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Hom - 弱 Hopf 代数上的 Hom-smash 余积

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摘 要: 本文研究了在 Hom-Hopf 代数上引入 Hom- 弱 Hopf 代数的问题. 利用建立弱左 H-Hom- 余模双代数的方法, 获得了 Hom-smash 余积的代数结构, 并证明了 Hom-smash 余积是 Hom- 余代数和 Hom- 弱 Hopf 代数, 推广了由 Molnar 定义的 smash 余积 Hopf 代数.

关键词: Hom - 弱 Hopf 代数; 弱左 H-Hom - 余模双代数; Hom-smash 余积
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1 引言

近年来, Hom - 结构 (Hom-Lie 代数、Hom - 代数、Hom - 余代数、Hom-Hopf 代数、Hom - 模、Hom - 余模和 Hom-Hopf 模) 得到了广泛的研究. 简而言之, Hom - 型结构把之前结构中的恒等映射替换为广义的扭曲映射 α. 随着 Hom - 代数研究的深入, 一些学者在文献 [1–5] 中又陆续引入了 Hom - 双代数、Hom-Hopf 代数和拟三角 Hom-Hopf 代数等概念, 并给出了一些重要的性质. 在文献 [6, 7] 中, 作者定义了 Hom-ω-smash 积和 Hom-ω-smash 余积, 并分别研究了它们的拟三角结构和辫化结构.

弱 Hopf 代数是由 Bohm 和 Nill 等人定义的 (见文献 [8]), 是通常的双代数和 Hopf 代数 (见文献 [9]) 的推广. 关于弱 Hopf 代数最简单的例子是群代数. 弱 Hopf 代数与 Hopf 代数有着相似的构成, 只是用更弱的条件去代替余乘法运算的保单位性和余单位运算的保乘法性. 因此, 弱 Hopf 代数的结构远比 Hopf 代数复杂.

本文的主要目的是在弱 Hopf 代数上引入扭曲映射 α, 定义 Hom - 弱 Hopf 代数概念. 然后, 在 Hom - 弱 Hopf 代数和余模结构的基础上, 建立弱左 H-Hom - 余模双代数并通过它构造 Hom-smash 余积, 并证明 Hom-smash 余积是 Hom - 余代数和 Hom - 弱 Hopf 代数, 推广了由 Molnar 定义的 smash 余积 Hopf 代数.

本文的所有工作都在域 k 上进行的. 所讨论的张量积和线性映射均指域 k 上的. 文中将使用 Sweedler 关于余代数余乘法的记号, 即对于 H 中的任意元 $h, \triangle(h) = \sum h_1 \otimes h_2$.

2 Hom - 弱 Hopf 代数

在这一节, 首先介绍 Hom - 代数和 Hom - 余代数的概念, 然后给出 Hom - 弱双代数和 Hom - 弱 Hopf 代数的定义, 并给出一些关于 Hom - 弱双代数的性质.

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定义 2.1 [1,2] 设 A 是线性空间, 并且有线性映射 $\mu: A\otimes A\rightarrow A$, $\alpha: A\rightarrow A$ 和 $\eta: k\rightarrow A$, 如果四元组 (A,μ,η,α) 对任意 $a,b,c\in A$ 满足下列条件

$$\alpha(ab)=\alpha(a)\alpha(b),\quad \alpha(\eta(1))=\eta(1), \tag{2.1}$$

$$\alpha(a)(bc)=(ab)\alpha(c),\quad a\eta(1)=\alpha(a)=\eta(1)a, \tag{2.2}$$

则称四元组 (A,μ,η,α) 为一个 Hom - 代数. 同时, 称 $\eta(1)$ 为 A 的弱单位元和记 $\eta(1)=1_A$.

定义 2.2 [2,3] 设 C 是线性空间, 并且有线性映射 $\triangle: C\rightarrow C\otimes C$,$\alpha: C\rightarrow C$ 和 $\varepsilon: C\rightarrow k$, 如果四元组 $(C,\triangle,\varepsilon,\alpha)$ 对任意 $c\in C$ 满足下列条件

$$\triangle(\alpha(c)) = \sum\alpha(c_1)\otimes\alpha(c_2),\quad \varepsilon(\alpha(c))=\varepsilon(c), \tag{2.3}$$

$$\sum\alpha(c_1)\otimes\triangle(c_2) = \sum\triangle(c_1)\otimes\alpha(c_2),\quad \sum c_1\varepsilon(c_2)=\alpha(c)=\sum\varepsilon(c_1)c_2, \tag{2.4}$$

则称四元组 $(C,\triangle,\varepsilon,\alpha)$ 为一个 Hom - 余代数.

定义 2.3 如果 (H,μ,η,α) 是一个 Hom - 代数, $(H,\triangle,\varepsilon,\alpha)$ 是一个 Hom - 余代数, 且代数和余代数结构满足下列相容性:

$$\triangle(xy) = \triangle(x)\triangle(y), \tag{2.5}$$

$$\varepsilon((xy)\alpha(z)) = \sum\varepsilon(xy_1)\varepsilon(y_2\alpha(z)), \tag{2.6}$$

$$\varepsilon(\alpha(x)(yz)) = \sum\varepsilon(\alpha(x)y_2)\varepsilon(y_1z), \tag{2.7}$$

$$(\triangle\otimes\alpha)\triangle(1) = \sum 1_1\otimes 1_2 1_{1'}\otimes\alpha(1_{2'}), \tag{2.8}$$

$$(\alpha\otimes\triangle)\triangle(1) = \sum\alpha(1_1)\otimes 1_{1'}1_2\otimes 1_{2'}, \tag{2.9}$$

则称六元组 $(H,\mu,\eta,\triangle,\varepsilon,\alpha)$ 为一个 Hom - 弱双代数, 并简记为 (H,α).

定义 2.4 设 (H,α) 是一个 Hom - 弱双代数, $S: H\rightarrow H$ 是一个线性映射, 如果满足

$$S\circ\alpha = \alpha\circ S, \tag{2.10}$$

$$\sum x_1S(x_2) = \sum\varepsilon(1_1\alpha^2(x))1_2, \tag{2.11}$$

$$\sum S(x_1)x_2 = \sum 1_1\varepsilon(\alpha^2(x)1_2), \tag{2.12}$$

$$\sum(S(x_{11})x_{12})S(\alpha^2(x_2)) = S(\alpha^4(x)), \tag{2.13}$$

则称 (H,α) 是一个 Hom - 弱 Hopf 代数, 并称 S 是 Hom - 弱 Hopf 代数 (H,α) 的对极映射.

注 2.5 由式 (2.11) 和 (2.12), 容易得到

$$\sum(x_{11}S(x_{12}))\alpha^2(x_2)=\alpha^4(x),$$

$$\sum\alpha^2(x_1)(S(x_{21})x_{22})=\alpha^4(x).$$

对于式 (2.13) 的合理性证明如下:

$$\begin{aligned}
\sum(S(x_{11})x_{12})S(\alpha^2(x_2)) &= \sum(1_1\varepsilon(\alpha^2(x_1)1_2))S(\alpha^2(x_2))\\
&= \sum(\alpha(1_1)\varepsilon((\alpha(x_1)1_{1'})\alpha(1_2)))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(x_1)(1_{1'}1_2)))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(x_1)1_{1'2})\varepsilon(1_{1'1}1_2))S(\alpha(x_2)1_{2'})\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(1_{1'})1_2))S(\varepsilon(\alpha^2(x_1)\alpha(1_{2'1}))(\alpha(x_2)1_{2'2}))\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(1_{1'})1_2))S(\varepsilon(\alpha(x_1)1_{2'1})(\alpha(x_2)1_{2'2}))\\
&= \sum(\alpha(1_1)\varepsilon(\alpha^2(1_{1'})1_2))S(\alpha^2(x)\alpha(1_{2'}))\\
&= \sum(\alpha^2(1_1)\varepsilon(\alpha(1_{1'})\alpha(1_2)))S(\alpha^2(x)1_{2'})\\
&= \sum(\alpha^2(1_1)\varepsilon(1_{21}))S(\alpha^2(x)1_{22})\\
&= \sum\alpha^2(1_1)S(\alpha^2(x)\alpha(1_2))\\
&= \sum(\alpha(1_1)S(\alpha(1_2)))S(\alpha^3(x))\\
&= \sum(\varepsilon(1_1\alpha^3(1))1_2)S(\alpha^3(x))\\
&= S(\alpha^4(x)).
\end{aligned}$$

注 2.6 Hom - 弱 Hopf 代数既不满足结合律也不满足余结合律, 且不再具有余乘法运算的保单位性和余单位运算的保乘法性, 但当扭曲映射 $\alpha = Id$ 时, 它就是弱 Hopf 代数; 但当余单位 ε 是代数映射时, Hom - 弱 Hopf 代数就是 Hom-Hopf 代数. 相对于 (余) 结合性, Hom - 弱 Hopf 代数也有 Hom -(余) 结合性, 即 $\mu\circ(\alpha\otimes\mu) = \mu\circ(\mu\otimes\alpha)$ 和 $(\alpha\otimes\triangle)\circ\triangle = (\triangle\otimes\alpha)\circ\triangle$. 因此, Hom - 弱 Hopf 代数的非 (余) 结合性的程度是由扭曲映射 α 偏离恒等映射的距离决定的.

对 Hom - 弱双代数 H, 定义映射 $\sqcap^L, \sqcap^R : H \to H$ 如下:

$$\sqcap^L(x) = \sum\varepsilon(1_1\alpha^2(x))1_2, \quad \sqcap^R(x) = \sum 1_1\varepsilon(\alpha^2(x)1_2).$$

用 H^L 表示映射 $\sqcap^L$ 的像集 $\sqcap^L(H), H^R$ 表示映射 $\sqcap^R$ 的像集 $\sqcap^R(H)$.

设 $(H, \mu, 1, \triangle, \varepsilon, S, \alpha)$ 是一个有限维的 Hom - 弱 Hopf 代数, H^* 是 H 的线性对偶空间, 则 $(H^*, \triangle^*, \widehat{1}, \mu^*, \widehat{\varepsilon}, S^*, \alpha^*)$ 是一个 Hom - 弱 Hopf 代数, 并对任意的 $x, y \in H$ 和 $\phi, \psi \in H^*$, 有

$$\begin{aligned}
&\langle\mu^*(\phi), h\otimes g\rangle = \langle\phi, \mu(h, g)\rangle, \quad \langle\triangle^*(\phi, \psi), h\rangle = \langle\phi\otimes\psi, \triangle(h)\rangle,\\
&\langle\widehat{1}, x\rangle = \varepsilon(x), \quad \widehat{\varepsilon}(\phi) = \langle\phi, 1\rangle,\\
&S^*(\phi) = \phi\circ S, \quad \alpha^*(\phi) = \phi\circ\alpha.
\end{aligned}$$

利用 Hom - 弱双代数的定义和上面的对偶关系, 容易得到下面的一些命题, 至于 Hom - 弱双代数的更多性质, 作者将另文讨论.

命题 2.7 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$\sqcap^L\circ\sqcap^L=\alpha^3\circ\sqcap^L=\sqcap^L\circ\alpha^3,$$
$$\sqcap^R\circ\sqcap^R=\alpha\circ\sqcap^R=\sqcap^R\circ\alpha.$$

命题 2.8 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$\sum 1_1\otimes\sqcap^L(1_2)=\sum 1_1\otimes\alpha^3(1_2),$$
$$\sum \sqcap^R(1_1)\otimes 1_2=\sum \alpha(1_1)\otimes 1_2.$$

命题 2.9 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$\sqcap^L(x\sqcap^L(y))=\sqcap^L(x\alpha^3(y)),$$
$$\sqcap^R(\sqcap^R(x)y)=\sqcap^R(\alpha(x)y).$$

命题 2.10 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$\triangle(\sqcap^L(x))=\sum 1_1\sqcap^L(x)\otimes 1_2,$$
$$\triangle(\sqcap^R(x))=\sum \alpha(1_1)\otimes\sqcap^R(\alpha(x))1_2.$$

命题 2.11 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$\sum x_1\otimes\sqcap^L(x_2)=\sum 1_1x\otimes\alpha^4(1_2),$$
$$\sum \sqcap^R(x_1)\otimes x_2=\sum \alpha^3(1_1)\otimes x1_2.$$

命题 2.12 设 (H,α) 是一个 Hom - 弱双代数, 则有如下结论:

$$x\sqcap^L(y)=\sum \varepsilon(x_1\alpha^4(y))x_2,$$
$$\sqcap^R(x)y=\sum y_1\varepsilon(\alpha^3(x)y_2).$$

3 Hom - 弱 smash 余积

关于 Hom - 模、Hom - 余模、Hom - 模代数和 Hom - 余模代数的相关定义可参阅文献 [4, 5]. 下面, 给出弱 H-Hom - 余模双代数的概念.

设 (H,β) 是 Hom - 弱双代数, (C,α) 是 Hom - 双代数. 如果有线性映射 $\rho: C\to H\otimes C$, $\rho(c)=\sum c_{(-1)}\otimes c_{(0)}$, 使得对任意 $c,d\in C$, 有

$$(\triangle_H\otimes\alpha)\circ\rho=(\beta\otimes\rho)\circ\rho,\quad(\beta\otimes\alpha)\circ\rho=\rho\circ\alpha,\tag{3.1}$$
$$(\mu_H\otimes I_C\otimes I_C)(I_H\otimes\tau\otimes I_C)(\rho\otimes\rho)\triangle_C=(\beta^2\otimes\triangle_C)\rho,\tag{3.2}$$
$$(\sqcap^L\otimes\varepsilon_C)\circ\rho=(I_H\otimes\varepsilon_C)\circ\rho,\quad(\varepsilon_H\otimes I_C)\circ\rho=\alpha,\tag{3.3}$$
$$\rho(1_C)=(\sqcap^R\otimes I_C)\circ\rho(1_C),\quad\rho(cd)=\rho(c)\rho(d).\tag{3.4}$$

这里 τ 是扭曲映射, 如果 (C,α) 满足条件 (3.1)–(3.3), 则称 (C,α) 是一个弱左 H-Hom - 余模余代数; 如果 (C,α) 满足条件 (3.1) 和 (3.4), 则称 (C,α) 是一个弱左 H-Hom - 余模代数; 如果 (C,α) 满足条件 (3.1)–(3.4), 则称 (C,α) 是一个弱左 H-Hom - 余模双代数. 如果 $\alpha=I_C$ 和 $\beta=I_H$, 则弱左 H-Hom - 余模双代数是弱 Hopf 代数上的弱左 H- 余模双代数. 如果 (H,β) 和 (C,α) 是 Hom - 双代数, 则弱左 H-Hom - 余模双代数是左 H-Hom - 余模双代数.

设 (H,β) 是 Hom - 弱双代数, (C,α) 是弱左 H-Hom - 余模余代数, 且 α,β 都可逆. 为方便, 分别记 1_H 为 1, 1_C 为 $\widehat{1}$. 定义线性映射 $\chi: C\otimes H\to C\otimes H$ 如下:

$$\chi(c\otimes h)=\sum\varepsilon_H(c_{(-1)}h_1)\alpha^{-1}(c_{(0)})\otimes\beta^{-1}(h_2).$$

则有

$$\begin{aligned}\chi^2(c\otimes h)&=\sum\varepsilon_H(c_{(-1)}h_1)\chi(\alpha^{-1}(c_{(0)})\otimes\beta^{-1}(h_2))\\&=\sum\varepsilon_H(c_{(-1)}h_1)\varepsilon_H(\beta^{-1}(c_{(0)(-1)})\beta^{-1}(h_{21}))\alpha^{-2}(c_{(0)(0)})\otimes\beta^{-2}(h_{22})\\&=\sum\varepsilon_H(\beta^{-1}(c_{(-1)1}h_{11}))\varepsilon_H(\beta^{-1}(c_{(-1)2}h_{12}))\alpha^{-1}(c_{(0)})\otimes\beta^{-1}(h_2)\\&=\sum\varepsilon_H(c_{(-1)}h_1)\alpha^{-1}(c_{(0)})\otimes\beta^{-1}(h_2)\\&=\chi(c\otimes h).\end{aligned}$$

所以 χ 是一个投射. 因此, 定义 $C\times H$, 作为向量空间有 $C\times H=(C\otimes H)/\ker\chi$, 令线性映射 $\gamma=\alpha\otimes\beta: C\otimes H\to C\otimes H$, 并定义弱余单位和余乘分别为

$$\begin{aligned}\varepsilon_{C\times H}(c\times h)&=\varepsilon_C\otimes\varepsilon_H(c\times h)=\varepsilon_C(c)\varepsilon_H(h)\widehat{1}\times 1,\\\triangle_{C\times H}(c\times h)&=\sum c_1\times\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1)\otimes\alpha^{-1}(c_{2(0)})\times h_2,\end{aligned}$$

则称四元组 $(C\times H,\triangle_{C\times H},\varepsilon_{C\times H},\gamma)$ 是一个 Hom-smash 余积, 并记为 $(C\times H,\gamma)$, 如果它是一个 Hom - 余代数.

定理 3.1 设 (H,β) 是 Hom - 弱双代数, (C,α) 是弱左 H-Hom - 余模余代数, 且 α,β 都可逆, 而 $C\times H$ 的定义如上, 若对任意的 $c\in C, h\in H$, 满足

$$\sum c_{(0)}\otimes c_{(-1)}h=\sum c_{(0)}\otimes hc_{(-1)},\tag{3.5}$$

则四元组 $(C\times H,\triangle_{C\times H},\varepsilon_{C\times H},\gamma)$ 是一个 Hom-smash 余积.

证 由于 (C,α) 是弱左 H-Hom - 余模余代数, 所以对任意的 $c\times h\in C\times H$, 有

$$\begin{aligned}&(\triangle_{C\times H}\otimes\gamma)\triangle_{C\times H}(c\times h)\\=&\sum c_{11}\times\beta^{-2}(c_{12(-1)})\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1)\\&\otimes\alpha^{-1}(c_{12(0)})\times(\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2\otimes c_{2(0)}\times\beta(h_2)\\\overset{(2.4)}{=}&\sum\alpha(c_1)\times\beta^{-2}(c_{21(-1)})(\beta^{-4}(c_{22(-1)1})\beta^{-1}(h_1))\\&\otimes\alpha^{-1}(c_{21(0)})\times\beta^{-3}(c_{22(-1)2})\beta^{-1}(h_{21})\otimes\alpha^{-1}(c_{22(0)})\times h_{22}\end{aligned}$$

$$\begin{aligned}
&\overset{(3.1)}{=} && \sum \alpha(c_1)\times\beta^{-2}(c_{21(-1)})(\beta^{-3}(c_{22(-1)})\beta^{-1}(h_1))\\
& && \otimes\alpha^{-1}(c_{21(0)})\times\beta^{-3}(c_{22(0)(-1)})\beta^{-1}(h_{21})\otimes\alpha^{-2}(c_{22(0)(0)})\times h_{22}\\
&\overset{(2.2)}{=} && \sum \alpha(c_1)\times\beta^{-3}(c_{21(-1)}c_{22(-1)})h_1\\
& && \otimes\alpha^{-1}(c_{21(0)})\times\beta^{-3}(c_{22(0)(-1)})\beta^{-1}(h_{21})\otimes\alpha^{-2}(c_{22(0)(0)})\times h_{22}\\
&\overset{(3.2)}{=} && \sum \alpha(c_1)\times\beta^{-1}(c_{2(-1)})h_1\\
& && \otimes\alpha^{-1}(c_{2(0)1})\times\beta^{-3}(c_{2(0)2(-1)})\beta^{-1}(h_{21})\otimes\alpha^{-2}(c_{2(0)2(0)})\times h_{22}\\
&= && (\gamma\otimes\triangle_{C\times H})\triangle_{C\times H}(c\times h).
\end{aligned}$$

直接计算可得 $\triangle_{C\times H}\gamma=(\gamma\otimes\gamma)\triangle_{C\times H},\varepsilon_{C\times H}\gamma=\varepsilon_{C\times H}$. 由 (3.3)、(3.5) 式和 $\sum c_{(0)}\otimes\varepsilon_H(c_{(-1)}h_1)h_2=\alpha(c)\otimes\beta(h)$. 易证 $(\varepsilon_{C\times H}\otimes I_{C\times H})\triangle_{C\times H}=\gamma$ 及 $(I_{C\times H}\otimes\varepsilon_{C\times H})\triangle_{C\times H}=\gamma$ 成立. 这证明 $(C\times H,\triangle_{C\times H},\varepsilon_{C\times H},\gamma)$ 是一个 Hom-smash 余积.

定理 3.2 设 (H,β) 和 (C,α) 是 Hom - 弱双代数, 且 (C,α) 是弱左 H-Hom - 余模双代数, 其中 α,β 都可逆, 并满足 (3.5) 式, 则 Hom-smash 余积 $(C\times H,\gamma)$ 是 Hom - 弱双代数, 其中 $(C\times H,\gamma)$ 的 Hom - 代数结构是张量积 Hom - 代数.

若此时 (H,β) 和 (C,α) 是 Hom - 弱 Hopf 代数, 其弱对极分别为 S_H 和 S_C, 则 Hom-smash 余积 $(C\times H,\gamma)$ 也是 Hom - 弱 Hopf 代数, 其弱对极为

$$S_{C\times H}(c\times h)=\sum S_C(\alpha^{-1}(c_{(0)}))\times S_H(\beta^{-2}(c_{(-1)})\beta^{-1}(h)).$$

证 显然, Hom-smash 余积 $(C\times H,\gamma)$ 是 Hom - 代数和 Hom - 余代数. 由题设知, 需要证明 Hom-smash 余积 $(C\times H,\gamma)$ 满足定义 2.3 的 (2.5)–(2.9) 项. 对任意的 $c,d\in C,h,g\in H$, 由 (2.2) 和 (3.5) 式及条件 $\rho(cd)=\rho(c)\rho(d)$, 可得下式

$$\begin{aligned}
& \triangle_{C\times H}(cd\times hg)\\
={}& \sum c_1d_1\times\beta^{-2}(c_{2(-1)}d_{2(-1)})\beta^{-1}(h_1g_1)\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \sum c_1d_1\times\beta^{-1}(c_{2(-1)})\beta^{-2}(d_{2(-1)}(h_1g_1))\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \sum c_1d_1\times\beta^{-1}(c_{2(-1)})\beta^{-2}((\beta^{-1}(d_{2(-1)})h_1)\beta(g_1))\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \sum c_1d_1\times\beta^{-1}(c_{2(-1)})\beta^{-2}((h_1\beta^{-1}(d_{2(-1)}))\beta(g_1))\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \sum c_1d_1\times\beta^{-1}(c_{2(-1)})\beta^{-2}(\beta(h_1)(\beta^{-1}(d_{2(-1)})g_1))\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \sum c_1d_1\times(\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))(\beta^{-2}(d_{2(-1)})\beta^{-1}(g_1))\otimes\alpha^{-1}(c_{2(0)}d_{2(0)})\times h_2g_2\\
={}& \triangle_{C\times H}(c\times h)\triangle_{C\times H}(d\times g).
\end{aligned}$$

对于余单位的弱乘运算, 事实上, 有

$$\begin{aligned}
& \sum\varepsilon_{C\times H}((a\times h)(b\times g)_1)\varepsilon_{C\times H}((b\times g)_2\gamma(c\times k))\\
={}& \sum\varepsilon_{C\times H}(ab_1\times h(\beta^{-2}(b_{2(-1)})\beta^{-1}(g_1)))\varepsilon_{C\times H}(\alpha^{-1}(b_{2(0)})\alpha(c)\times g_2\beta(k))\\
={}& \sum\varepsilon_C(ab_1)\varepsilon_H(h(\beta^{-2}(b_{2(-1)})\beta^{-1}(g_1)))\varepsilon_C(\alpha^{-1}(b_{2(0)})\alpha(c))\varepsilon_H(g_2\beta(k))
\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.2)}{=} \sum \varepsilon_C(ab_1)\varepsilon_H((\beta^{-1}(h)\beta^{-2}(b_{2(-1)}))g_1)\varepsilon_C(\alpha^{-1}(b_{2(0)})\alpha(c))\varepsilon_H(g_2\beta(k))\\
&\overset{(3.5)}{=} \sum \varepsilon_C(ab_1)\varepsilon_H((\beta^{-2}(b_{2(-1)})\beta^{-1}(h))g_1)\varepsilon_C(\alpha^{-1}(b_{2(0)})\alpha(c))\varepsilon_H(g_2\beta(k))\\
&\overset{(2.3)}{=} \sum \varepsilon_C(ab_1)\varepsilon_H((\beta^{-1}(b_{2(-1)})h)\beta(g_1))\varepsilon_C(\alpha^{-1}(b_{2(0)})\alpha(c))\varepsilon_H(g_2\beta(k))\\
&\overset{(2.6)}{=} \sum \varepsilon_C(ab_1)\varepsilon_H(\beta^{-1}(b_{2(-1)})h_1)\varepsilon_H(h_2\beta(g_1))\varepsilon_C(\alpha^{-1}(b_{2(0)})\alpha(c))\varepsilon_H(g_2\beta(k))\\
&= \sum \varepsilon_C(ab_1)\varepsilon_H(\beta(h)\beta(g_1))\varepsilon_C(b_2\alpha(c))\varepsilon_H(g_2\beta(k))\\
&\overset{(2.6)}{=} \sum \varepsilon_C((ab)\alpha(c))\varepsilon_H((hg)\beta(k))\\
&= \varepsilon_{C\times H}(((a\times h)(b\times g))\gamma(c\times k)).
\end{aligned}
$$

同理可得

$$
\varepsilon_{C\times H}(\gamma(a\times h)((b\times g)(c\times k))) = \sum \varepsilon_{C\times H}(\gamma(a\times h)(b\times g)_2)\varepsilon_{C\times H}((b\times g)_1(c\times k)).
$$

下面计算单位的余乘运算

$$
\begin{aligned}
&(\triangle_{C\times H}\otimes\gamma)\triangle_{C\times H}(\widehat{1}\times 1)\\
= &\sum \widehat{1}_{11}\times\beta^{-2}(\widehat{1}_{12(-1)})(\beta^{-3}(\widehat{1}_{2(-1)1})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{12(0)})\times\beta^{-2}(\widehat{1}_{2(-1)2})\beta^{-1}(1_{12})\otimes\widehat{1}_{2(0)}\times\beta(1_2)\\
\overset{(2.8)}{=} &\sum \widehat{1}_1\times\beta^{-2}((\widehat{1}_2\widehat{1}_{1'})_{(-1)})(\beta^{-3}(\widehat{1}_{2'(-1)1})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}((\widehat{1}_2\widehat{1}_{1'})_{(0)})\times\beta^{-2}(\widehat{1}_{2'(-1)2})\beta^{-1}(1_{12})\otimes\widehat{1}_{2'(0)}\times\beta(1_2)\\
\overset{(3.4)}{=} &\sum \widehat{1}_1\times\beta^{-2}(\widehat{1}_{2(-1)}\widehat{1}_{1'(-1)})(\beta^{-3}(\widehat{1}_{2'(-1)1})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{1'(0)})\times\beta^{-2}(\widehat{1}_{2'(-1)2})\beta^{-1}(1_{12})\otimes\widehat{1}_{2'(0)}\times\beta(1_2)\\
\overset{(3.1)}{=} &\sum \widehat{1}_1\times\beta^{-2}(\widehat{1}_{2(-1)}\widehat{1}_{1'(-1)})(\beta^{-2}(\widehat{1}_{2'(-1)})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{1'(0)})\times\beta^{-2}(\widehat{1}_{2'(0)(-1)})\beta^{-1}(1_{12})\otimes\alpha^{-1}(\widehat{1}_{2'(0)(0)})\times\beta(1_2)\\
\overset{(2.2)}{=} &\sum \widehat{1}_1\times\beta^{-1}(\widehat{1}_{2(-1)})(\beta^{-3}(\widehat{1}_{1'(-1)}\widehat{1}_{2'(-1)})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{1'(0)})\times\beta^{-2}(\widehat{1}_{2'(0)(-1)})\beta^{-1}(1_{12})\otimes\alpha^{-1}(\widehat{1}_{2'(0)(0)})\times\beta(1_2)\\
\overset{(3.2)}{=} &\sum \widehat{1}_1\times\beta^{-1}(\widehat{1}_{2(-1)})(\beta^{-1}(\widehat{1}_{(-1)})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{(0)1})\times\beta^{-2}(\widehat{1}_{(0)2(-1)})\beta^{-1}(1_{12})\otimes\alpha^{-1}(\widehat{1}_{(0)2(0)})\times\beta(1_2)\\
\overset{(3.4)}{=} &\sum \widehat{1}_1\times\beta^{-1}(\widehat{1}_{2(-1)})(\beta^{-1}(\sqcap^R(\widehat{1}_{(-1)}))\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{(0)1})\times\beta^{-2}(\widehat{1}_{(0)2(-1)})\beta^{-1}(1_{12})\otimes\alpha^{-1}(\widehat{1}_{(0)2(0)})\times\beta(1_2)\\
\overset{(2.6)}{=} &\sum \widehat{1}_1\times\beta^{-1}(\widehat{1}_{2(-1)})(\beta^{-1}(1_{1'})\varepsilon_H(\varepsilon_H(\beta(\widehat{1}_{(-1)})1_{1''})1_{2''}1_{2'})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)}\widehat{1}_{(0)1})\times\beta^{-2}(\widehat{1}_{(0)2(-1)})\beta^{-1}(1_{12})\otimes\alpha^{-1}(\widehat{1}_{(0)2(0)})\times\beta(1_2)\\
= &\sum \widehat{1}_1\times\beta^{-1}(\widehat{1}_{2(-1)})(\beta^{-1}(1_{1'})\varepsilon_H(\beta(1)1_{2'})\beta^{-2}(1_{11}))\\
&\otimes\alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'}\times\beta^{-1}(\widehat{1}_{2'(-1)})\beta^{-1}(1_{12})\otimes\widehat{1}_{2'(0)}\times\beta(1_2)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(2.4)}{=} \sum \widehat{1}_1 \times \beta^{-1}(\widehat{1}_{2(-1)})\beta^{-1}(1_{11}) \\
&\quad \otimes \alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'} \times \beta^{-1}(\widehat{1}_{2'(-1)})\beta^{-1}(1_{12}) \otimes \widehat{1}_{2'(0)} \times \beta(1_2) \\
&\overset{(2.8)}{=} \sum \widehat{1}_1 \times \beta^{-1}(\widehat{1}_{2(-1)})\beta^{-1}(1_1) \\
&\quad \otimes \alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'} \times \beta^{-1}(\widehat{1}_{2'(-1)})\beta^{-1}(1_2 1_{1'}) \otimes \widehat{1}_{2'(0)} \times \beta(1_{2'}) \\
&\overset{(2.2)}{=} \sum \widehat{1}_1 \times \beta^{-2}(\widehat{1}_{2(-1)})\beta^{-1}(1_1) \\
&\quad \otimes \alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'} \times (\beta^{-2}(\widehat{1}_{2'(-1)})\beta^{-1}(1_2))1_{1'} \otimes \widehat{1}_{2'(0)} \times \beta(1_{2'}) \\
&\overset{(3.5)}{=} \sum \widehat{1}_1 \times \beta^{-2}(\widehat{1}_{2(-1)})\beta^{-1}(1_1) \\
&\quad \otimes \alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'} \times (\beta^{-1}(1_2)\beta^{-2}(\widehat{1}_{2'(-1)}))1_{1'} \otimes \widehat{1}_{2'(0)} \times \beta(1_{2'}) \\
&\overset{(2.2)}{=} \sum \widehat{1}_1 \times \beta^{-2}(\widehat{1}_{2(-1)})\beta^{-1}(1_1) \\
&\quad \otimes \alpha^{-1}(\widehat{1}_{2(0)})\widehat{1}_{1'} \times 1_2(\beta^{-2}(\widehat{1}_{2'(-1)})\beta^{-1}(1_{1'})) \otimes \widehat{1}_{2'(0)} \times \beta(1_{2'}) \\
&= \sum (\widehat{1} \times 1)_1 \otimes (\widehat{1} \times 1)_2 (\widehat{1} \times 1)_{1'} \otimes \gamma((\widehat{1} \times 1)_{2'}).
\end{aligned}
$$

同理可得

$$(\gamma \otimes \triangle_{C\times H}) \triangle_{C\times H} (\widehat{1} \times 1) = \sum \gamma((\widehat{1} \times 1)_1) \otimes (\widehat{1} \times 1)_{1'}(\widehat{1} \times 1)_2 \otimes (\widehat{1} \times 1)_{2'}.$$

因此 Hom-smash 余积 $(C \otimes H, \gamma)$ 是 Hom - 弱双代数.

最后, 设 (H, β) 和 (C, α) 是 Hom - 弱 Hopf 代数, 由于 $S_{C\times H} \circ \gamma = \gamma \circ S_{C\times H}$ 直接验证可得. 现需证明 $S_{C\times H}$ 满足定义 2.4 中的 (2.11)–(2.13) 项. 设任意 $c \in C$ 和 $h \in H$, 有

$$
\begin{aligned}
& I * S_{C\times H}(c \times h) \\
= {} & \sum (c_1 \times \beta^{-2}(c_{2(-1)})\beta^{-1}(h_1)) S_{C\times H}(\alpha^{-1}(c_{2(0)}) \times h_2) \\
= {} & \sum c_1 S_C(\alpha^{-2}(c_{2(0)(0)})) \times (\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1)) S_H(\beta^{-3}(c_{2(0)(-1)})\beta^{-1}(h_2)) \\
\overset{(2.2)}{=} {} & \sum c_1 S_C(\alpha^{-2}(c_{2(0)(0)})) \times \beta^{-1}(c_{2(-1)})(\beta^{-2}(h_1 S_H(h_2)) S_H(\beta^{-3}(c_{2(0)(-1)}))) \\
\overset{(3.5)}{=} {} & \sum c_1 S_C(\alpha^{-2}(c_{2(0)(0)})) \times \beta^{-1}(c_{2(-1)})(S_H(\beta^{-3}(c_{2(0)(-1)})) \sqcap^L (\beta^{-2}(h))) \\
\overset{(2.2)}{=} {} & \sum c_1 S_C(\alpha^{-2}(c_{2(0)(0)})) \times (\beta^{-2}(c_{2(-1)}) S_H(\beta^{-3}(c_{2(0)(-1)}))) \sqcap^L (\beta^{-1}(h)) \\
\overset{(3.1)}{=} {} & \sum c_1 S_C(\alpha^{-1}(c_{2(0)})) \times \beta^{-3}(c_{2(-1)1} S_H(c_{2(-1)2})) \sqcap^L (\beta^{-1}(h)) \\
\overset{(3.5)}{=} {} & \sqcap^L(c) \times \sqcap^L(h) \\
= {} & \sum \varepsilon_C(\widehat{1}_1 \alpha^2(c))\widehat{1}_2 \times \varepsilon_H(1_1 \beta^2(h))1_2 \\
= {} & \sum \varepsilon_C(\widehat{1}_1 \alpha^2(c))\alpha^{-1}(\widehat{1}_{2(0)}) \times \varepsilon_H(\beta^{-1}(\varepsilon_H(\beta^{-1}(\widehat{1}_{2(-1)})1_{11})1_{12})\beta^2(h))1_2 \\
\overset{(2.6)}{=} {} & \sum \varepsilon_C(\widehat{1}_1 \alpha^2(c))\alpha^{-1}(\widehat{1}_{2(0)}) \times \varepsilon_H((\beta^{-2}(\widehat{1}_{2(-1)})\beta^{-1}(1_1))\beta^2(h))1_2 \\
= {} & \sum \varepsilon_{C\times H}(\widehat{1}_1 \alpha^2(c) \times (\beta^{-2}(\widehat{1}_{2(-1)})\beta^{-1}(1_1))\beta^2(h))\alpha^{-1}(\widehat{1}_{2(0)}) \times 1_2 \\
= {} & \sum \varepsilon_{C\times H}((\widehat{1} \times 1)_1 \gamma^2(c \times h))(\widehat{1} \times 1)_2 = \sqcap^L(c \times h).
\end{aligned}
$$

因此 $I * S_{C\times H}(c\times h)=\sqcap^L(c\times h)$. 同理可得 $S_{C\times H}*I(c\times h)=\sqcap^R(c\times h)$.

$$
\begin{aligned}
& (S_{C\times H}((c\times h)_{11})(c\times h)_{12})S_{C\times H}(\gamma^2((c\times h)_2))\\
= & \sum(S_{C\times H}(c_{11}\times\beta^{-2}(c_{12(-1)})\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1))\\
& (\alpha^{-1}(c_{12(0)})\times(\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2))S_{C\times H}(\alpha(c_{2(0)})\times\beta^2(h_2))\\
= & \sum(S_C(\alpha^{-1}(c_{11(0)}))\alpha^{-1}(c_{12(0)}))S_C(c_{2(0)(0)})\\
& \times(S_H(\beta^{-2}(c_{11(-1)})\beta^{-1}(\beta^{-2}(c_{12(-1)})\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1)))\\
& (\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2)S_H(\beta^{-1}(c_{2(0)(-1)})\beta(h_2))\\
\overset{(2.2)}{=} & \sum(S_C(\alpha^{-1}(c_{11(0)}))\alpha^{-1}(c_{12(0)}))S_C(c_{2(0)(0)})\\
& \times(S_H(\beta^{-3}(c_{11(-1)}c_{12(-1)})\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1))\\
& (\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2)S_H(\beta^{-1}(c_{2(0)(-1)})\beta(h_2))\\
\overset{(3.2)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(c_{2(0)(0)})\\
& \times(S_H(\beta^{-1}(c_{1(-1)})\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1))\\
& (\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2)S_H(\beta^{-1}(c_{2(0)(-1)})\beta(h_2))\\
\overset{(3.5)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(c_{2(0)(0)})\\
& \times((S_H(\beta^{-1}(c_{1(-1)}))S_H(\beta^{-1}((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1)))\\
& (\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2)S_H(\beta^{-1}(c_{2(0)(-1)})\beta(h_2))\\
\overset{(2.2)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(c_{2(0)(0)})\\
& \times(S_H(c_{1(-1)})\beta^{-1}(S_H((\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_1)(\beta^{-2}(c_{2(-1)})\beta^{-1}(h_1))_2))\\
& S_H(\beta^{-1}(c_{2(0)(-1)})\beta(h_2))\\
\overset{(3.1)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(\alpha(c_{2(0)}))\\
& \times(S_H(c_{1(-1)})\beta^{-1}(S_H((\beta^{-3}(c_{2(-1)1})\beta^{-1}(h_1))_1)(\beta^{-3}(c_{2(-1)1})\beta^{-1}(h_1))_2))\\
& S_H(\beta^{-1}(c_{2(-1)2})\beta(h_2))\\
\overset{(2.2)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(\alpha(c_{2(0)}))\times\beta(S_H(c_{1(-1)}))\\
& \beta^{-1}((S_H((\beta^{-3}(c_{2(-1)1})\beta^{-1}(h_1))_1)(\beta^{-3}(c_{2(-1)1})\beta^{-1}(h_1))_2)\\
& S_H(\beta^{-1}(c_{2(-1)2})\beta(h_2)))\\
\overset{(2.13)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(\alpha(c_{2(0)}))\times\beta(S_H(c_{1(-1)}))\\
& \beta^{-1}(S_H(\beta(c_{2(-1)})\beta^3(h)))\\
& \overset{(3.5)}{=}\sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(\alpha(c_{2(0)}))\times S_H(\beta(c_{1(-1)})(c_{2(-1)}\beta^2(h)))\\
\overset{(2.2)}{=} & \sum(S_C(\alpha^{-1}(c_{1(0)1}))\alpha^{-1}(c_{1(0)2}))S_C(\alpha(c_{2(0)}))\times S_H((c_{1(-1)}c_{2(-1)})\beta^3(h))\\
\overset{(3.2)}{=} & \sum(S_C(\alpha^{-1}(c_{(0)11}))\alpha^{-1}(c_{(0)12}))S_C(\alpha(c_{(0)2}))\times S_H(\beta^2(c_{(-1)})\beta^3(h))\\
\overset{(2.13)}{=} & \sum S_C(\alpha^3(c_{(0)}))\times S_H(\beta^2(c_{(-1)})\beta^3(h))\\
= & S_{C\times H}(\gamma^4(c\times h)).
\end{aligned}
$$

所以 Hom-smash 余积 $(C\times H,\gamma)$ 是 Hom - 弱 Hopf 代数.

注 3.3 如果线性映射 α 和 β 是恒等映射, 即对任意的 $c\in C$ 和 $h\in H$, 有 $\gamma(c\otimes h)=c\otimes h$, 则 Hom-smash 余积 $(C\times H,\gamma)$ 是由文献 [8] 定义的弱 Hopf 代数. 如果 (C,α) 和 (H,β) 是 Hom-Hopf 代数, 则 Hom-smash 余积 $(C\times H,\gamma)$ 是 Hom-Hopf 代数, 并可得文献 [7] 中的例 2.11, 并推广了文献 [10] 中由 Molnar 定义的 smash 余积 Hopf 代数.

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HOM-SMASH COPRODUCTS OVER HOM-WEAK HOPF ALGEBRAS

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Abstract: In this paper, we study the concept of weak Hopf algebras over Hom-Hopf algebras. Using the method of establishing weak left H-Hom-comodule bialgebra, we construct Hom-smash coproduct and demonstrate that Hom-smash coproduct is a Hom-coalgebra and Hom-weak Hopf algebra, which generalizes smash coproduct Hopf algebra introduced by Molnar.

Keywords: Hom-weak Hopf algebra; weak left H-Hom-comodule bialgebra; Hom-smash coproduct

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