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COMPOSITION OPERATORS ASSOCIATED WITH F_p SPACES

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Abstract: In this paper, we study composition operators associated with F_p space. By the means of functional analysis, we study some necessary and sufficient conditions for composition operators C_φ to be bounded and compact from F_p (resp. $F_{p,0}$) to $\mathcal{B}$. In addition, we also characterize the isometric composition operators from $\mathcal{B}$ to F_p for $1 \le p < \infty$ and show that no composition operator on $F_{p,0}$ for $0 < p < \infty$ is Fredholm, which extend some previous results.

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1 Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane, $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$ and $\mathbb{T}$ the boundary of the unit disk. For $0 < p < \infty$, $f \in H(\mathbb{D})$, we set, as usual,

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}$$

and

$$M_\infty(r, f) = \sup_{0 \le t \le 2\pi} |f(re^{it})|.$$

Let F_p denote the space of all functions $f \in H(\mathbb{D})$ satisfying $|||f|||_{F_p}^p = |f(0)|^p + ||f||_{F_p}^p < \infty$, where $||f||_{F_p} = \sup\limits_{0<r<1} (1 - r^2) M_p(r, f')$.

$F_{p,0}$ is the little version of F_p, which consists of all $f \in H(\mathbb{D})$ satisfying

$$\lim_{r \to 1^-} (1 - r^2) M_p(r, f') = 0.$$

For $0 < p \le \infty$ and $0 < q \le \infty$, we write $\mathcal{B}(p, q)$ for the space of those $f \in H(\mathbb{D})$ such that

$$K_{p,q}(f) = \left(\int_0^1 M_p^q(r, f')(1 - r)^{q-1} dr \right)^{\frac{1}{q}} < \infty, \quad \text{if } q < \infty$$

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and

$$K_{p,\infty}(f) = \sup_{0<r<1} (1-r)M_p(r, f') < \infty, \ \text{ if } q = \infty.$$

Refer to [6] for more details on $\mathcal{B}(p,q)$. Recall that a function $f \in H(\mathbb{D})$ belongs to the classical Bloch space $\mathcal{B}$ if

$$||f||_{\mathcal{B}} = \sup_{z\in\mathbb{D}} \big(1-|z|^2\big)|f'(z)| < \infty,$$

and $\mathcal{B}$ is a Banach space equipped with the norm

$$|||f|||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} \big(1-|z|^2\big)|f'(z)|.$$

Notice in [1] that F_∞ coincides with the space $\mathcal{B}$. We also know that $M_p(r,f)$ increases with respect to p from [7]. It follows from the definitions above that for $0 < p < q < \infty$,

$$\mathcal{B} = F_\infty \subset F_q \subset F_p = \mathcal{B}(p, \infty).$$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic map, then the composition operator C_φ is defined by

$$C_\varphi : f \to f \circ \varphi. \tag{1.1}$$

Composition operators were studied by numerous authors in many subspaces of $H(\mathbb{D})$. Among others, Madigan and Matheson characterized continuity and compactness of composition operators on the classical Bloch space $\mathcal{B}$ in [10]. Madigan [9] discussed the boundedness and w-compactness of composition operators on holomorphic Lipschitz spaces $\mathcal{A}_\alpha$. Xiao studied composition operators between Bloch-type spaces in [17] and Yang worked about composition operators from $F(p,q,s)$ spaces to the nth weighted-type spaces on the unit disk in [18]. Good general references of composition operators on classical spaces of holomorphic functions on the unit disk are [2, 14].

Motivated by these papers, here we study the boundedness and compactness of composition operators from F_p (resp. $F_{p,0}$) to $\mathcal{B}$. As an appilcation, we characterize the boundedness and compactness of composition operators on F_p. In addition, we also describe the isometric composition operators from $\mathcal{B}$ to F_p for $1 \le p < \infty$, and show that no composition operators on $F_{p,0}$ is Fredholm for $0 \le p < \infty$.

Throughout the paper we use the same letter C to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants C will be often specified in the parenthesis. We use the notation $X \lesssim Y$ or $Y \gtrsim X$ for nonnegative quantities X and Y to mean $X \le CY$ for some inessential constant $C > 0$. Similarly, we use the notation $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2 Some Lemmas

The first lemma below is essential due to Hardy and Littlewood (refer to [3]).

Lemma 2.1 For $0 < p < q \le \infty$, there exists a positive constant C_{pq} depending only on p and q such that for each $f \in H(\mathbb{D})$ and each $r \in (0,1)$,

$$M_q\big(r, f\big) \le C_{pq} M_p\left(\frac{1+r}{2}, f\right)\big(1-r\big)^{\frac{1}{q}-\frac{1}{p}}.$$

With this lemma, we get the following corollary.

Corollary 2.2 For $0 < p < \infty$, there exists a positive constant C depending only on p such that for any $f \in F_p$,

$$|f'(z)| \le \frac{C}{(1-|z|^2)^{1+\frac{1}{p}}} |||f|||_{F_p}. \tag{2.1}$$

Proof If we replace f by f' in Lemma 2.1 and let $q=\infty$, then for every $f \in F_p$, there exists $C > 0$ depending only on p such that

$$\begin{aligned}
|f'(z)| &\le \sup_{0\le t\le 2\pi} |f'(|z|e^{it})| = M_\infty\big(|z|, f'\big)\\
&\le C_p M\left(\frac{1+|z|}{2}, f'\right)\big(1-|z|\big)^{-\frac{1}{p}}\\
&= C_p \frac{\big(1-(\frac{1+|z|}{2})^2\big) M_p\big(\frac{1+|z|}{2}, f'\big)}{\big(1-(\frac{1+|z|}{2})^2\big)\big(1-|z|\big)^{\frac{1}{p}}}\\
&\le \frac{C}{\big(1-|z|^2\big)^{1+\frac{1}{p}}} ||f||_{F_p},
\end{aligned}$$

combining with $||f||_{F_p} \le |||f|||_{F_p}$, which completes the proof.

Corollary 2.3 Suppose that $0 < p < \infty$, then there is a positive constant C satisfying such that for any $f \in F_p$ and $z \in \mathbb{D}$,

$$|f(z)| \le \frac{C}{(1-|z|^2)^{\frac{1}{p}}} |||f|||_{F_p}. \tag{2.2}$$

Proof Suppose $f \in F_p$, we have

$$f(z) = f(0) + z\int_0^1 f'(tz)dt. \tag{2.3}$$

Then by Corollary 2.2 and $|f(0)| \le |||F|||_{F_p}$, we can easily get

$$|f(z)| \le C_p(1+p2^{\frac{1}{p}})(1-|z|^2)^{-\frac{1}{p}} |||f|||_{F_p},$$

which completes the proof.

Corollary 2.4 For any $z \in \mathbb{D}$, $\lim\limits_{|z|\to 1^-} ||\delta_z||_{F_{p,0}^*} \to \infty$. Moreover, $\frac{\delta_z}{||\delta_z||_{F_{p,0}^*}} \to 0$ weak* in $F_{p,0}^*$ when $|z| \to 1^-$.

Proof For each $\zeta \in \mathbb{T}$, we consider the function $f_\zeta(z) = \log(1-\overline{\zeta}z)$ which belongs to $\mathcal{B}$. Recall that $||\delta_z||_{\mathcal{B}^*} \approx \log(\frac{2}{1-|z|^2})$ and $F_p^* \subset \mathcal{B}^*$ for $0<p<\infty$. Thus for each $\delta_z \in F_p^*$, we have

$$\log\big(\frac{2}{1-|z|^2}\big) \lesssim ||\delta_z||_{\mathcal{B}^*} \lesssim ||\delta_z||_{F_p^*} \lesssim \frac{1}{1-|z|^2},$$

the last inequality comes from Corollary 2.3. Then it is obvious that $||\delta_z||_{F_p^*} \to \infty$ as $|z| \to 1^-$. We know from [5] that for little Bloch space $\mathcal{B}_0$, $||\delta_z||_{\mathcal{B}_0^*} \to \infty$ when $|z| \to 1^-$. Moreover, for any $z \in \mathbb{D}$,

$$||\delta_z||_{\mathcal{B}_0^*} \le ||\delta_z||_{F_{p,0}^*} \le ||\delta_z||_{F_p^*}.$$

Then $||\delta_z||_{F_{p,0}^*} \to \infty$ when $|z| \to 1^-$. For any holomorphic polynomial P, we have

$$\lim_{|z|\to 1^-} \frac{|\delta_z(P)|}{||\delta_z||_{F_{p,0}^*}} \le \lim_{|z|\to 1^-} \frac{\sup\{|P(z)| : z \in \mathbb{D}\}}{||\delta_z||_{F_{p,0}^*}} = 0.$$

Then the proof is complete due to every $f \in F_{p,0}$ can be approached by a sequence of holomorphic polynomials,

By Corollary 2.4, we know that the closed unit disk $B(F_{p,0})$ of $F_{p,0}$ endowed with the compact open topology co is relatively compact and the evaluation function at $z \in \mathbb{D}$, $\delta_z : f \to f(z)$ is a bounded linear functional on $F_{p,0}$. So $F_{p,0}$ has a predual space X by the Dixmier-Ng theorem in [13]. Moreover, it can be seen that the closed linear span of the set $\{\delta_z : z \in \mathbb{D}\}$ in $F_{p,0}^*$ coincides with this predual space X. Hence elements in X are also continuous for the compact-open topology on bounded subset of $F_{p,0}$. Moreover, it is easy to see that $C_\varphi^*(\delta_z) = \delta_{\varphi(z)}$ for $z \in \mathbb{D}$.

For our next lemma, we need more notation. $H_{v_p}^\infty (0<p<\infty)$ is defined in [5]:

$$H_{v_p}^\infty = \big\{ f \in H(\mathbb{D}) : ||f||_{H_{vp}^\infty} = \sup_{z\in\mathbb{D}} v_p(z)|f(z)| < \infty \big\},$$

where $v_p(z) = (1-|z|^2)^p$ is the standard weight.

Lemma 2.5 For $0<p<\infty$, the map $\delta : \mathbb{D} \to F_p^*$, defined by $\delta : z \mapsto \delta_z$, is continuous with respect to pseudo-hyperbolic distance metric.

Proof By Corollary 2.3, we obtain that $(1-|z|^2)^{\frac{1}{p}}|f(z)| \lesssim |||f|||_{F_p}$, then $F_p \subset H_{v_1}^\infty$ and $||f||_{H_{v_1}^\infty} \lesssim |||f|||_{F_p}$ for all $f \in F_p$. Therefore, by Lemma 4 from [15], there exists a constant $C>0$ such that

$$|f(z)-f(w)| \le C||f||_{H_{v_1}^\infty} \max\left\{ \frac{\rho(z,w)}{1-|z|^2}, \frac{\rho(z,w)}{1-|w|^2} \right\}$$

for all $f \in H_{v_1}^\infty$ and all $z, w \in \mathbb{D}$, where $\rho(z,w)$ is the pseudo-hyperbolic distance metric between z and w. Hence

$$||\delta_z - \delta_w||_{F_p^*} \le C \max\left\{ \frac{\rho(z,w)}{1-|z|^2}, \frac{\rho(z,w)}{1-|w|^2} \right\}$$

for all $z, w \in \mathbb{D}$, which gives the continuity of δ.

3 Boundedness and Compactness

In the section, we will characterize the boundedness and compactness of composition operators C_φ from F_p to $\mathcal{B}$. For simplicity, in the sequel, we write

$$F(p,\varphi,z)=\frac{1-|z|^2}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+1}}|\varphi'(z)|$$

and

$$G(p,\varphi,z)=\big(1-|z|^2\big)^p\frac{1}{2\pi}\int_0^{2\pi}\frac{\big|\varphi'\big(|z|e^{it}\big)\big|^p}{\big|1-|\varphi(|z|e^{it})|^2\big|^{p+1}}dt.$$

Theorem 3.6 Let φ be a holomorphic self-map of $\mathbb{D}$ and $0<p<\infty$. Then the following statements are equivalent.

(i) $\sup\limits_{z\in\mathbb{D}}F(p,\varphi,z)<\infty$.

(ii) $C_\varphi:F_p\to\mathcal{B}$ is bounded.

(iii) $C_\varphi:F_{p,0}\to\mathcal{B}$ is bounded.

Proof (i)⇒(ii) If $\sup_{z\in\mathbb{D}}F(p,\varphi,z)<\infty$, then it follows from Corollary 2.2 that

$$\begin{aligned}||C_\varphi f||_{\mathcal{B}}&=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)|f'\big(\varphi(z)\big)\varphi'(z)|\\&\lesssim\sup_{z\in\mathbb{D}}\frac{\big(1-|z|^2\big)|\varphi'(z)|}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+1}}||f||_{F_p}\\&\lesssim\sup_{z\in\mathbb{D}}F(p,\varphi,z)\cdot||f||_{F_p},\end{aligned}$$

which implies that $C_\varphi:F_p\to\mathcal{B}$ is bounded.

(ii)⇒(iii) This implication is clear.

(iii)⇒(i) Assume that $C_\varphi:F_{p,0}\to\mathcal{B}$ is bounded. We have known from [7] that for $0<p<\infty$, $f_0(z)=\frac{p}{(1-z)^{\frac{1}{p}}}\in F_p$. Now fix $z_0\in\mathbb{D}$ such that $w=\varphi(z_0)\neq0$ and let $f_w(z)=\frac{p}{\overline{w}(1-\overline{w}z)^{\frac{1}{p}}}-\frac{p}{\overline{w}}$. Then $f_w'(z)=\frac{1}{(1-\overline{w}z)^{\frac{1}{p}+1}}$, and $f_w\in F_{p,0}$. Moreover,

$$\begin{aligned}||f_w||_{F_p}^p&=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\big|f_w'(|z|e^{it})\big|^pdt\\&=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\big|1-|w||z|e^{i(\theta+t)}\big|^{p+1}}dt\\&\le\sup_{z\in\mathbb{D}}\big(1-(|w||z|)^2\big)^p\cdot\frac{1}{2\pi}\int_\theta^{\theta+2\pi}\frac{1}{\big|1-|w||z|e^{it}\big|^{p+1}}dt\\&=\sup_{z\in|w|\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-|z|e^{it}|^{p+1}}dt\\&\le\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\big|1-|z|e^{it}\big|^{p+1}}dt\\&=||f_0||_{F_p}^p.\end{aligned}\tag{3.1}$$

By the boundedness of C_φ and $f_w(0)=0$, we have

$$|f_w(\varphi(0))| + ||C_\varphi f_w||_{\mathcal{B}} \lesssim ||f_w||_{F_p}.$$

Then

$$\begin{aligned}\infty > ||f_0||_{F_p} \geq ||f_w||_{F_p} \gtrsim ||C_\varphi f_w||_{\mathcal{B}} \\ &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|f_w'(\varphi(z))\varphi'(z)\right| \geq \frac{\left(1-|z_0|^2\right)}{\left(1-|\varphi(z_0)|^2\right)^{\frac{1}{p}+1}}\left|\varphi'(z_0)\right| \\ &= F(p,\varphi,z_0),\end{aligned}$$

which deduces that

$$\sup_{z\in\mathbb{D}\setminus\{z|\varphi(z)=0\}} F(p,\varphi,z) < \infty. \tag{3.2}$$

Next we consider the situation $z_0 \in \mathbb{D}$ such that $\varphi(z_0) = 0$. Since $f \equiv z \in F_{p,0}$, then $\varphi = C_\varphi f \in \mathcal{B}$. Therefor,

$$\begin{aligned}\sup_{\{z|\varphi(z)=0\}} F(p,\varphi,z) &= \sup_{\{z|\varphi(z)=0\}} \left(1-|z|^2\right)\left|\varphi'(z)\right| \\ &\leq ||\varphi||_{\mathcal{B}} < \infty.\end{aligned} \tag{3.3}$$

Combining (3.2) with (3.3), we obtain (i), which completes the proof.

Theorem 3.7 Let φ be a holomorphic self-map of $\mathbb{D}$ and $0 < p < \infty$. Then the following statements are equivalent:

(i) $\sup_{z\in\mathbb{D}} G(p,\varphi,z) < \infty$.

(ii) $C_\varphi : F_p \to F_p$ is bounded.

(iii) $C_\varphi : F_{p,0} \to F_p$ is bounded.

Proof The proof of the theorem is similar to that of Theorem 3.6. First, we prove

(i)$\Rightarrow$(ii) If $\sup_{z\in\mathbb{D}} G(p,\varphi,z) < \infty$, then by Corollary 2.2, we have

$$\begin{aligned}||C_\varphi f||_{F_p}^p &= \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^p \cdot \frac{1}{2\pi}\int_0^{2\pi} \left|f'\left(\varphi(|z|e^{it})\right)\right|^p \cdot \left|\varphi'\left(|z|e^{it}\right)\right|^p dt \\ &\lesssim \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^p \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|\varphi'\left(|z|e^{it}\right)\right|^p}{\left(1-\left|\varphi(|z|e^{it})\right|^2\right)^{p+1}} dt \cdot ||f||_{F_p}^p \\ &= \sup_{z\in\mathbb{D}} G(p,\varphi,z)\cdot ||f||_{F_p}^p,\end{aligned}$$

which implies that $C_\varphi : F_p \to F_p$ is bounded.

(ii)$\Rightarrow$(iii) This is obvoius.

(iii)$\Rightarrow$(i) Assume that $C_\varphi : F_{p,0} \to F_p$ is bounded. Fix $z_0 \in \mathbb{D}$ such that $w = \varphi(z_0) \neq 0$ and take the same test functions as that used in Theorem 3.1,

$$f_w(z) = \frac{p}{\overline{w}(1-\overline{w}z)^{\frac{1}{p}}} - \frac{p}{\overline{w}} \quad \text{and} \quad f_0(z) = \frac{p}{(1-z)^{\frac{1}{p}}}.$$

By the boundedness of C_φ and $f_w(0)=0$, we have

$$|f_w(\varphi(0))| + ||C_\varphi f_w||_{F_p} \lesssim ||f_w||_{F_p}.$$

Then it follows from (3.1) that

$$\begin{aligned}\infty > ||f_0||_{F_p}^p &\geq ||f_w||_{F_p}^p \gtrsim ||C_\varphi f_w||_{F_p}^p \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1-|z_0|^2\right)^p}{\left(1-\left|\varphi(|z_0|e^{it})\right|^2\right)^{p+1}} \left|\varphi'\left(|z_0|e^{it}\right)\right|^p dt \\ &= G(p,\varphi,z_0), \end{aligned} \tag{3.4}$$

which implies that

$$\sup_{z\in\mathbb{D}\setminus\{z|\varphi(z)=0\}} G(p,\varphi,z) < \infty. \tag{3.5}$$

On the other hand, by taking he test function $f\equiv z$, the boundedness of $C_\varphi : F_{p,0}\to F_p$ gives that $\varphi = C_\varphi f\in F_p$. Then

$$\begin{aligned}\sup_{\{z|\varphi(z)=0\}} G(p,\varphi,z) &= \sup_{\{z|\varphi(z)=0\}} \left(1-|z|^2\right)^p \cdot \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi'\left(|z|e^{it}\right)\right|^p dt \\ &\leq ||\varphi||_{F_p}^p < \infty. \end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6) gives (i), which completes the proof.

Given two linear metric spaces X and Y, a linear operator $T : X \to Y$ is called to be compact if T maps every bounded subset of X into a relatively compact subset of Y. Equivalently, T is compact if and only if for any bounded sequence $\{f_n\}$ in X, there exists a subsequence $\{f_{n_k}\}$ such that $\{Tf_{n_k}\}$ converges in Y. Since when $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$, it is easy to see that C_φ is compact, so we always suppose $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$ when studying the compactness of C_φ.

Theorem 3.8 Let $0<p<\infty$ and φ be a holomorphic self-map of $\mathbb{D}$ such that $C_\varphi : F_p\to\mathcal{B}$ is bounded. Then the following statements are equivalent.

(i) $\lim\limits_{|\varphi(z)|\to 1} F(p,\varphi,z) = 0$.

(ii) $C_\varphi : F_p \to \mathcal{B}$ is compact.

(iii) $C_\varphi : F_{p,0}\to\mathcal{B}$ is compact.

Proof (i)⇒(ii) Since $C_\varphi : F_p\to\mathcal{B}$ is bounded, we have that $\sup_{z\in\mathbb{D}} F(p,\varphi,z) < \infty$ from Theorem 3.6. Let $\{f_n\}$ be a bounded sequence in F_p converging to 0 uniformly on any compact subset of $\mathbb{D}$. Then in order to show that $C_\varphi : F_p\to\mathcal{B}$ is compact, by Corollary 2.3 [16], it suffices to verify that

$$\lim_{n\to\infty} ||C_\varphi f_n||_{\mathcal{B}} = 0. \tag{3.7}$$

Set $M = \sup_n ||f_n||_{F_p} < \infty$. By the assumption, we obtain that, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that whenever $|\varphi(z)| > \delta$, we have $F(p,\varphi,z) < \frac{\varepsilon}{2M}$. Then

$$\begin{aligned}\big(1-|z|^2\big)\big|f_n'(\varphi(z))\varphi'(z)\big| &\lesssim \frac{1-|z|^2}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+1}}\big|\varphi'(z)\big|||f_n||_{F_p}\\ &\le MF(p,\varphi,z).\end{aligned}$$

It follows that

$$\big(1-|z|^2\big)\big|f_n'\big(\varphi(z)\big)\varphi'(z)\big| \le \frac{\varepsilon}{2}, \text{ if } \big|\varphi(z)\big| > \delta. \tag{3.8}$$

If $\big|\varphi(z)\big| \le \delta$, then

$$\big(1-|z|^2\big)\big|f_n'\big(\varphi(z)\big)\varphi'(z)\big| \lesssim ||\varphi||_{\mathcal{B}}\big|f_n'\big(\varphi(z)\big)\big|,$$

which implies that $\big(1-|z|^2\big)\big|f_n'\big(\varphi(z)\big)\varphi'(z)\big| \to 0$ uniformly as $n \to \infty$. Then for n large enough,

$$\big(1-|z|^2\big)\big|f_n'\big(\varphi(z)\big)\varphi'(z)\big| < \frac{\varepsilon}{2}, \text{ if } |\varphi(z)| \le \delta. \tag{3.9}$$

Hence, combining (3.8) and (3.9), we get $||C_\varphi f_n||_{\mathcal{B}} \le \varepsilon$ for sufficiently large n, i.e (3.7) holds.

(ii)⇒(iii) The implication is clear.

(iii)⇒(i) Let $C_\varphi : F_{p,0} \to \mathcal{B}$ be compact. If $\lim\limits_{|\varphi(z)|\to 1} F(p,\varphi,z) \ne 0$, there would be a positive constant ε_0 and a sequence $\{z_n\} \subset \mathbb{D}$ such that $F(p,\varphi,z_n) \ge \varepsilon_0$ and $|\varphi(z_n)| > 1-n^{-1}$. We may assume that $w_n = \varphi(z_n)$ tends to some point $w_0 \in \partial\mathbb{D}$. Here we can suppose that $w_n \ne 0$. We put

$$f_n(z) = \frac{p}{\overline{w_n}(1-\overline{w_n}z)^{\frac{1}{p}}} - \frac{p}{\overline{w_n}}.$$

Then $\{f_n\}$ is bounded in $F_{p,0}$ and f_n converges uniformly to

$$f_0(z) = \frac{p}{\overline{w_0}(1-\overline{w_0}z)^{\frac{1}{p}}} - \frac{p}{\overline{w_0}}$$

on any compact subset of $\mathbb{D}$. According to these constructions, we obtain

$$\begin{aligned}||C_\varphi f_n - C_\varphi f_0||_{\mathcal{B}} &= \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big|\big(C_\varphi f_n\big)'(z) - \big(C_\varphi f_0\big)'(z)\big|\\ &\ge \big(1-|z_n|^2\big)\big|\big(f_n'\big(\varphi(z_n)\big) - f_0'\big(\varphi(z_n)\big)\big)\varphi'(z_n)\big|\\ &\gtrsim F(p,\varphi,z_n)\left|1-\left(\frac{1-|w_n|^2}{1-\overline{w}_0 w_n}\right)^{\frac{1}{p}+1}\right|\\ &\gtrsim \varepsilon_0\left|1-\left(1+\frac{w_n(\overline{w}_0-\overline{w}_n)}{1-\overline{w}_0 w_n}\right)^{\frac{1}{p}+1}\right|\\ &\gtrsim \varepsilon_0\left|1-\left|1+\frac{w_n(\overline{w}_0-\overline{w}_n)}{1-\overline{w}_0 w_n}\right|^{\frac{1}{p}+1}\right|.\end{aligned}$$

Due to the compactness of $C_\varphi : F_{p,0} \to \mathcal{B}$, we get $||C_\varphi f_n - C_\varphi f_0||_{\mathcal{B}} \to 0$. Consequently,

$$\lim_{n\to\infty} \Big|1 - \Big|1 + \frac{w_n(\overline{w}_0 - \overline{w}_n)}{1 - \overline{w}_0 w_n}\Big|^{\frac{1}{p}+1}\Big| = 0,$$

which violates

$$\lim_{n\to\infty} \Big|\frac{w_n(\overline{w}_0 - \overline{w}_n)}{1 - \overline{w}_0 w_n}\Big| \to 1.$$

This contradiction completes the proof.

Similarly, we have the following characterization of compact composition operator $C_\varphi : F_p \to F_p$.

Theorem 3.9 Let $0 < p < \infty$ and φ be a holomorphic self-map of $\mathbb{D}$ such that $C_\varphi : F_p \to F_p$ is bounded. Then the following statements are equivalent:

(i) $\lim\limits_{|\varphi(z)|\to 1} G(p, \varphi, z) = 0$.

(ii) $C_\varphi : F_p \to F_p$ is compact.

(iii) $C_\varphi : F_{p,0} \to F_p$ is compact.

Proof First we prove (iii)⇒(i). Assume $\lim\limits_{|\varphi(z)|\to 1} G(p, \varphi, z) \neq 0$. Then there exists a constant $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subset \mathbb{D}$ such that $G(p, \varphi, z_n) \geq \varepsilon_0$ and $|\varphi(z_n)| > 1 - n^{-1}$ for all n. Again assume that $w_n = \varphi(z_n)$ tends to some point $w_0 \in \partial\mathbb{D}$. Here we can suppose that $w_n \neq 0$. Because

$$\left|\frac{w_n(\overline{w_0} - \overline{w_n})}{1 - \overline{w_0} w_n}\right| \to 1,$$

then

$$\left|1 - \big(1 + \frac{w_n(\overline{w_0} - \overline{w_n})}{1 - \overline{w_0} w_n}\big)^{\frac{1}{p}+1}\right| \nrightarrow 0.$$

Passing to a subsequence if necessary we can assume that there exists another positive number ε_0' such that

$$\left|1 - \Big|1 + \frac{w_n(\overline{w_0} - \overline{w_n})}{1 - \overline{w_0} w_n}\Big|^{\frac{1}{p}+1}\right| > \varepsilon_0'. \tag{3.10}$$

Define

$$f_n(z) = \frac{p}{\overline{w_n}\big(1 - \overline{w_n} z\big)^{\frac{1}{p}}} - \frac{p}{\overline{w_n}}.$$

Then $f_n \in F_{p,0}$ and converges uniformly to

$$f_0(z) = \frac{p}{\overline{w_0}\big(1 - \overline{w_0} z\big)^{\frac{1}{p}}} - \frac{p}{\overline{w_0}}$$

on any compact subset of $\mathbb{D}$. Therefore, by (3.10),

$$
\begin{aligned}
&||C_\varphi f_n - C_\varphi f_0||_{F_p}^p\\
=&\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\big|f_n'\big(\varphi(|z|e^{it})\big)-f_0'\big(\varphi(|z|e^{it})\big)|^p\cdot|\varphi'(|z|e^{it})\big|^p dt\\
\geq&\big(1-|z_n|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{\big|1-|w_n|^2\big|^{\frac{1}{p}+1}}-\frac{1}{\big|1-\overline{w_0}w_n\big|^{\frac{1}{p}+1}}\right|^p\big|\varphi'\big(|z_n|e^{it}\big)\big|^p dt\\
=&\big(1-|z_n|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\big|\varphi'(|z_n|e^{it})\big|^p}{\big|1-|\varphi\big(|z_n|e^{it}\big)\big|^2|^{p+1}}\cdot\left|1-\big(\frac{1-|w_n|^2}{1-\overline{w_0}w_n}\big)^{\frac{1}{p}+1}\right|^p dt\\
=&\big(1-|z_n|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\big|\varphi'(|z_n|e^{it})\big|^p}{\big|1-\big|\varphi\big(|z_n|e^{it}\big)\big|^2\big|^{p+1}}\cdot\left|1-\big(1+\frac{w_n(\overline{w_0}-\overline{w_n})}{1-\overline{w_0}w_n}\big)^{\frac{1}{p}+1}\right|^p dt\\
\gtrsim&\big(1-|z_n|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\big|\varphi'\big(|z_n|e^{it}\big)\big|^p}{\big|1-\big|\varphi\big(|z_n|e^{it}\big)\big|^2\big|^{p+1}}\cdot\left|1-\big|1+\frac{w_n(\overline{w_0}-\overline{w_n})}{1-\overline{w_0}w_n}\big|^{\frac{1}{p}+1}\right|^p dt\\
>&\varepsilon_0\varepsilon_0'^p.
\end{aligned}
$$

Then $C_\varphi f_n$ does not converge to $C_\varphi f_0$ in norm. Hence C_φ is not compact. This contradiction completes the proof of (iii)$\Rightarrow$(i).

(i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) can be proved by the means used in Theorem 3.8 and we omit the details here.

4 Isometry and Fredholmness

Many spaces of holomorphic functions in the unit disk $\mathbb{D}$ posses plenty of isometries (see [4, 8, 11]). In this note, we describe isometric composition operators from $\mathcal{B}$ to F_p for $1\leq p<\infty$.

Theorem 4.10 Let φ be a holomorphic self-map of the disk and $1\leq p<\infty$, then the composition operator $C_\varphi:\mathcal{B}\to F_p$ is an isometry if and only if φ is a rotation.

Proof The sufficiency is obvious. We only need to prove the necessity. To this end, suppose C_φ is an isometry from $\mathcal{B}$ to F_p, $1<p<\infty$. We first claim that $\varphi(0)=0$. Indeed, by the Schwarz-Pick lemma, for every $f\in\mathcal{B}$,

$$
\begin{aligned}
||C_\varphi f||_{F_p}^p&=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^p\cdot\frac{1}{2\pi}\int_0^{2\pi}\big|f'\big(\varphi(|z|e^{it})\big)\varphi'\big(|z|e^{it}\big)\big|^p dt\\
&=\sup_{z\in\mathbb{D}}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-|z|^2}{1-\big|\varphi\big(|z|e^{it}\big)\big|^2}\varphi'(|z|e^{it})\cdot\big(1-\big|\varphi(|z|e^{it})\big|^2\big)f'\big(\varphi(|z|e^{it})\big)\right|^p dt\\
&\leq||f||_{\mathcal{B}}^p.
\end{aligned}\tag{4.1}
$$

Since C_φ is isometric, then $|||C_\varphi(f)|||_{F_p}=|||f|||_{\mathcal{B}}$, which implies that

$$|f(\varphi(0))|^p+||C_\varphi f||_{F_p}^p=(|f(0)|+||f||_{\mathcal{B}})^p\geq|f(0)|^p+||f||_{\mathcal{B}}^p,\tag{4.2}$$

where the last inequality comes from $1 \le p < \infty$. It follows that $|f(\varphi(0))| \ge |f(0)|$ for all $f \in \mathcal{B}$. Write $\varphi(0) = a$, and choose

$$f(z) = \varphi_a(z) = \frac{a - z}{1 - \overline{a}z},$$

which interchanges the origin and the point a, $0 = |\varphi_a(a)| = |\varphi_a(\varphi(0))| \ge |\varphi_a(0)| = |a|$. Hence $\varphi(0) = 0$, which gives the claim. Thus, if C_φ is an isometry, then $||C_\varphi f||_{F_p} = ||f||_{\mathcal{B}}$ for all $f \in \mathcal{B}$. Now choose $f(z) \equiv z$, by the definition of F_p and the Schwarz-Pick lemma, we have

$$\begin{aligned} 0 &= ||z||_{\mathcal{B}}^p - ||C_\varphi z||_{F_p}^p \\ &= 1 - \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^p \cdot \frac{1}{2\pi} \int_0^{2\pi} \left|\varphi'(|z|e^{it})\right|^p dt \\ &\ge 1 - \sup_{z\in\mathbb{D}} \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \left|\varphi(|z|e^{it})\right|^2\right)^p dt \ge 0. \end{aligned}$$

Hence the inequality above actually is equality. So the equality in Schwarz-Pick lemma holds. Therefore, φ must be a disk automorphism. Recall that φ fixes the origin, it follows that φ is actually a rotation.

Finally, we consider the Fredholmness of composition operators on $F_{p,0}$. For a linear metric space X, recall that a bounded linear operator T on X is said to be Fredholm if both the dimension of its kernel and the codimension of its image are finite. This occurs if and only if T is invertible modulo the compact operators, that is, there is a bounded operator S such that both $TS - I$ and $ST - I$ are compact. We also notice that an operator is Fredholm if and only if its dual is Fredholm (see for example [12]).

The following result first gives a necessary condition of the Fredholm composition operators.

Theorem 4.11 Let φ be a holomorphic self map of $\mathbb{D}$. If C_φ is Fredholm on $F_{p,0}$, then φ is an automorphism.

Proof It is only needed to prove that φ is injective and onto. First, note that φ cannot be a constant mapping. Otherwise $\varphi(z) \equiv a$, we have $(z-a)^n \in \ker C_\varphi$ and $\dim\ker C_\varphi = \infty$, which contradicts the Fredholmness of C_φ.

Assume φ is not one to one. So there exist $z_1, z_2 \in \mathbb{D}$, $z_1 \ne z_2$ with $\varphi(z_1) = \varphi(z_2)$. Select the neighborhoods U, V of z_1, z_2 respectively such that $U \cap V = \emptyset$. $\varphi(U) \cap \varphi(V)$ is a nonempty and open set due to φ is open by the Open Mapping Theorem, so there exist infinite sequences $\{z_n^1\} \subseteq U$, $\{z_n^2\} \subseteq V$ such that $\varphi(z_n^1) = \varphi(z_n^2) = \omega_n$ which are distinct. Hence $C_\varphi^*\delta_{z_n^1} = \delta_{\varphi(z_n^1)} = \delta_{\varphi(z_n^2)} = C_\varphi^*\delta_{z_n^2}$, namely, $C_\varphi^*(\delta_{z_n^1} - \delta_{z_n^2}) = 0$, where $\delta_z : f \to f(z)$ is the evaluation function, which is a bounded linear functional on $F_{p,0}$. Since $F_{p,0}$ contains all polynomials, we have each evaluation function is not a linear combination of other evaluation functions, so the sequence $\{\delta_{z_n^1} - \delta_{z_n^2}\}$ is linearly independent in the kernel of the adjoint operator C_φ^*. It is worth pointing out that C_φ^* is also Fredholm. It is a contradiction, so φ is injective.

We now show that φ is onto. Assume that φ is not onto. Then we can find $z_0 \in \mathbb{D} \cap \partial\varphi(\mathbb{D})$ and $\{z_n\} \subseteq \mathbb{D}$ such that $\varphi(z_n) \to z_0$ as $n \to \infty$. Further, we get, by the Open Mapping Theorem, that $|z_n| \to 1$ as $n \to \infty$. For arbitrary $f \in F_{p,0}$,

$$C_\varphi^* \delta_{z_n} f = \delta_{\varphi(z_n)} f = f \circ \varphi(z_n) \to f(z_0) = \delta_{z_0} f,$$

we get $\delta_{\varphi(z_n)} \xrightarrow{w^*} \delta_{z_0}$ and $\{\delta_{\varphi(z_n)}\}$ is uniformly bounded . Again, it is obvious that $||\delta_{z_n}|| \to \infty$ as $n \to \infty$. Therefore, $||\frac{\delta_{\varphi(zn)}}{||\delta_{zn}||}|| = ||\frac{C_\varphi^*\delta_{zn}}{||\delta_{zn}||}|| \to 0$. On the other hand, since C_φ^* is also Fredholm, there are operators K and S on $F_{p,0}^*$, with K compact and S bounded, such that $SC_\varphi^* = I + K$. Thus $\frac{\delta_{zn}}{||\delta_{zn}||} + \frac{K\delta_{zn}}{||\delta_{zn}||} \to 0$. Because K is compact and $\{\frac{\delta_{zn}}{||\delta_{zn}||}\}$ is bounded, there exists subsequence $\{\frac{\delta_{zn_k}}{||\delta_{zn_k}||}\}$ such that $\frac{K\delta_{zn_k}}{||\delta_{zn_k}||} \to h$, $\frac{\delta_{zn_k}}{||\delta_{zn_k}||} \to -h$, which means $||h|| = 1$. Moreover, $F_{p,0}$ is the closure of all polynomials with respect to the norm $|||\cdot|||_{F_{p,0}}$, which gives $\frac{\delta_{zn_k}}{||\delta_{zn_k}||} \xrightarrow{w^*} 0$. This implies that $\frac{\delta_{zn_k}}{||\delta_{zn_k}||} \xrightarrow{w^*} -h = 0$. This is a contradiction. So φ is onto. Thus φ is a Möbius transformation, which completes the proof.

Remark By Theorem 3.7, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} G(p,\varphi_a,z) = \sup_{z\in\mathbb{D}} \big(1-|z|^2\big)^p \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{\big(1-|a|^2\big)^p \big|1-\overline{a}|z|e^{it}\big|^2}{\big||1-\overline{a}|z|e^{it}|^2 - |a-|z|e^{it}|^2\big|^{p+1}} dt \\
=&\sup_{z\in\mathbb{D}} \big(1-|z|^2\big)^p \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{\big(1-|a|^2\big)^p \big(1+|a|^2|z|^2 - 2\Re(\overline{a}z)\big)}{\big|1+|a|^2|z|^2-2\Re(\overline{a}z)-|a|^2-|z|^2+2\Re(\overline{a}z)\big|^{p+1}} dt \\
=&\sup_{z\in\mathbb{D}} \frac{1}{(1-|a|^2)(1-|z|^2)} \int_0^{2\pi} \big(1+|a|^2|z|^2-2\Re(\overline{a}z)\big) dt \\
=&\sup_{z\in\mathbb{D}} \frac{1+|a|^2|z|^2}{(1-|a|^2)(1-|z|^2)} \to \infty,
\end{aligned}$$

where $\varphi_a = \frac{a-z}{1-\overline{a}z}$. Then the composition operator $C_{\varphi_a} : F_{p,0} \to F_p$ is unbounded. Therefore, if φ is an automorphism, then $C_\varphi : F_{p,0} \to F_{p,0}$ is unbounded. Thus C_φ is never invertible and Fredholm, which is different from many other classical function spaces, such as Hardy spaces, where the Fredholmness and the invertibility of the composition operators are equivalent, at the same time, they are equivalent to the induced symbol φ is an automorphism. Then we end the paper by summarizing the following result.

Theorem 4.12 There is no Fredholm composition operator on $F_{p,0}$.

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F_p 空间上相关的复合算子

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摘要: 本文研究了F_p 空间上的复合算子的几个问题. 应用泛函分析的方法研究了F_p (相应地, $F_{p,0}$) 空间到Bloch 空间的复合算子的有界性和紧性的若干充分和必要条件. 此外, 也刻画了当$1 \le p < \infty$ 时从Bloch空间到F_p 空间的等距复合算子并且证明了当$0 < p < \infty$时$F_{p,0}$ 上的复合算子不具有Fredholm性.

关键词: 复合算子; 紧性; 等距; Fredholm性

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