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PERIODIC SOLUTIONS OF DAMPED IMPULSIVE SYSTEMS

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Abstract: In this paper, we study the existence of periodic solutions of damped impulsive problems via variational method. By presenting a new approach, we obtain the critical points of impulsive systems with periodic boundary under some assumptions, which generalizes the known results and enriches the research methods of impulsive problems.

Keywords: critical points; impulses; periodic boundary; constraint

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1 Introduction

Impulsive differential equations arising from the real world describe the dynamic of processes in which sudden discontinuous jumps occur. In recent years, impulsive problems attracted the attention of a lot of researchers and in consequence the number of papers related to this topic is huge, see [1–6] and their references. For a second order differential equation $x'' + f(t, x, x') = 0$, one usually considers impulses in the position x and the velocity x'. However, in the motion of spacecraft one has to consider instantaneous impulses dependent only on the position, and the result in jump is discontinuous in velocity, but with no change in position [7, 8]. Let $t_0 = 0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 2\pi$. Recently, the following Dirichlet boundary value problems with impulses

$$x'' + g(t)x' + f(t, x) = 0 \text{ a.e. } t \in [0, 2\pi], \tag{1.1}$$

$$\Delta x'(t_j) := x'(t_j^+) - x'(t_j^-) = I_j(x(t_j)), \quad j = 1, 2, \cdots, p, \tag{1.2}$$

$$x(0) = x(2\pi) \tag{1.3}$$

were studied by variational method in [9, 10], where $f : [0, 2\pi] \times R \to R$ is continuous, $g \in C[0, 2\pi]$, and the impulse functions $I_j : R \to R$ is continuous for every j. After that, impulsive problems (1.1)–(1.2) with periodic boundary

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{1.4}$$

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was also investigated in [6] when $g(t) \equiv 0$.

Generally, people are used to obtain the critical points of impulsive problems via Mountain pass theorem or Saddle point theorem. In this paper, we use Lagrange multipliers theorem, that is conditional extremum theory, to investigate impulsive problems (1.1)–(1.2) with periodic boundary

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \tag{1.5}$$

The organization of the paper is as follows. In Section 2, we give variational structure of impulsive problem (1.1)–(1.2)–(1.5). In Section 3, critical points corresponding to periodic solutions of impulsive problems (1.1)–(1.2) are obtained by constrain theory.

2 Variational Structure

In this section, we always assume that $f : R \times R \to R$ is 2π-periodic in t for all $x \in R$ and satisfies the following carathéodory assumptions:

(1) for every $x \in R$, $f(\cdot, x)$ is measurable on $[0, 2\pi]$;

(2) for a.e. $t \in [0, 2\pi]$, $f(t, \cdot)$ is continuous on R;

(3) there exist $a \in C(R^+, R^+)$ and $b \in L^1(0, 2\pi; R^+)$ such that $|F(t, x)| + |f(t, x)| \leq a(|x|)b(t)$ for all $x \in R$ and a.e. $t \in [0, 2\pi]$, where $F(t, x) = \int_0^x f(t, s)\mathrm{d}s$.

We also assume that $g \in L^1(0, 2\pi; R)$ is 2π-periodic with $\int_0^{2\pi} g(s)\mathrm{d}s = 0$, and the impulse functions $I_j : R \to R$ is continuous for every j.

Multiplying equation (1.1) by $e^{G(t)}$, we can see that impulsive problem (1.1)+(1.2)+(1.5) is equivalent to

$$(e^{G(t)}x')' + e^{G(t)}f(t, x) = 0 \text{ a.e. } t \in [0, 2\pi] \tag{2.1}$$

with (1.2)+(1.5), where $G(t) = \int_0^t g(s)\mathrm{d}s$.

We now investigate impulsive system (2.1)+(1.2)+(1.5). Define Hilbert space

$$H^1_{2\pi} = \{x : [0, 2\pi] \to R \mid x(0) = x(2\pi), \int_0^{2\pi} (x'^2 + x^2)\mathrm{d}t < +\infty\}$$

with the norm $\|x\| = (\int_0^{2\pi} (x'^2 + x^2)\mathrm{d}t)^{1/2}$. Consider the functional $\varphi(x)$ defined on $H^1_{2\pi}$ by

$$\varphi(x) = \frac{1}{2}\int_0^{2\pi} e^{G(t)}|x'(t)|^2\mathrm{d}t - \int_0^{2\pi} e^{G(t)}F(t, x)\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)} \int_0^{x(t_j)} I_j(s)\mathrm{d}s.$$

Proposition 2.1 Under our assumptions, functional $\varphi(x)$ is weakly lower semi-continuous on $H^1_{2\pi}$.

Proof First, it is easy to see that functional $\int_0^{2\pi} e^{G(t)}|x'(t)|^2\mathrm{d}t$ is convex continuous. Consequently, by Mazur Theorem, $\int_0^{2\pi} e^{G(t)}|x'(t)|^2\mathrm{d}t$ is weakly lower semi-continuous on $H^1_{2\pi}$.

On the other hand, by Proposition 1.2 in [13], we know that if sequence $\{x_k\}$ converges weakly to x in $H^1_{2\pi}$, then $\{x_k\}$ converges uniformly to x on $[0,2\pi]$. Hence $\int_0^{2\pi} e^{G(t)}F(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{x(t_j)} I_j(s)\mathrm{d}s$ is weakly continuous on $H^1_{2\pi}$. Thus we complete the proof.

The following result is evident.

Proposition 2.2 Under our assumptions, $\varphi(x)$ is continuously differentiable on $H^1_{2\pi}$, and for every $v \in H^1_{2\pi}$, one has

$$\langle \varphi'(x), v\rangle = \int_0^{2\pi} e^{G(t)}x'(t)v'(t)\mathrm{d}t - \int_0^{2\pi} e^{G(t)}f(t,x)v\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)}I_j(x(t_j))v(t_j).$$

Proposition 2.3 Under our assumptions, if $x \in H^1_{2\pi}$ is a critical point of φ, then x is one 2π-periodic solution of problem (2.1)+(1.2)+(1.5).

Proof Let x be a critical point of φ in $H^1_{2\pi}$, then for every $v \in H^1_{2\pi}$ we have

$$\langle \varphi'(x), v\rangle = \int_0^{2\pi} e^{G(t)}x'v'\mathrm{d}t - \int_0^{2\pi} e^{G(t)}f(t,x)v\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)}I_j(x(t_j))v(t_j) = 0. \tag{2.2}$$

We now check that x satisfies (2.1)+(1.2)+(1.5).

Since $x \in H^1_{2\pi}$, we have $x(0) = x(2\pi)$. Evidently, the Sobolev space $H^1_0(0,2\pi) \subseteq H^1_{2\pi}$. For any fixed $j \in \{0,1,2,\cdots,p\}$, let $H^1_0(t_j,t_{j+1}) = \{v \in H^1_0(0,2\pi) : v(t) = 0, \forall t \in [0,t_j] \cup [t_{j+1},2\pi]\}$. Then

$$\int_{t_j}^{t_{j+1}} e^{G(t)}x'v'\mathrm{d}t - \int_{t_j}^{t_{j+1}} e^{G(t)}f(t,x)v\mathrm{d}t = 0, \forall\ v \in H^1_0(t_j,t_{j+1}).$$

It implies that $(e^{G(t)}x')' + e^{G(t)}f(t,x) = 0$ a.e. $t \in [t_j,t_{j+1}]$. Hence x satisfies

$$(e^{G(t)}x')' + e^{G(t)}f(t,x) = 0 \quad \text{a.e. } t \in\ [0,2\pi]. \tag{2.3}$$

That is, x satisfies equation (2.1).

Take $v \in H^1_0(0,2\pi)$ and multiply (2.3) by v, then integrate between 0 and 2π. (2.3) gives that $\int_0^{2\pi} (e^{G(t)}x')'v\mathrm{d}t + \int_0^{2\pi} e^{G(t)}f(t,x)v\mathrm{d}t = 0$. That is $\sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} (e^{G(t)}x')'v\mathrm{d}t + \int_0^{2\pi} e^{G(t)}f(t,x)v\mathrm{d}t =$

0. By integration by parts, we have

$$-\int_0^{2\pi} e^{G(t)}x'v'\mathrm{d}t-\sum_{j=1}^{p}e^{G(t_j)}\Delta x'(t_j)v(t_j)+\int_0^{2\pi}e^{G(t)}f(t,x)v\mathrm{d}t=0.$$

Combining with (2.2), which implies that

$$\sum_{j=1}^{p}e^{G(t_j)}\Delta x'(t_j)v(t_j)=\sum_{j=1}^{p}e^{G(t_j)}I_j(x(t_j))v(t_j),\ \forall v\in H_0^1(0,2\pi).$$

Hence

$$\Delta x'(t_j)=I_j(x(t_j)) \text{ for every } j=1,2,\cdots,p. \tag{2.4}$$

This is just condition (1.2).

On the other hand, in (2.2), let $v=1$, then

$$-\int_0^{2\pi}e^{G(t)}f(t,x)\mathrm{d}t+\sum_{j=1}^{p}e^{G(t_j)}I_j(x(t_j))=0. \tag{2.5}$$

Moreover, integrating (2.3) between 0 and 2π, we get $\int_0^{2\pi}(e^{G(t)}x')'\mathrm{d}t+\int_0^{2\pi}e^{G(t)}f(t,x)\mathrm{d}t=0.$ It gives that

$$-\sum_{j=1}^{p}e^{G(t_j)}(x'(t_j^+)-x'(t_j^-))+e^{G(2\pi)}x'(2\pi)-e^{G(0)}x'(0)=-\int_0^{2\pi}e^{G(t)}f(t,x)\mathrm{d}t. \tag{2.6}$$

At last, by (2.4), (2.5), (2.6) and $G(2\pi)=G(0)$ since $\int_0^{2\pi}g=0$, one has $x'(0)=x'(2\pi)$.

Thus we complete the proof.

Remark 2.4 Since $g\in L^1(0,2\pi;R)$ with $\int_0^{2\pi}g(s)\mathrm{d}s=0$, $G(t)$ is absolutely continuous and 2π-periodic, from which one has that $e^{G(t)}$ is continuous , 2π-periodic and positive function. Hence, from the viewpoint of variational functional φ, there are no difference between problem (1.1)–(1.2)–(1.5) and equation $x''+f(t,x)=0$ with (1.2)–(1.5). Therefore, with a similar proof as [6], we can obtain critical points by saddle point theorem using similar conditions.

3 Critical Points in Constraints

The following Lagrange multipliers theorem is well known (see Theorem 2.1 in [11] or Theorem 3.1.31 in [12]).

Lemma 3.1 Let $\varphi\in C^1(H^1_{2\pi},R)$ and $M=\{x\in H^1_{2\pi}:\psi_j(x)=0, j=1,\cdots,n\}$, where $\psi_j\in C^1(H^1_{2\pi},R)$, $j=1,\cdots,n$, and $\psi_1'(x),\cdots,\psi_n'(x)$ are linearly independent for each $x \in H^1_{2\pi}$. Then if $u \in M$ is a critical point of $\varphi|_M$, there exist $\lambda_j \in R$ $j = 1, \cdots, n$, such that

$$\varphi'(u) = \sum_{j=1}^{n} \lambda_j \psi_j'(u). \tag{3.1}$$

We now give the following minimization principle in constraint M.

Lemma 3.2 (see Theorem 1.1 in [11]) Let M be a weakly closed subset of a Hilbert space X. Suppose a functional $\varphi : M \to R$ is

(i) weakly lower semi-continuous,

(ii) $\varphi(u) \to +\infty$ as $\|u\| \to \infty$, $u \in M$,

then φ is bounded from below and there exists $u_0 \in M$ such that $\varphi(u_0) = \inf\limits_M \varphi$.

Using the above lemmas, the author of [11] consider the following Neumann problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

for some suitable $\Omega \subset R^N$ and $f(u)$ under natural constraints (see [11]). Inspired by his work, in this section, we take our attention to find the critical points of functional φ over a set of constraints $M \subseteq H^1_{2\pi}$.

For $x \in H^1_{2\pi}$, let $\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)\mathrm{d}t$, $\tilde{x}(t) = x(t) - \bar{x}$ and $\tilde{H}^1_{2\pi} = \{x \in H^1_{2\pi} | \ \bar{x} = 0\}$, then one has

$$\|\tilde{x}\|^2_\infty \le \frac{\pi}{6}\int_0^{2\pi} x'^2(t)\mathrm{d}t \tag{3.2}$$

and

$$\int_0^{2\pi} \tilde{x}^2(t)\mathrm{d}t \le \int_0^{2\pi} x'^2(t)\mathrm{d}t. \tag{3.3}$$

By (3.3), we have

$$(\int_0^{2\pi} x'^2(t)\mathrm{d}t)^{1/2} \le \|\tilde{x}\| \le \sqrt{2}(\int_0^{2\pi} x'^2(t)\mathrm{d}t)^{1/2}. \tag{3.4}$$

It is easy to see that $H^1_{2\pi} = R \oplus \tilde{H}^1_{2\pi}$.

Besides those conditions given to $f(t,x), g(t)$ and $I_j(x), j = 1, \cdots, k$ in Section 2, we also assume that there exist constants $\alpha, \beta > 0$, $\xi_j \in R$, $j = 1, \cdots, k$, such that

$$f(t,x)f(t,-x) < 0, \quad \text{a.e. } t \in [0, 2\pi], \forall |x| > \alpha, \tag{3.5}$$

$$f_2'(t,x) \triangleq \frac{\partial f(t,x)}{\partial x} > 0, \text{ a.e. } t \in [0, 2\pi], \tag{3.6}$$

$$I_j(\xi_j) = 0, \qquad -\beta < I_j'(x) \le 0, \ j = 1, \cdots, k, \tag{3.7}$$

$$a(x) \le x^2 + o(x^\eta), \tag{3.8}$$

where constant $0 \le \eta < 2$ and the function a is from carathéodory assumption (3).

For convenience, we denote $A = \max\{e^{G(t)}\}$ and $B = \min\{e^{G(t)}\}$, then $A, B > 0$.

Theorem 3.3 If above assumptions hold and $6B - A(2\|b\|_1 + p\beta)\pi > 0$, then problem (1.1)–(1.2)–(1.5) has at least one solution.

Remark 3.4 We only need to prove that problem (2.1)–(1.2)–(1.5) has at least one solution.

Consider the subset M of $H^1_{2\pi}$ defined by

$$M = \{x \in H^1_{2\pi} : \int_0^{2\pi} e^{G(t)} f(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} I_j(x(t_j)) = 0\}.$$

Since $\int_0^{2\pi} e^{G(t)} F(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{x(t_j)} I_j(s)\mathrm{d}s$ is weakly continuous, one obtains that the set M is weakly closed.

Let functional $\Gamma(x) = \int_0^{2\pi} e^{G(t)} f(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} I_j(x(t_j))$. For $\forall\ v \in H^1_{2\pi}$, we have

$$(\Gamma'(x), v) = \int_0^{2\pi} e^{G(t)} f_2'(t,x) v\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} I_j'(x(t_j)) v(t_j).$$

Then by conditions (3.6) and (3.7), one has $\Gamma'(x) \ne 0$, which indicates that $\Gamma'(x)$ linearly independent for each $x \in H^1_{2\pi}$.

Remark 3.5 It is easy to see that, by conditions (3.5)–(3.7), we have that, $\forall\ u \in \tilde{H}^1_{2\pi}$, there exists a unique $c \in R$ such that $u + c \in M$. In fact, $\forall\ u \in \tilde{H}^1_{2\pi}$, one has that u is continuous and the function $\tilde{\Gamma}(c) = \int_0^{2\pi} e^{G(t)} f(t, u+c)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} I_j(u(t_j) + c)$ defined on R is continuous and strictly increasing, moreover, $\tilde{\Gamma}(-\infty) < 0, \tilde{\Gamma}(+\infty) > 0$.

Lemma 3.6 Under our assumptions, $x \in H^1_{2\pi}$ is a critical point of φ if and only if $x \in M$ and x is a critical point of $\varphi|_M$.

Proof If $x \in H^1_{2\pi}$ is a critical point of φ, by choosing $v = 1$ in (2.2), we have

$$\int_0^{2\pi} e^{G(t)} f(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} I_j(x(t_j)) = 0,$$

i.e., $x \in M$, and hence x is a critical point of $\varphi|_M$.

On the other hand, if x is a critical point of $\varphi|_M$, by Lemma 3.1, there exists $\lambda \in R$

such that for every $v \in H^1_{2\pi}$,

$$\int_0^{2\pi} e^{G(t)}x'(t)v'(t)\mathrm{d}t - \int_0^{2\pi} e^{G(t)}f(t,x)v\mathrm{d}t + \sum_{j=1}^{p} e^{G(t_j)}I_j(x(t_j))v(t_j)$$

$$= \lambda(\int_0^{2\pi} e^{G(t)}f_2'(t,x)v\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)}I_j'(x(t_j))v(t_j)). \tag{3.9}$$

Choosing $v=1$ and observing that $x \in M$, we have

$$\lambda(\int_0^{2\pi} e^{G(t)}f_2'(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)}I_j'(x(t_j))) = 0,$$

which follows that $\lambda = 0$ since $f_2' > 0$ and $I_j' \le 0$. Putting it into (3.9), one has $\varphi'(x) = 0$.

Thus we complete the proof.

To functional

$$\Phi(x) = \int_0^{2\pi} e^{G(t)}F(t,x)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{x(t_j)} I_j(s)\mathrm{d}s,$$

we have the following results.

Lemma 3.7 Under our assumptions, we have

(i) $\Phi(u+c) \le \Phi(u)$, $\forall\ u+c \in M$, where $u \in \tilde{H}^1_{2\pi}$, $c \in R$.

(ii) Let $u_n + c_n \in M$, where $u_n \in \tilde{H}^1_{2\pi}$ and $c_n \in R$. Then if $\|u_n + c_n\| \to \infty$, one has $\|u_n\| \to \infty$.

Proof First, by conditions $f_2' > 0$ and $I_j' \le 0, j = 1, \cdots, k$, one has the convexity of $F(t,\cdot)$ and $-\int_0^{x(t_j)} I_j(s)\mathrm{d}s$, that is $F(t,u) \ge F(t,u+c) + f(u+c)(u-(u+c))$ and

$$-\int_0^{u(t_j)} I_j(s)\mathrm{d}s \ge -\int_0^{u(t_j)+c} I_j(s)\mathrm{d}s - I_j(u(t_j)+c)(u(t_j)-(u(t_j)+c)).$$

The above two inequalities give that

$$\begin{aligned}\int_0^{2\pi} e^{G(t)}F(t,u)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)} I_j(s)\mathrm{d}s \ \ge\ & \int_0^{2\pi} e^{G(t)}F(t,u+c)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)} \int_0^{u(t_j)+c} I_j(s)\mathrm{d}s \\ & -c\int_0^{2\pi} e^{G(t)}f(t,u+c)\mathrm{d}t + c\sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j)+c),\end{aligned}$$

which follows (i).

Next, we turn to prove (ii). Define functional $\Psi : \tilde{H}^1_{2\pi} \times R \to R$ by the following

$$\Psi(u,c) = \int_0^{2\pi} e^{G(t)}f(t,u+c)\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)}I_j(u(t_j)+c).$$

Since $f_2' > 0$ and $I_j' \leq 0$, $\Psi(u,\cdot)$ is strictly increasing. From Remark 3.5, we know that, $\forall u \in \tilde{H}^1_{2\pi}$, there exists a unique $c = c(u) \in R$ such that $u + c \in M$. By contradiction, we assume that, going to a subsequence if necessary, $u_n + c_n \in M$, $\|u_n + c_n\| \to \infty$ and $\{\|u_n\|\}$ is bounded. Then we may assume $u_n \rightharpoonup v$ weakly in $H^1_{2\pi}$ and $c_n \to +\infty$ (similar analysis for $c_n \to -\infty$). Since $u_n \rightharpoonup v$ weakly in $H^1_{2\pi}$, then by the Proposition 1.2 in [13], one has $u_n \to v$ uniformly on $[0, 2\pi]$.

Because of the strict increase of $\Psi(u,\cdot)$, when n is big enough, we have

$$\begin{aligned} 0 &= \Psi(v, c(v)) < \Psi(v, c_n) = \Psi(v, c_n) - \Psi(v_n, c_n) \\ &= \int_0^{2\pi} e^{G(t)}[f(t, v + c_n) - f(t, v_n + c_n)]\mathrm{d}t - \sum_{j=1}^{p} e^{G(t_j)}[I_j(v(t_j) + c_n) - I_j(v_n(t_j) + c_n)] \\ &\leq A\|v_n - v\|_\infty \int_0^{2\pi} \eta(t)\mathrm{d}t + pA\beta\|v_n - v\|_\infty \to 0 \end{aligned}$$

as $n \to \infty$. It is contradictory.

Thus we complete the proof.

Proof of Theorem 3.3 Without loss of generality, we may assume that $a(x) \leq x^2$ in condition (3.8) and $\xi_j = 0, j = 1, 2, \cdots, p$ in condition (3.7). Then under our assumptions, one has

$$F(t,x) \leq b(t)x^2, \quad |\int_0^x I_j(s)\mathrm{d}s| \leq \frac{1}{2}\beta x^2.$$

It implies that

$$\begin{aligned} |\Phi(u)| &\leq A\int_0^{2\pi} |u(t)|^2 b(t)\mathrm{d}t + \frac{1}{2}\sum_{j=1}^{p} A\beta u^2(t_j) \\ &\leq A\|b\|_1\|u\|^2_\infty + \frac{pA\beta}{2}\|u\|^2_\infty \leq (A\|b\|_1 + \frac{pA\beta}{2})\frac{\pi}{6}\|u'(t)\|^2_2 \end{aligned}$$

by (3.2) if $u \in \tilde{H}^1_{2\pi}$. Hence $\forall\ u + c \in M$, where $u \in \tilde{H}^1_{2\pi}$ and $c \in R$, using (i) of Lemma 3.7, we have

$$\begin{aligned} \varphi(u + c) &= \frac{1}{2}\int_0^{2\pi} e^{G(t)}|u'(t)|^2\mathrm{d}t - \Phi(u + c) \geq \frac{1}{2}\int_0^{2\pi} e^{G(t)}|u'(t)|^2\mathrm{d}t - \Phi(u) \\ &\geq \frac{B}{2}\|u'(t)\|^2_2 - (A\|b\|_1 + \frac{pA\beta}{2})\frac{\pi}{6}\|u'(t)\|^2_2 \geq \frac{6B - A(2\|b\|_1 + p\beta)\pi}{12}\|u'(t)\|^2_2. \end{aligned}$$

Since $6B - A(2\|b\|_1 + p\beta)\pi > 0$, by (3.4) and (ii) of Lemma 3.7, we have $\varphi(u + c) \to +\infty$ as $\|u + c\| \to \infty$, $u + c \in M$.

On the other hand, M is weakly closed and φ is weakly lower semi-continuous, therefore by Lemma 3.2, there exists at least one critical point $x \in M$ of $\varphi|_M$. Then by Lemma 3.6, we complete the proof.

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带阻尼项的脉冲系统的周期解

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摘要: 本文利用变分法研究了带阻尼项的脉冲系统的周期解. 采用一种新的方法, 在一些条件下证明了带周期边界条件的脉冲系统存在临界点. 本文不仅推广了已有的结果而且还丰富了研究脉冲系统的方法.

关键词: 临界点; 脉冲; 周期边界; 限制

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