Vol. 36 (2016)
No. 2

数 学 杂 志
J. of Math. (PRC)

BISYMMETRIC MINIMAL RANK SOLUTIONS AND ITS OPTIMAL APPROXIMATION TO A CLASS OF MATRIX EQUATION

FENG Tian-xiang

(*Department of Basic, Dongguan Polytechnic, Dongguan 523808, China*)

Abstract: In this paper, the Bisymmetric maximal and minimal rank solutions to the matrix equation $AX = B$ and their optimal approximation are considered. By applying the matrix rank method, the necessary and sufficient conditions for the existence of the maximal and minimal rank solutions with Bisymmetric to the equation. The expressions of such solutions to this equation are also given when the solvability conditions are satisfied. In addition, in corresponding the minimal rank solution set to the equation, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided.

Keywords: matrix equation; bisymmetric matrix; maximal rank; minimal rank; optimal approximate solution

2010 MR Subject Classification: 15A06; 15A29

Document code: A **Article ID:** 0255-7797(2016)02-0285-08

1 Introduction

Throughout this paper, let $R^{n\times m}$ be the set of all $n \times m$ real matrices, $SR^{n\times m}$ be the set of all $n \times m$ real symmetric matrices, $OR^{n\times n}$ be the set of all $n \times n$ orthogonal matrices. Denote by I_n the identity matrix with order n. For matrix A, A^T, A^+, $\|A\|$ and $r(A)$ represent its transpose, Moore-Penrose inverse, Frobenius norm and rank, respectively. For a matrix A, the two matrices L_A and R_A stand for the two orthogonal projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by A.

Definition 1 A real symmetric matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be a Bisymmetric matrix if $a_{ij} = a_{n+1-j,n+1-i}, i, j = 1, 2, \cdots, n$. The set of all $n \times n$ Bisymmetric matrices is denoted by $BSR^{n\times n}$.

In this paper, we consider the Bisymmetric extremal rank solutions of the matrix equation

$$AX = B, \tag{1.1}$$

* **Received date:** 2013-08-29 **Accepted date:** 2013-12-10

Foundation item: Supported by Scientific Research Fund of Dongguan Polytechnic (2012a01).

Biography: Feng Tianxiang (1961–), male, professor, major in computational mathematics and mathematics education.

where X and B are given matrices in $R^{n\times m}$. In 1972, Mitra [1] considered solutions with fixed ranks for the matrix equations $AX = B$ and $AXB = C$. In 1984, Mitra [2] gave common solutions of minimal rank of the pair of complex matrix equations $AX = C, XB = D$. In 1987, Uhlig [3] presented the extremal ranks of solutions to the matrix equation $AX = B$. In 1990, Mitra studied the minimal ranks of common solutions to the pair of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$ over a general field in [4]. In 2003, Tian (see [5, 6]) investigated the extremal rank solutions to the complex matrix equation $AXB = C$ and gave some applications. Xiao et al. [7, 8] considered the symmetric and anti-symmetric minimal rank solution to equation $AX = B$. The Bisymmetric maximal and minimal rank solutions of the matrix equation (1.1), however, has not been considered yet. In this paper, we will discuss this problem.

We also consider the matrix nearness problem

$$\min_{A\in S_m} \left\|A-\tilde{A}\right\|, \tag{1.2}$$

where $\tilde{A}$ is a given matrix in $R^{n\times m}$ and S_m is the minimal rank solution set of eq. (1.1).

2 Some Lemmas

Denote by e_i be the ith column of I_n and set $S_n = (e_n, e_{n-1}, \cdots, e_1)$. It is easy to see that

$$S_n^T = S_n, \quad S_n^T S_n = I.$$

Let $k = [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the maximum integer which is not greater than $\frac{n}{2}$. Define D_n as

$$D_n = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ S_k & -S_k \end{bmatrix} (n=2k), \quad D_n = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{bmatrix} (n=2k+1), \tag{2.1}$$

then it is easy verified that the above matrices D_n are orthogonal matrices.

Lemma 1 [9] Let $A \in R^{n\times n}$ and D_n with the forms of (2.1), then A is the Bisymmetric matrix if and only if there exist $A_2 \in SR^{(n-k)\times(n-k)}$ and $A_3 \in SR^{k\times k}$, whether n is odd or even, such that

$$A = D_n \begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} D_n^T. \tag{2.2}$$

Here, we always assume $k = [\frac{n}{2}]$.

Given matrix $X_1, B_1 \in R^{n\times m}$, the singular value decomposition of X_1 be

$$X_1 = U_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} V_1^T = U_{11}\Sigma_1 V_{11}^T, \tag{2.3}$$

where $U_1 = [U_{11}, U_{12}] \in OR^{n\times n}$, $U_{11} \in R^{n\times r_1}$, $V_1 = [V_{11}, V_{12}] \in OR^{m\times m}$, $V_{11} \in R^{m\times r_1}$, $r_1 = r(X_1)$, $\Sigma_1 = \mathrm{diag}(\sigma_1, \cdots, \sigma_{r_1})$, $\sigma_1 \geq \cdots \geq \sigma_{r_1} > 0$.

Let $A_{11}=U_{11}^TB_1V_{11}\Sigma_1^{-1}$, $A_{12}=U_{12}^TB_1V_{11}\Sigma_1^{-1}$, $G_1=A_{12}L_{A_{11}}$, the singular value decomposition of G_1 be

$$G_1=P_1\begin{bmatrix}\Gamma_1 & 0\\ 0 & 0\end{bmatrix}Q_1^T=P_{11}\Gamma_1Q_{11}^T, \tag{2.4}$$

where $P_1=[P_{11},P_{12}]\in OR^{(n-r_1)\times(n-r_1)}$, $P_{11}\in R^{(n-r_1)\times s_1}$, $Q_1=[Q_{11},Q_{12}]\in OR^{r_1\times r_1}$, $Q_{11}\in R^{r_1\times s_1}$, $s_1=r(G_1)$, $\Gamma_1=\mathrm{diag}(\gamma_1,\cdots,\gamma_{s_1})$, $\gamma_1\geq\cdots\geq\gamma_{s_1}>0$.

Lemma 2 [7] Given matrices $X_1,B_1\in R^{n\times m}$. Let the singular value decompositions of X_1 and G_1 be (2.3), (2.4), respectively. Then the matrix equation $A_1X_1=B_1$ has a symmetric solution A_1 if and only if

$$X_1^TB_1=B_1^TX_1,\ \ B_1X_1^+X_1=B_1. \tag{2.5}$$

In this case, let Ω_1 be the set of all symmetric solutions of equation $A_1X_1=B_1$, then the extreme ranks of A_1 are as follows:

(1) The maximal rank of A_1 is

$$\max_{A_1\in\Omega_1}r(A_1)=n+r(B_1)-r(X_1). \tag{2.6}$$

The general expression of A_1 satisfying (2.6) is

$$A_1=A_0+U_{12}N_1U_{12}^T, \tag{2.7}$$

where $A_0=B_1X_1^++(B_1X_1^+)^+R_{X_1}+R_{X_1}B_1X_1^+(X_1X_1^+B_1X_1^+)^+(B_1X_1^+)^TR_{X_1}$ and $N_1\in SR^{(n-r_1)\times(n-r_1)}$ is chosen such that $r(R_{G_1}N_1R_{G_1})=n+r(X_1^TB_1)-r(B_1)-r(X_1)$.

(2) The minimal rank of A_1 is

$$\min_{A_1\in\Omega_1}r(A_1)=2r(B_1)-r(X_1^TB_1). \tag{2.8}$$

The general expression of A_1 satisfying (2.8) is

$$A_1=A_0+U_{12}P_{11}P_{11}^TM_1P_{11}P_{11}^TU_{12}^T, \tag{2.9}$$

where $A_0=B_1X_1^++(B_1X_1^+)^+R_{X_1}+R_{X_1}B_1X_1^+(X_1X_1^+B_1X_1^+)^+(B_1X_1^+)^TR_{X_1}$ and $M_1\in SR^{(n-r_1)\times(n-r_1)}$ is arbitrary.

3 Bisymmetric Extremal Rank Solutions to $AX=B$

Assume D_n with the form of (2.1). Let

$$D_n^TX=\begin{bmatrix}X_2\\ X_3\end{bmatrix},\ \ D_n^TB=\begin{bmatrix}B_2\\ B_3\end{bmatrix}, \tag{3.1}$$

where $X_2 \in R^{(n-k)\times m}$, $X_3 \in R^{k\times m}$, $B_2 \in R^{(n-k)\times m}$, $B_3 \in R^{k\times m}$, and the singular value decomposition of matrices X_2, X_3 are, respectively,

$$X_2 = U_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V_2^T = U_{21}\Sigma_2 V_{21}^T, \tag{3.2}$$

where $U_2 = [U_{21}, U_{22}] \in OR^{(n-k)\times(n-k)}$, $U_{21} \in R^{(n-k)\times r_2}$, $V_2 = [V_{21}, V_{22}] \in OR^{m\times m}$, $V_{21} \in R^{m\times r_2}$, $r_2 = r(X_2)$, $\Sigma_2 = \mathrm{diag}(\alpha_1, \cdots, \alpha_{r_2})$, $\alpha_1 \geq \cdots \geq \alpha_{r_2} > 0$.

$$X_3 = U_3 \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} V_3^T = U_{31}\Sigma_3 V_{31}^T, \tag{3.3}$$

where $U_3 = [U_{31}, U_{32}] \in OR^{k\times k}$, $U_{31} \in R^{k\times r_3}$, $V_3 = [V_{31}, V_{32}] \in OR^{m\times m}$, $V_{31} \in R^{m\times r_3}$, $r_3 = r(X_3)$, $\Sigma_3 = \mathrm{diag}(\beta_1, \cdots, \beta_{r_3})$, $\beta_1 \geq \cdots \geq \beta_{r_3} > 0$.

Let $A_{21} = U_{21}^T B_2 V_{21} \Sigma_2^{-1}$, $A_{22} = U_{22}^T B_2 V_{21} \Sigma_2^{-1}$, $G_2 = A_{22} L_{A_{21}}$, $A_{31} = U_{31}^T B_3 V_{31} \Sigma_3^{-1}$, $A_{32} = U_{32}^T B_3 V_{31} \Sigma_3^{-1}$, $G_3 = A_{32} L_{A_{31}}$, the singular value decomposition of matrices G_2, G_3 are, respectively,

$$G_2 = P_2 \begin{bmatrix} \Gamma_2 & 0 \\ 0 & 0 \end{bmatrix} Q_2^T = P_{21}\Gamma_2 Q_{21}^T, \tag{3.4}$$

where $P_2 = [P_{21}, P_{22}] \in OR^{(n-k-r_2)\times(n-k-r_2)}$, $P_{21} \in R^{(n-k-r_2)\times s_2}$, $Q_2 = [Q_{21}, Q_{22}] \in OR^{r_2\times r_2}$, $Q_{21} \in R^{r_2\times s_2}$, $s_2 = r(G_2)$, $\Gamma_2 = \mathrm{diag}(\zeta_1, \cdots, \zeta_{s_2})$, $\zeta_1 \geq \cdots \geq \zeta_{s_2} > 0$.

$$G_3 = P_3 \begin{bmatrix} \Gamma_3 & 0 \\ 0 & 0 \end{bmatrix} Q_3^T = P_{31}\Gamma_3 Q_{31}^T, \tag{3.5}$$

where $P_3 = [P_{31}, P_{32}] \in OR^{(k-r_3)\times(k-r_3)}$, $P_{31} \in R^{(k-r_3)\times s_3}$, $Q_3 = [Q_{31}, Q_{32}] \in OR^{r_3\times r_3}$, $Q_{31} \in R^{r_3\times s_3}$, $s_3 = r(G_3)$, $\Gamma_3 = \mathrm{diag}(\xi_1, \cdots, \xi_{s_3})$, $\xi_1 \geq \cdots \geq \xi_{s_3} > 0$.

Now we can establish the existence theorems as follows.

Theorem 1 Let $X, B \in R^{n\times m}$ be known. Suppose D_n with the form of (2.1), $D_n^T X$, $D_n^T B$ have the partition forms of (3.1), and the singular value decompositions of the matrices X_2, X_3 and G_2, G_3 are given by (3.2), (3.3) and (3.4), (3.5), respectively. Then the equation (1.1) has a Bisymmetric solution A if and only if

$$X_2^T B_2 = B_2^T X_2, \quad B_2 X_2^+ X_2 = B_2, \quad X_3^T B_3 = B_3^T X_3, \quad B_3 X_3^+ X_3 = B_3. \tag{3.6}$$

In this case, let Ω be the set of all Bisymmetric solutions of equation (1.1), then the extreme ranks of A are as follows:

(1) The maximal rank of A is

$$\max_{A\in\Omega} r(A) = n + r(B_2) + r(B_3) - r(X_2) - r(X_3). \tag{3.7}$$

The general expression of A satisfying (3.7) is

$$A = D_n \begin{bmatrix} A_2 + U_{22}N_2U_{22}^T & 0 \\ 0 & A_3 + U_{32}N_3U_{32}^T \end{bmatrix} D_n^T \tag{3.8}$$

where $A_i = B_iX_i^+ + (B_iX_i^+)^+R_{X_i} + R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^TR_{X_i}$, $i = 2, 3$, and $N_2 \in SR^{(n-k-r_2)\times(n-k-r_2)}$, $N_3 \in SR^{(k-r_3)\times(k-r_3)}$ are chosen such that

$$r(R_{G_2}N_2R_{G_2}) = n - k + r(X_2^TB_2) - r(B_2) - r(X_2),$$
$$r(R_{G_3}N_3R_{G_3}) = k + r(X_3^TB_3) - r(B_3) - r(X_3).$$

(2) The minimal rank of A is

$$\min_{A\in\Omega} r(A) = 2r(B_2) + 2r(B_3) - r(X_2^TB_2) - r(X_3^TB_3). \tag{3.9}$$

The general expression of A satisfying (3.9) is

$$A = D_n \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^TM_2P_{21}P_{21}^TU_{22}^T & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^TM_3P_{31}P_{31}^TU_{32}^T \end{bmatrix} D_n^T, \tag{3.10}$$

where $A_i = B_iX_i^+ + (B_iX_i^+)^+R_{X_i} + R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^TR_{X_i}, i = 2, 3$ and $M_2 \in SR^{(n-k-r_2)\times(n-k-r_2)}$, $M_3 \in SR^{(k-r_3)\times(k-r_3)}$ are arbitrary.

Proof Suppose the matrix equation (1.1) has a solution A which is Bisymmetric, then it follows from Lemma 1 that there exist $A_2 \in SR^{(n-k)\times(n-k)}$, $A_3 \in SR^{k\times k}$ satisfying

$$A = D_n \begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} D_n^T \quad \text{and} \quad AX = B. \tag{3.11}$$

By (3.1), that is

$$\begin{bmatrix} A_2 & 0 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \tag{3.12}$$

i.e.,

$$A_2X_2 = B_2, \quad A_3X_3 = B_3. \tag{3.13}$$

Therefore by Lemma 2, (3.6) hold, and in this case, let Ω be the set of all bisymmetric solutions of equation (1.1), we have

(1) By (3.11),

$$\max_{A\in\Omega} r(A) = \max_{\substack{A_2X_2=B_2 \\ A_2^T=A_2}} r(A_2) + \max_{\substack{A_3X_3=B_3 \\ A_3^T=A_3}} r(A_3). \tag{3.14}$$

By Lemma 2,

$$\max_{\substack{A_2X_2=B_2 \\ A_2^T=A_2}} r(A_2) = n - k + r(B_2) - r(X_2), \quad \max_{\substack{A_3X_3=B_3 \\ A_3^T=A_3}} r(A_3) = k + r(B_3) - r(X_3). \tag{3.15}$$

Taking (3.15) into (3.14) yields (3.7). According to the general expression of the solution in Lemma 2, it is easy to verify the rest of part in (1).

(2) The proof is very similar to that of (1) By (3.1) and Lemma 1, so we omit it.

4 The Expression of the Optimal Approximation Solution to the Set of the Minimal Rank Solution

From (3.10), when the solution set $S_m = \{A \mid AX = B, A \in BSR^{n\times n}, r(A) = \min_{Y\in\Omega} r(Y)\}$ is nonempty, it is easy to verify that S_m is a closed convex set, therefore there exists a unique solution $\hat{A}$ to the matrix nearness problem (1.2).

Theorem 2 Given matrix $\tilde{A}$, and the other given notations and conditions are the same as in Theorem 1. Let

$$D_n^T \tilde{A} D_n = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}_{11} \in R^{(n-k)\times(n-k)}, \quad \tilde{A}_{22} \in R^{k\times k}, \tag{4.1}$$

and we denote

$$U_2^T(\tilde{A}_{11} - A_2)U_2 = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}, \quad \tilde{B}_{11} \in R^{r_2\times r_2}, \quad \tilde{B}_{22} \in R^{(n-k-r_2)\times(n-k-r_2)}, \tag{4.2}$$

$$U_3^T(\tilde{A_{22}} - A_3)U_3 = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix}, \quad \tilde{C}_{11} \in R^{r_3\times r_3}, \quad \tilde{C}_{22} \in R^{(k-r_3)\times(k-r_3)}. \tag{4.3}$$

If S_m is nonempty, then problem (1.2) has a unique $\hat{A}$ which can be represented as

$$\hat{A} = D_n \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^T\tilde{B}_{22}P_{21}P_{21}^TU_{22}^T & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^T\tilde{C}_{22}P_{31}P_{31}^TU_{32}^T \end{bmatrix} D_n^T, \tag{4.4}$$

where $\tilde{B}_{22}$, $\tilde{C}_{22}$ are the same as in (4.2), (4.3).

Proof When S_m is nonempty, it is easy to verify from (3.10) that S_m is a closed convex set. Problem (1.2) has a unique solution $\hat{A}$ by [10]. By Theorem 1, for any $A \in S_m$, A can be expressed as

$$A = D_n \begin{bmatrix} A_2 + U_{22}P_{21}P_{21}^TM_2P_{21}P_{21}^TU_{22}^T & 0 \\ 0 & A_3 + U_{32}P_{31}P_{31}^TM_3P_{31}P_{31}^TU_{32}^T \end{bmatrix} D_n^T, \tag{4.5}$$

where $A_i = B_iX_i^+ + (B_iX_i^+)^+R_{X_i} + R_{X_i}B_iX_i^+(X_iX_i^+B_iX_i^+)^+(B_iX_i^+)^TR_{X_i}, \quad i = 2, 3$, and $M_2 \in SR^{(n-k-r_2)\times(n-k-r_2)}$, $M_3 \in SR^{(k-r_3)\times(k-r_3)}$ are arbitrary.

Using the invariance of the Frobenius norm under orthogonal transformations, and $P_{21}P_{21}^T + P_{22}P_{22}^T = I$, $P_{31}P_{31}^T + P_{32}P_{32}^T = I$, where $P_{21}P_{21}^T$, $P_{22}P_{22}^T$, $P_{31}P_{31}^T$, $P_{32}P_{32}^T$ are orthogonal projection matrices, and $P_{21}P_{21}^TP_{22}P_{22}^T=0$, $P_{31}P_{31}^TP_{32}P_{32}^T=0$, we have

$$\begin{aligned}
\|\tilde{A}-A\|^2 &= \left\|D_n^T\tilde{A}D_n-\begin{bmatrix} A_2+U_{22}P_{21}P_{21}^TM_2P_{21}P_{21}^TU_{22}^T & 0 \\ 0 & A_3+U_{32}P_{31}P_{31}^TM_3P_{31}P_{31}^TU_{32}^T \end{bmatrix}\right\|^2 \\
&= \left\|\tilde{A}_{11}-A_2-U_{22}P_{21}P_{21}^TM_2P_{21}P_{21}^TU_{22}^T\right\|^2+\left\|\tilde{A}_{12}\right\|^2 \\
&\quad +\left\|\tilde{A}_{22}-A_3-U_{32}P_{31}P_{31}^TM_3P_{31}P_{31}^TU_{32}^T\right\|^2+\left\|\tilde{A}_{21}\right\|^2 \\
&= \left\|U_2^T(\tilde{A}_{11}-A_2)U_2-\begin{bmatrix} 0 & 0 \\ 0 & P_{21}P_{21}^TM_2P_{21}P_{21}^T \end{bmatrix}\right\|^2+\left\|\tilde{A}_{12}\right\|^2 \\
&\quad +\left\|U_3^T(\tilde{A}_{22}-A_3)U_3-\begin{bmatrix} 0 & 0 \\ 0 & P_{31}P_{31}^TM_3P_{31}P_{31}^T \end{bmatrix}\right\|^2+\left\|\tilde{A}_{21}\right\|^2 \\
&= \left\|\tilde{A}_{12}\right\|^2+\left\|\tilde{A}_{21}\right\|^2+\left\|\tilde{B}_{11}\right\|^2+\left\|\tilde{B}_{12}\right\|^2+\left\|\tilde{B}_{21}\right\|^2+\left\|\tilde{C}_{11}\right\|^2+\left\|\tilde{C}_{12}\right\|^2+\left\|\tilde{C}_{21}\right\|^2 \\
&\quad +\left\|\tilde{B}_{22}-P_{21}P_{21}^TM_2P_{21}P_{21}^T\right\|^2+\left\|\tilde{C}_{22}-P_{31}P_{31}^TM_3P_{31}P_{31}^T\right\|^2 \\
&= \left\|\tilde{A}_{12}\right\|^2+\left\|\tilde{A}_{21}\right\|^2+\left\|\tilde{B}_{11}\right\|^2+\left\|\tilde{B}_{12}\right\|^2+\left\|\tilde{B}_{21}\right\|^2+\left\|\tilde{C}_{11}\right\|^2+\left\|\tilde{C}_{12}\right\|^2+\left\|\tilde{C}_{21}\right\|^2 \\
&\quad +\left\|\tilde{B}_{22}P_{22}P_{22}^T\right\|^2+\left\|\tilde{B}_{22}P_{21}P_{21}^T-P_{21}P_{21}^TM_2P_{21}P_{21}^T\right\|^2 \\
&\quad +\left\|\tilde{C}_{22}P_{31}P_{31}^T\right\|^2+\left\|\tilde{C}_{22}P_{31}P_{31}^T-P_{31}P_{31}^TM_3P_{31}P_{31}^T\right\|^2 \\
&= \left\|\tilde{A}_{12}\right\|^2+\left\|\tilde{A}_{21}\right\|^2+\left\|\tilde{B}_{11}\right\|^2+\left\|\tilde{B}_{12}\right\|^2+\left\|\tilde{B}_{21}\right\|^2+\left\|\tilde{C}_{11}\right\|^2+\left\|\tilde{C}_{12}\right\|^2+\left\|\tilde{C}_{21}\right\|^2 \\
&\quad +\left\|\tilde{B}_{22}P_{22}P_{22}^T\right\|^2+\left\|P_{22}P_{22}^T\tilde{B}_{22}P_{21}P_{21}^T\right\|^2+\left\|P_{21}P_{21}^T\tilde{B}_{22}P_{21}P_{21}^T-P_{21}P_{21}^TM_2P_{21}P_{21}^T\right\|^2 \\
&\quad +\left\|\tilde{C}_{22}P_{32}P_{32}^T\right\|^2+\left\|P_{32}P_{32}^T\tilde{C}_{22}P_{31}P_{31}^T\right\|^2+\left\|P_{31}P_{31}^T\tilde{C}_{22}P_{31}P_{31}^T-P_{31}P_{31}^TM_3P_{31}P_{31}^T\right\|^2.
\end{aligned}$$

Therefore, $\min\limits_{A\in S_m}\|\tilde{A}-A\|$ is equivalent to

$$\min_{M_2\in SR^{(n-k)\times(n-k)}}\left\|P_{21}P_{21}^T\tilde{B}_{22}P_{21}P_{21}^T-P_{21}P_{21}^TM_2P_{21}P_{21}^T\right\|, \tag{4.6}$$

$$\min_{M_3\in SR^{k\times k}}\left\|P_{31}P_{31}^T\tilde{C}_{22}P_{31}P_{31}^T-P_{31}P_{31}^TM_3P_{31}P_{31}^T\right\|. \tag{4.7}$$

Obviously, the solutions of (4.6), (4.7) can be written as

$$M_2=\tilde{B}_{22}+P_{22}P_{22}^T\tilde{M}_2P_{22}P_{22}^T,\quad \forall\ \tilde{M}_2\in SR^{(n-k-r_2)\times(n-k-r_2)}, \tag{4.8}$$

$$M_3=\tilde{C}_{22}+P_{32}P_{32}^T\tilde{M}_3P_{32}P_{32}^T,\quad \forall\ \tilde{M}_3\in SR^{(k-r_3)\times(k-r_3)}. \tag{4.9}$$

Substituting (4.8), (4.9) into (4.5), then we get that the unique solution to problem (1.2) can be expressed in (4.4). The proof is completed.

References

[1] Mitra S K. Fixed rank solutions of linear matrix equations[J]. Sankhya Ser. A., 1972, 35: 387–392.

[2] Mitra S K. The matrix equation $AX=C,XB=D$[J]. Linear Algebra Appl., 1984, 59: 171–181.

[3] Uhlig F. On the matrix equation $AX = B$ with applications to the generators of controllability matrix[J]. Linear Algebra Appl., 1987, 85: 203–209.

[4] Mitra S K. A pair of simultaneous linear matrix equations $A_1X_1B_1 = C_1, A_2X_2B_2 = C_2$ and a matrix programming problem[J]. Linear Algebra Appl., 1990, 131: 107–123.

[5] Tian Y G, Wiens D P. On equality and proportionality of ordinary least squares, weighted least squares and best linear unbiased estimators in the general linear model[J]. Linear Algebra Appl., Statist. Probab. Lett., 2006, 76: 1265-1272.

[6] Tian Y G. The minimal rank of the matrix expression $A - BX - YC$[J]. Missouri. J. Math. Sci., 2002, 14: 40–48.

[7] Xiao Q F, Hu X Y, Zhang L. The symmetric minimal rank solution of the matrix equation $AX = B$ and the optimal approximation[J]. Electron. J. Linear Algebra, 2009, 18: 264–271.

[8] Xiao Q F, Hu X Y, Zhang L. The rank-constrained anti-symmetric solution of the matrix equation AX=B and the optimal approximation[J]. J. Math., 2013, 33(5): 803–810.

[9] Xie D X, Zhang L, Hu X Y. The solvability conditions for the inverse problem of bisymmetric nonnegative definite matrices[J]. J. Comput. Math., 2000, 18(6): 597–608.

[10] Zhang L. The approximation on the closed convex cone and its numerical application[J]. Hunan Ann. Math., 1986, 6: 43–48(in Chinese).

一类矩阵方程的双对称定秩解及其最佳逼近

冯天祥

(东莞职业技术学院基础课部, 广东 东莞 523808)

摘要: 本文研究了矩阵方程$AX = B$ 的双对称最大秩和最小秩解问题. 利用矩阵秩的方法, 获得了矩阵方程$AX = B$有最大秩和最小秩解的充分必要条件以及解的表达式, 同时对于最小秩解的解集合, 得到了最佳逼近解.

关键词: 矩阵方程; 双对称矩阵; 最大秩; 最小秩; 最佳逼近解

MR(2010)**主题分类号**: 15A06; 15A29 **中图分类号**: O241.6