\int_t^T f(s,Y(s),Z(s),\mathbb{E}^{\mathcal{F}_s}Y(s+\theta(s)))\mathrm{d}s-\int_t^T Z(s)\mathrm{d}W(s), & t\in[0,T],\\ Y(t)=\xi(t), & t\in[T,T+K]. \end{cases} \tag{1.2}$$

Set $g(t,y,z,\mu(r))\doteq f(t,y,z,E^{\mathcal{F}_t}\mu(r))$, where $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\mu(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R})$, $r\in[t,T+K]$. We get the minimal solution of this type of ABSDEs with continuous coefficients. Furthermore, we give an application of the minimal-solution theorem.

2 Main Reaults

Before starting our main results, we give some necessary notions and hypotheses.

2.1 Preliminaries

Let $(\Omega,\mathcal{F},P)$ be a complete probability space, and let $(W(t))_{t\in[0,T]}$ be a d-dimensional standard Brownian motion on $(\Omega,\mathcal{F},P)$. Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the natural filtration generated by W.

Now, we give the definitions of some spaces and norms, which will be used later. For $x,y\in\mathbb{R}^k$, we denote by $|x|$ the Euclidean norm of x, and denote by (x,y) the Euclidean inner product. For $a\in\mathbb{R}^{k\times d}$, let $|a|\doteq\sqrt{\mathrm{Tr}aa^*}$.

- $L^2(\mathcal{F}_T;\mathbb{R})\doteq\{\mathbb{R}$-valued $\mathcal{F}_T$-measurable random variables such that $\mathbb{E}[|\xi|^2]<\infty\}$;
- $L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)\doteq\{\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted random processes such that

$$\mathbb{E}\int_0^T|\varphi(t)|^2\mathrm{d}t<\infty\};$$

- $S^2_{\mathcal{F}}(0,T;\mathbb{R})\doteq\{$continuous processes in $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ such that $\mathbb{E}[\sup\limits_{t\in[0,T]}|\varphi(t)|^2]<\infty\}$.

We also need the following assumptions.

(H1) Assume that for all $t\in[0,T]$, $g(t,\omega,y,z,\mu):[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^2(\mathcal{F}_r;\mathbb{R})\to L^2(\mathcal{F}_t;\mathbb{R})$, where $r\in[t,T+K]$, and g satisfies the following conditions

$$\mathbb{E}\left[\int_0^T|g(t,0,0,0)|^2\mathrm{d}s\right]<\infty.$$

(H2) For all $t\in[0,T],y\in\mathbb{R},z\in\mathbb{R}^d,\mu(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}),r\in[t,T+K]$, we have

$$|g(t,y,z,\mu(r))|\leqslant u_1(t)(f_t+|y|+\mathbb{E}^{\mathcal{F}_t}[(\mu(r))^-])+u_2(t)|z|,$$

where $f_t=g(t,0,0,0)$ is a adapted process which satisfies $\mathbb{E}\displaystyle\int_0^T|f_t|^2\mathrm{d}t<\infty$. $u_1(t)$ and $u_2(t)$ are nonnegative, deterministic real functions and satisfy

$$\int_0^\infty(u_1(t)+u_1^2(t))\mathrm{d}t+\int_0^\infty u_2^2(t)\mathrm{d}t<\infty,$$

moreover, $u_1(t)\leqslant u_1(t+\theta(t))$, $\theta(t)$ satisfies (i) and (ii).

(H3) For any $t\in[0,T],y\in\mathbb{R},z\in\mathbb{R}^d$, $g(t,y,z,\cdot)$ is increasing, and for fixed $t\in[0,T]$, $g(t,\cdot,\cdot,\cdot)$ satisfies the following jointly continuous condition: $y,y^n\in\mathbb{R}$, $z,z^n\in\mathbb{R}^d$, $\mu(\cdot),\mu^n(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}),r\in[t,T+K]$, and if $|y^n-y|\to0$, $|z^n-z|\to0$, $\mathbb{E}\sup\limits_{s\in[t,T+K]}|\mu^n(s)-\mu(s)|^2\to0,n\to\infty$, we have $|g(t,y^n,z^n,\mu^n(r))-g(t,y,z,\mu(r))|\to0,n\to\infty$.

Lemma 2.1 Set

$$\begin{aligned}g_n(t,y,z,\mu(r))=\inf_{(u,q)\in\mathbb{R}^{d+1};\ \nu\in L^2(\mathcal{F}_r;R)}\Big[&g(t,u,q,\nu(r))+u_1(t)n|u-y|\\&+u_2(t)n|q-z|+u_1(t)n\mathbb{E}^{\mathcal{F}_t}(\mu(r)-\nu(r))^+\Big],\end{aligned}$$

then $g_n(t,y,z,\mu(r))$ has the following properties.

(a) Linear growth: for any $t\in[0,T], y\in\mathbb{R}, z\in\mathbb{R}^d, \mu(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}), r\in[t,T+K]$, we have

$$|g_n(t,y,z,\mu(r)|\leqslant u_1(t)(f_t+|y|+\mathbb{E}^{\mathcal{F}_t}[(\mu(r))^-])+u_2(t)|z|.$$

(b) Monotone property in n : for any $t\in[0,T], y\in\mathbb{R}, z\in\mathbb{R}^d, \mu(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}), r\in[t,T+K]$, $g_n(t,y,z,\mu(r))\leqslant g_{n+1}(t,y,z,\mu(r))\leqslant g(t,y,z,\mu(r))$, and $g_n(t,y,z,\cdot)$ is increasing.

(c) Lipschitz condition: for any $t\in[0,T], y,y'\in\mathbb{R}, z,z'\in\mathbb{R}^d, \mu(\cdot),\mu'(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}), r\in[t,T+K]$,

$$|g_n(t,y,z,\mu(r))-g_n(t,y',z',\mu'(r))|\leqslant u_1(t)|y-y'|+u_2(t)|z-z'|+u_1(t)\mathbb{E}^{\mathcal{F}_t}|\mu(r)-\mu'(r)|.$$

(d) Strong convergence: if $(y^n,z^n)\to(y,z)$, $y,y^n\in\mathbb{R}, z,z^n\in\mathbb{R}^d$, $\mu(\cdot),\mu^n(\cdot)\in L^2(\mathcal{F}_r;\mathbb{R}), r\in[t,T+K]$ and $\mathbb{E}\sup\limits_{s\in[t,T+K]}|\mu^n(s)-\mu(s)|^2\to 0$, we have

$$g_n(t,y^n,z^n,\mu^n(r))\to g(t,y,z,\mu(r)), n\to\infty.$$

Proof We use the similar method as used in [3, 6] to prove (a), (b) and (c) are obvious. We only need to prove (d). By the definition of infimum, for each $n\in\mathbb{N}$, $n>1$, there exist $u^n\in\mathbb{R}, q^n\in\mathbb{R}^d, \nu^n\in L^2(\mathcal{F}_t;\mathbb{R})$, $r\in[t,T+K]$, such that

$$\begin{aligned}
&g(t,y^n,z^n,\mu^n(r))\geqslant g_n(t,y^n,z^n,\mu^n(r))\\
\geqslant\ &g(t,u^n,q^n,\nu^n(r))+u_1(t)n|u^n-y^n|+u_2(t)n|q^n-z^n|+u_1(t)n\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+-\frac{1}{n}\\
\geqslant\ &-[u_1(t)(f_t+|y^n|+\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-)+u_2(t)|z^n|]-[u_1(t)(f_t+|u^n|+\mathbb{E}^{\mathcal{F}_t}(\nu^n(r))^-)+u_2(t)|q^n|]\\
&+[u_1(t)(f_t+|y^n|+\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-)+u_2(t)|z^n|]+u_1(t)n|u^n-y^n|+u_2(t)n|q^n-z^n|\\
&+u_1(t)n\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+-\frac{1}{n}\\
\geqslant\ &-[u_1(t)(f_t+|y^n|+\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^+)+u_2(t)|z^n|]-[u_1(t)|u^n-y^n|+u_2(t)|q^n-z^n|\\
&+u_1(t)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^-]+u_1(t)n|u^n-y^n|+u_2(t)n|q^n-z^n|\\
&+u_1(t)n\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+-\frac{1}{n}\\
\geqslant\ &-[u_1(t)(f_t+|y^n|+\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-)+u_2(t)|z^n|]+u_1(t)(n-1)|u^n-y^n|+u_2(t)(n-1)|q^n-z^n|\\
&+u_1(t)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+-\frac{1}{n}.
\end{aligned}\tag{2.1}$$

For the above proof, we apply the triangle inequality $a^\pm-b^\pm\leqslant(a-b)^\pm$ and $a^-=(-a)^+$.

Thus we have

$$\begin{aligned}
&u_1(t)(n-1)|u^n-y^n|+u_2(t)(n-1)|q^n-z^n|+u_1(t)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+-\frac{1}{n}\\
\leqslant\ &2[u_1(t)(f_t+|y^n|+\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-)+u_2(t)|z^n|].
\end{aligned}\tag{2.2}$$

Since $\mathbb{E}[|\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-|^2]\leqslant\mathbb{E}[\mathbb{E}^{\mathcal{F}_t}|(\mu^n(r))|^2]\leqslant\mathbb{E}|\mu^n(r)|^2<\infty$, then when $n\in\mathbb{N}$, $n>1$, we derive

$$\begin{aligned}
&\limsup_{n\to\infty}u_1(t)(n-1)|u^n-y^n|<\infty,\quad \limsup_{n\to\infty}u_2(t)(n-1)|q^n-z^n|<\infty,\\
&\limsup_{n\to\infty}u_1(t)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r)-\nu^n(r))^+<\infty,
\end{aligned}$$

and $\lim\limits_{n\to\infty} u^n = y$, $\lim\limits_{n\to\infty} q^n = z$. By (2.2), we have

$$\begin{aligned}
&\mathbb{E}[|2u_1(t)(f_t + |y^n| + \mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-) + 2u_2(t)|z^n| + \frac{1}{n}|^2] \\
\leqslant\ & 4\mathbb{E}[4u_1^2(t)|y^n|^2 + 4u_2^2(t)|z^n|^2 + 4u_1^2(t)|\mathbb{E}^{\mathcal{F}_t}(\mu^n(r))^-|^2 + (2u_1(t)f_t + \frac{1}{n})^2] \\
\leqslant\ & 4\mathbb{E}[4u_1^2(t)|y^n|^2 + 4u_2^2(t)|z^n|^2 + 4u_1^2(t)|(\mu^n(r))^-|^2] + C' < \infty,
\end{aligned}$$

therefore

$$\limsup_{n\to\infty} \mathbb{E}[(u_1(t)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+)^2] < \infty. \tag{2.3}$$

For an appropriate $A > 0$, there exists a $\mathbb{N} > 0$, such that for any $n > \mathbb{N}$,

$$u_1(t)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+ \leqslant A$$

and

$$(\sup_{t\in[0,T]} u_1(t) + 1)(n-1)\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+ \leqslant A.$$

Then

$$\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+ \leqslant \frac{A}{(n-1)(\sup\limits_{t\in[0,T]} u_1(t) + 1)}. \tag{2.4}$$

By the above inequality, we know $\{\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+; n \in \mathbb{N}, n > 1\}$ is bounded in $L^2(\mathcal{F}_t;\mathbb{R})$, with (2.4), we get

$$\lim_{n\to\infty} \mathbb{E}[(\mathbb{E}^{\mathcal{F}_t}(\mu^n(r) - \nu^n(r))^+)^2] = 0.$$

From (2.4), we also have $\mathbb{E}^{\mathcal{F}_t}[\nu^n(r)] \geqslant \mathbb{E}^{\mathcal{F}_t}[\mu^n(r)] - \frac{A}{(n-1)(\sup\limits_{t\in[0,T]} u_1(t)+1)}$.

On the other hand, since $g(t,y,z,\cdot)$ is increasing and (2.3), we have

$$g(t,y^n,z^n,\mu^n(r)) \geqslant g_n(t,y^n,z^n,\mu^n(r)) \geqslant g\left(t,u^n,q^n,\mu^n(r) - \frac{A}{(n-1)(\sup\limits_{t\in[0,T]} u_1(t) + 1)}\right) - \frac{1}{n}.$$

Since g is continuous in $L^2(\mathcal{F};\mathbb{R})$, we have

$$\lim_{n\to\infty} g(t,u^2,q^2,\mu^n(r) - \frac{A}{(n-1)(\sup\limits_{t\in[0,T]} u_1(t) + 1)}) = g(t,y,z,\mu).$$

From assumption (H3), we obtain $\lim\limits_{n\to\infty} g_n(t,y^n,z^n,\mu^n(r)) = g(t,y,z,\mu(r))$.

Consider the following equations

$$\begin{cases} Y^n(t) = \xi(T) + \displaystyle\int_t^T g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\mathrm{d}s - \int_t^T Z^n(s)\mathrm{d}W(s), & t\in[0,T], \\ Y^n(t) = \xi(t), & t\in[T,T+K]. \end{cases}$$

$$\begin{cases} U(t) = \xi(T) + \displaystyle\int_t^T l(s,U(s),V(s),U(s+\theta(s)))\mathrm{d}s - \int_t^T V(s)\mathrm{d}W(s), & t\in[0,T], \\ U(t) = \xi(t), & t\in[T,T+K], \end{cases}$$

where $l(t,y,z,\mu(r)) = C(f_t+|y|+|z|+\mathbb{E}^{\mathcal{F}_t}(\mu(r))^-)$, by the comparison theorem in [4], for any $t\in[0,T+K]$, $n\geqslant m, m,n\in N, U(t)\geqslant Y^n(t)\geqslant Y^m(t)$ a.e..

Before giving our main result, we give the following lemma.

Lemma 2.2 Assume that $\xi(\cdot)\in S^2_{\mathcal{F}}(T,T+K;\mathbb{R})$, then there exists a constant $M>0$ which only depends on $T+K$, L, $\mathbb{E}\sup\limits_{t\in[T,T+K]}|\xi(t)|^2$, $\int_0^\infty (u_1(t)+u_2^2(t))\mathrm{d}t$, $\mathbb{E}\int_0^T |f_t|^2\mathrm{d}t$ such that

$$\mathbb{E}\sup_{t\in[0,T+K]}|Y^n(t)|^2\mathrm{d}t\leqslant M,\quad \mathbb{E}\int_0^T|Z^n(t)|^2\mathrm{d}t\leqslant M.$$

Proof Using Itô's formula to $|Y^n(t)|^2$, we have

$$\begin{aligned}\mathbb{E}|\xi(T)|^2 = {} & \mathbb{E}|Y^n(t)|^2 - 2\mathbb{E}\int_t^T Y^n(s)g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\mathrm{d}s\\ & +\mathbb{E}\int_t^T|Z^n(s)|^2\mathrm{d}s.\end{aligned}\tag{2.5}$$

Thus by (H1)–(H3), (i), (ii) in introduction and Lemma 2.1 (b), Young's inequality, Fubini's lemma, $(a+b+c)^2\leqslant C(a^2+b^2+c^2)$, Hölder's inequality, we have

$$\begin{aligned}
& \mathbb{E}|Y^n(t)|^2+\mathbb{E}\int_t^T|Z^n(s)|^2\mathrm{d}s\\
= {} & \mathbb{E}|\xi(T)|^2+2\mathbb{E}\int_t^T Y^n(s)g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\mathrm{d}s\\
\leqslant {} & \mathbb{E}|\xi(T)|^2+2\mathbb{E}\int_t^T Y^n(s)(u_1(s)f_s+u_1(s)|Y^n(s)|+u_2(s)|Z^n(s)|\\
& +u_1(s)\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))|)\mathrm{d}s\\
\leqslant {} & \mathbb{E}|\xi(T)|^2+\mathbb{E}\int_t^T(3\beta+\frac{L}{\beta})u_1(s)|Y^n(s)|^2\mathrm{d}s+\mathbb{E}\int_t^T\frac{1}{\beta}|Z^n(s)|^2\mathrm{d}s+\mathbb{E}\int_t^T\frac{1}{\beta}u_1(s)f_s^2\mathrm{d}s\\
& +\mathbb{E}\int_T^{T+K}\frac{L}{\beta}u_1(s)|\xi(s)|^2\mathrm{d}s\\
\leqslant {} & C'_\beta+C'_\beta\mathbb{E}\int_t^T|Y^n(s)|^2\mathrm{d}s+\frac{1}{\beta}\mathbb{E}\int_t^T|Z^n(s)|^2\mathrm{d}s.
\end{aligned}\tag{2.6}$$

So choose a fixed $\beta>0$ such that $\frac{1}{\beta}=\frac{1}{2}$, we have

$$\mathbb{E}|Y^n(t)|^2+\frac{1}{2}\mathbb{E}\int_t^T|Z^n(s)|^2\mathrm{d}s\leqslant C'\{1+\mathbb{E}\int_t^T|Y^n(s)|^2\mathrm{d}s\}.\tag{2.7}$$

By Gronwall's lemma, we obtain

$$\sup_n\mathbb{E}\int_0^T|Y^n(s)|^2\mathrm{d}s<\infty.$$

Thus

$$\sup_n\mathbb{E}\int_0^T|Z^n(s)|^2\mathrm{d}s<\infty.$$

At last, by BDG inequality, there exists a constant M, which only depends on $T+K$, L, $\mathbb{E}\sup\limits_{t\in[T,T+K]}|\xi(t)|^2$, $\int_0^\infty(u_1(t)+u_2^2(t))\mathrm{d}t$, $\mathbb{E}\int_0^T|f_t|^2\mathrm{d}t$ such that

$$\mathbb{E}\sup_{t\in[0,T+K]}|Y^n(t)|^2\leqslant M,\quad \mathbb{E}\int_0^T|Z^n(t)|^2\mathrm{d}t\leqslant M.$$

Theorem 2.3 (Minimal-solution theorem) Under assumptions (H1)–(H3), (i), (ii), equation (1.2) has a minimal solution, that is if Y' is another solution of equation (1.2). Then for any given terminal value $\xi(\cdot)\in S^2_{\mathcal{F}}(T,T+K;\mathbb{R})$, we have

$$Y(t)\leqslant Y'(t),\quad \text{a.e.},\quad \text{for}\quad \text{all}\quad t\in[0,T+K].$$

Proof Due to for any $t\in[0,T+K]$, $n\geqslant m, m,n\in\mathbb{N}, U(t)\geqslant Y^n(t)\geqslant Y^m(t)$ a.e., there exists a stochastic process $\{Y(t),t\in[0,T+K]\}$ such that $Y^n(t)\uparrow Y(t), n\to\infty$. Due to the monotone convergence theorem, $\mathbb{E}\int_0^{T+K}|Y^n(t)-Y(t)|^2\mathrm{d}t\to 0, n\to\infty$. Using Itô's formula to $|Y^n(t)-Y^m(t)|^2$, we obtain

$$\begin{aligned}&\mathbb{E}|Y^n(t)-Y^m(t)|^2+\mathbb{E}\int_t^T|Z^n(s)-Z^m(s)|^2\mathrm{d}s\\ \leqslant\;& 2\mathbb{E}\int_t^T(Y^n(s)-Y^m(s))(g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\\ &-g_m(s,Y^m(s),Z^m(s),Y^m(s+\theta(s)))\mathrm{d}s.\end{aligned}$$

By Lemma 2.1, Lemma 2.2, we have

$$\begin{aligned}&\mathbb{E}\int_t^T|Z^n(s)-Z^m(s)|^2\mathrm{d}s\\ \leqslant\;& 2\left(\mathbb{E}\int_0^T|Y^n(s)-Y^m(s)|^2\mathrm{d}s\right)^{\frac12}\left(\mathbb{E}\int_0^T|g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\right.\\ &\left.-g_m(s,Y^m(s),Z^m(s),Y^m(s+\theta(s)))|^2\mathrm{d}s\right)^{\frac12}\\ \leqslant\;& C'\left(\mathbb{E}\int_0^T|Y^n(s)-Y^m(s)|^2\mathrm{d}s\right)^{\frac12}.\end{aligned}$$

Thus

$$\mathbb{E}\int_0^T|Z^n(s)-Z^m(s)|^2\mathrm{d}s\to 0, n,m\to\infty.$$

So there exists a $Z\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ such that $\lim\limits_{n\to\infty}\mathbb{E}\int_0^T|Z^n(s)-Z(s)|^2\mathrm{d}s=0$. Using Itô's formula, BDG inequality and Lemma 2.1, Lemma 2.2, we can easily obtain

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T+K]}|Y^n(t)-Y(t)|^2\mathrm{d}t=0.$$

Furthermore, there exists a subsequence of $\{n\}$, which we still denote this subsequence by $\{n\}$ such that

$$\mathbb{E}\int_0^{T+K}|Y^n(t)-Y(t)|\mathrm{d}t\leqslant\frac{1}{2^n},\quad \mathbb{E}\int_0^{T}|Z^n(t)-Z(t)|\mathrm{d}t\leqslant\frac{1}{2^n}.$$

By the linear growth, we get

$$\begin{aligned}&|g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))|\\ \leqslant\ & u_1(s)(f_s+|Y^n(s)|+\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))|)+u_2(s)|Z^n(s)|\\ \leqslant\ & u_1(s)(f_s+\sup_n|Y^n(s)|+\sup_n\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))|)+u_2(s)\sup_n|Z^n(s)|,\end{aligned}$$

while

$$\begin{aligned}&\mathbb{E}\int_0^T u_1(s)\sup_n\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))|\mathrm{d}s\leqslant\mathbb{E}\int_0^T u_1^2(s)\mathrm{d}s+\mathbb{E}\int_0^T\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))|^2\mathrm{d}s\\ \leqslant\ & 2\mathbb{E}\int_0^T\sup_n\mathbb{E}^{\mathcal{F}_s}|Y^n(s+\theta(s))-Y(s+\theta(s))|^2\mathrm{d}s+2\mathbb{E}\int_0^T\mathbb{E}^{\mathcal{F}_s}|Y(s+\theta(s))|^2\mathrm{d}s+C'\\ \leqslant\ & 2L\mathbb{E}\int_0^{T+K}\sum_{n=1}^{\infty}|Y^n(s)-Y(s)|\mathrm{d}s+2L\mathbb{E}\int_0^{T+K}|Y(s)|\mathrm{d}s+C'<\infty.\end{aligned}$$

Thus

$$\sup_n\mathbb{E}^{\mathcal{F}_t}|Y^n(t+\theta(t))|\in L^2([0,T],\mathrm{d}t).$$

Using the similar method, we get

$$\sup_n|Y^n(t)|\in L^2([0,T+K],\mathrm{d}t),\quad \sup_n|Z^n(t)|\in L^2([0,T+K],\mathrm{d}t).$$

Controled convergence theorem leads to

$$\int_0^T g_n(s,Y^n(s),Z^n(s),Y^n(s+\theta(s)))\mathrm{d}s\to\int_0^T g(s,Y(s),Z(s),Y(s+\theta(s)))\mathrm{d}s,n\to\infty.$$

By BDG inequality, we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_t^T Z^n(s)\mathrm{d}W(s)-\int_t^T Z(s)\mathrm{d}W(s)\right|^2\leqslant C'\mathbb{E}\int_t^T|Z^n(s)-Z(s)|^2\mathrm{d}s\to0,n\to\infty.$$

Thus there exists a subsequence, which we still denote by $\{n\}$ such that

$$\sup_{t\in[0,T]}\left|\int_t^T Z^n(s)\mathrm{d}W(s)-\int_t^T Z(s)\mathrm{d}W(s)\right|\to0,n\to\infty.$$

Then (Y,Z) is a solution of equation (1.2). Now, we are going to prove Y is a minimal solution of equation (1.2). Assume (Y',Z') is another solution of equation (1.2), by the comparison theorem in [4], we have $Y(t)\leqslant Y'(t)$ a.e. for any $t\in[0,T+K]$. The proof is completed.

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一类特殊的延迟倒向随机微分方程的最小解

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摘要: 本文研究一类特殊的延迟倒向随机微分方程最小解的相关问题. 当假设生成子满足连续性假设和类似线性增长条件时, 证明了最小解的存在性. 本文推广了最小解存在的一般假设条件, 这里假设要弱于之前的文献, 然而本文得到了更好的引理, 并且得到了相同的结论.

关键词: 延迟倒向随机微分方程; 最小解; 比较定理

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