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GROWTH AND COVERING THEOREMS FOR STRONGLY SPIRALLIKE MAPPINGS OF TYPE β

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Abstract: In this paper, we study strongly spirallike mappings of type β on the unit ball in complex Banach spaces. By using the definition and the geometrical characteristic of strongly spirallike mappings of type β, the growth and covering theorems for the above mappings are obtained. Combining zero of order k of strongly spirallike mappings of type β, the corresponding growth and covering theorems are also obtained. The results extend the corresponding results of spirallike mappings.

Keywords: strongly spirallike mappings of type β; zero of order k; growth and covering theorems

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1 Introduction and Lemmas

Growth and covering theorems for biholomorphic mappings are important parts in geometric function theory of several complex variables. Especially as the biholomorphic mappings were restricted with the geometric characteristic, many scholars started to discuss the problem. From the geometric characteristic of spirallike mappings of type β, Feng Shuxia [1] gave the definitions of almost spirallike mappings of type β and order α, spirallike mappings of type β and order α and strongly spirallike mappings of type β and order α, and obtained their growth and covering theorems. Liu X S [2] considered the order of zero and aiso obtained the growth and covering theorems for spirallike mappings of type β, almost starllike mappings of order α and starllike mappings of order α. And also there are many refining growth and covering theorems for other biholomorphic mappings (see [3–5]).

In 2001, Hamada and Kohr [6] firstly gave the definition of strongly spirallike mappings of type α on the unit ball B^n in $\mathbb{C}^n$, and later Xu Q H [4] generalized the definition on the unit ball B in complex Banach spaces. In this paper, we mainly discuss the growth and

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covering theorems for strongly spirallike mappings of type β on B, and where D is the unit disc.

Definition 1.1 [4] Suppose that f is a normalized locally biholomorphic mapping on B. If

$$|e^{-i\beta}\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)]-\frac{1+c^2}{1-c^2}\cos\beta+i\sin\beta|<\frac{2c}{1-c^2}\cos\beta, c\in(0,1),\beta\in(-\frac{\pi}{2},\frac{\pi}{2}),$$

then f is said to be a strongly spirallike mapping of type β, and if $\beta=0$, then f is said to be a strongly starllike mapping.

Remark From Definition 1.1 we know that the image of the unit ball under the mapping $e^{-i\beta}\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)]$ is in the circle where the center is $\frac{1+c^2}{1-c^2}\cos\beta-i\sin\beta$ and the radius is $\frac{2c}{1-c^2}\cos\beta$. Yet

$$\frac{1+c^2}{1-c^2}\cos\beta-\frac{2c}{1-c^2}\cos\beta=\frac{1-c}{1+c}\cos\beta>0,$$

so the circle must be in the right half-plain, and thus strongly spirallike mappings of type β must be spirallike mappings of type β.

Lemma 1.2 (see [7]) Suppose that $f:D\to D$ is holomorphic and $f(0)=0$, then $|f'(0)|<1$, and $|f(z)|\le|z|,\forall z\in D$.

Lemma 1.3 (see [8]) Suppose that $f:D\to D$ is holomorphic and $z=0$ is the zero of $f(z)$ of order $k(k\in\mathbb{N})$, then $|f(z)|\le|z|^k,\forall z\in D$.

Lemma 1.4 (see [1]) Suppose that $f:B\to X$ is a normalized biholomorphic spirallike mapping of type $\beta,\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$. If we denote $x(t)=f^{-1}(\exp(-e{-}i\beta t)f(x))$ $(0\le t<+\infty)$ for $x\in B\setminus 0$, then

(1) $\|x(t)\|$ strictly decreases monotonically for $t\in[0,+\infty)$;

(2) $\lim\frac{\|f(x(t))\|}{\|x(t)\|}=1, \frac{dx}{dt}(t)=-e^{-i\beta}[(Df(x(t)))^{-1}f(x(t))]$ for arbitrarily $t\in(0,+\infty)$;

(3) $\frac{d\|f(x(t))\|}{dt}=-\cos\beta\|f(x(t))\|, t\in(0,+\infty)$.

Lemma 1.5 (see [1]) Suppose that $x(t):[0,+\infty)\to X$ is differentiable in the point of $x\in(0,+\infty)$, and $\|x(t)\|$ is also differentiable in the point of s, then

$$\mathrm{Re}T_{x(t)}[\frac{dx}{dt}(s)]=\frac{\|dx(s)\|}{dt}, s\in[0,+\infty).$$

2 Main Results

Theorem 2.1 Suppose that f is a normalized biholomorphic strongly spirallike mapping of type β on B, $c\in(0,1)$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$, then

$$\frac{\|x\|}{(1+c\|x\|)^2}\le\|f(x)\|\le\frac{\|x\|}{(1-c\|x\|)^2},\ \ f(B)\supset\frac{1}{4}B.$$

When $\beta = 0$, i.e., f is a strongly starllike mapping on B, we have the same result.

Proof Fix $x \in B \setminus \{0\}$, and let $x_0 = \frac{x}{\|x\|}$, then $x_0 \in \partial B$. Let

$$g(\zeta) = \begin{cases} e^{-i\beta}\frac{1}{\zeta}T_{x_0}[(Df(\zeta x_0))^{-1}f(\zeta x_0)], & \zeta \in D \setminus \{0\}, \\ e^{-i\beta}, & \zeta = 0. \end{cases} \tag{1}$$

Since f is a normalized biholomorphic strongly spirallike mapping of type β on B, then by Definition 1.1, we have

$$|e^{-i\beta}\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] - (\frac{1+c^2}{1-c^2}\cos\beta - i\sin\beta)| < \frac{2c}{1-c^2}\cos\beta,$$

Suppose $x = \zeta x_0 = |\zeta|e^{i\theta}x_0, \zeta \neq 0$, then we have

$$|e^{-i\beta}\frac{1}{\zeta}T_{x_0}[(Df(\zeta x_0))^{-1}f(\zeta x_0)] - (\frac{1+c^2}{1-c^2}\cos\beta - i\sin\beta)| < \frac{2c}{1-c^2}\cos\beta,$$

i.e.,

$$|g(\zeta) - (\frac{1+c^2}{1-c^2}\cos\beta - i\sin\beta)| < \frac{2c}{1-c^2}\cos\beta, \zeta \in D \setminus \{0\}.$$

Furthermore, for $\zeta = 0$, the above inequality holds by eq. (1), thus the above inequality holds for arbitrarily $\zeta \in D$, so we have

$$|\frac{1-c^2}{2c\cos\beta}[g(\zeta) + i\sin\beta)] - \frac{1+c^2}{2c}| < 1, \zeta \in D.$$

Suppose

$$p(\zeta) = \frac{1-c^2}{2c\cos\beta}[g(\zeta) + i\sin\beta)] - \frac{1+c^2}{2c},$$

then we have $|p(\zeta)| < 1$ and $p(0) = -c$, Let $\varphi(\zeta) = \frac{p(\zeta) - p(0)}{1 - p(0)p(\zeta)}$, it is clear that $|\varphi(\zeta)| < 1$ and $\varphi(0) = 0$, so we have $|\varphi(\zeta)| < |\zeta|$ by Lemma 1.2, i.e.,

$$\left|\frac{\frac{1-c^2}{2c\cos\beta}[g(\zeta) + i\sin\beta)] - \frac{1+c^2}{2c} + c}{1 + c[\frac{1-c^2}{2c\cos\beta}[g(\zeta) + i\sin\beta)] - \frac{1+c^2}{2c}]}\right| < |\zeta|,$$

i.e.,

$$\left|\frac{(g(\zeta) + i\sin\beta)) - \cos\beta}{(g(\zeta) + i\sin\beta)) + \cos\beta}\right| < c|\zeta|.$$

Let $g(\zeta) + i\sin\beta = A + Bi(A, B \in \mathbb{R})$, and from the above inequality, then we have

$$(A - \frac{1+c^2|\zeta|^2}{1-c^2|\zeta|^2}\cos\beta)^2 + B^2 < \frac{4c^2|\zeta|^2}{(1-c^2|\zeta|^2)^2}\cos^2\beta,$$

so $g(\zeta)+i\sin\beta$ is in the circle taking a as the center and r as the radius in the complex plain, and here

$$a=\frac{1+c^2|\zeta|^2}{1-c^2|\zeta|^2}\cos\beta, r=\frac{2c|\zeta|}{1-c^2|\zeta|^2}\cos\beta.$$

Then $\mathrm{Re}a-r\leq\mathrm{Re}(g(\zeta)+i\sin\beta)\leq\mathrm{Re}a+r$, i.e.,

$$\frac{1-c|\zeta|}{1+c|\zeta|}\cos\beta\leq\mathrm{Re}g(\zeta)\leq\frac{1+c|\zeta|}{1-c|\zeta|}\cos\beta.$$

According to (1), and let $\zeta=\|x\|$, we obtain

$$\frac{\|x\|(1-c\|x\|)}{1+c\|x\|}\cos\beta\leq\mathrm{Re}\{e^{-i\beta}T_x[(Df(x))^{-1}f(x)]\}\leq\frac{\|x\|(1+c\|x\|)}{1-c\|x\|}\cos\beta. \tag{2}$$

Let

$$x(t)=f^{-1}(\exp(-e^{-i\beta}t)f(x))(0\leq t<+\infty),$$

by Lemma 1.4, then $\|x(t)\|$ is differentiable almost everywhere in $[0,+\infty)$, and

$$\frac{dx}{dt}(t)=-e^{-i\beta}[(Df(x(t)))^{-1}f(x(t))]. \tag{3}$$

From (2), (3) and Lemma 1.5, we obtain

$$-\frac{\|x(t)\|(1+c\|x(t)\|)}{1-c\|x(t)\|}\cos\beta\leq\mathrm{Re}T_{x(t)}(\frac{dx(t)}{dt})=\frac{d\|x(t)\|}{dt}\leq-\frac{\|x(t)\|(1-c\|x(t)\|)}{1+c\|x(t)\|}\cos\beta. \tag{4}$$

Also from Lemma 1.4, we have $\dfrac{d\|f(x(t))\|}{dt}=-\cos\beta\|f(x(t))\|$, then from (4) we obtain

$$\begin{aligned}&\frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}\frac{\|x(t)\|(1+c\|x(t)\|)}{1-c\|x(t)\|}\leq\frac{d\|x(t)\|}{dt}\\ \leq&\frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}\frac{\|x(t)\|(1-c\|x(t)\|)}{1+c\|x(t)\|}.\end{aligned} \tag{5}$$

For $\tau\geq0$, from the left of (5) we obtain

$$\int_0^\tau\frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}dt\leq\int_0^\tau\frac{1-c\|x(t)\|}{\|x(t)\|(1+c\|x(t)\|)}\frac{d\|x(t)\|}{dt}dt,$$

thus

$$\log\|f(x(\tau))\|-\log\|f(x)\|\leq\log\frac{\|x(\tau)\|}{(1+c\|x(\tau)\|)^2}-\log\frac{\|x\|}{(1+c\|x\|)^2},$$

i.e.,

$$\frac{\|f(x(\tau))\|}{\|f(x)\|}\leq\frac{\|x(\tau)\|}{(1+c\|x(\tau)\|)^2}\frac{(1+c\|x\|)^2}{\|x\|}.$$

Let $\tau\to+\infty$, from Lemma 1.4 we obtain $\|f(x)\|\geq\dfrac{\|x\|}{(1+c\|x\|)^2}$. Also from the right of (5) we can obtain $\|f(x)\|\leq\dfrac{\|x\|}{(1-c\|x\|)^2}$. Hence

$$\frac{\|x\|}{(1+c\|x\|)^2}\leq\|f(x)\|\leq\frac{\|x\|}{(1-c\|x\|)^2},$$

and so we have $f(B) \supset \frac{1}{4}B$, this completes the proof.

Theorem 2.2 Suppose that f is a normalized biholomorphic strongly spirallike mapping of type β on B, $c \in (0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $x = 0$ is the zero of $f(x) - x$ of order $k+1 (k \in \mathbb{N})$, then

$$\frac{\|x\|}{(1+c\|x\|^k)^{\frac{2}{k}}} \leq \|f(x)\| \leq \frac{\|x\|}{(1-c\|x\|^k)^{\frac{2}{k}}}, f(B) \supset \frac{1}{2^{\frac{2}{k}}}B.$$

When $\beta = 0$, that is, when f is a strongly starllike mapping on B, we have the same result.

Proof With the same method of the proof of Theorem 2.1, and from Lemma 1.3 we can obtain $|\varphi(\zeta)| < |\zeta|^k$. Replacing $|\zeta|$ with $|\zeta|^k$ in the proof of Theorem 2.1, so we can obtain

$$\begin{aligned} &\frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}\frac{\|x(t)\|(1+c\|x(t)\|^k)}{1-c\|x(t)\|^k} \leq \frac{d\|x(t)\|}{dt} \\ \leq &\frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}\frac{\|x(t)\|(1-c\|x(t)\|^k)}{1+c\|x(t)\|^k}. \end{aligned} \tag{6}$$

From the left of (6), we have

$$\int_0^\tau \frac{1}{\|f(x(t))\|}\frac{d\|f(x(t))\|}{dt}dt \leq \int_0^\tau \frac{1-c\|x(t)\|^k}{\|x(t)\|(1+c\|x(t)\|^k)}\frac{d\|x(t)\|}{dt}dt,$$

thus

$$\begin{aligned} & \log\|f(x(\tau))\| - \log\|f(x)\| \leq \log\frac{\|x(\tau)\|}{(1+c\|x(\tau)\|)^2} - \log\frac{\|x\|}{(1+c\|x\|)^2} \\ = \; & \log\{\|x(\tau)\|\exp\int_0^{\|x(\tau)\|}[\frac{1-cw^k}{1+cw^k}-1]\frac{dw}{w}\} - \log\{\|x\|\exp\int_0^{\|x\|}[\frac{1-cw^k}{1+cw^k}-1]\frac{dw}{w}\} \\ = \; & \log\{\|x(\tau)\|(1+c\|x(\tau)\|^k)^{-\frac{2}{k}}\} - \log\{\|x\|(1+c\|x\|^k)^{-\frac{2}{k}}\}, \end{aligned}$$

So we have

$$\frac{\|f(x(\tau))\|}{\|f(x)\|} \leq \frac{\|x(\tau)\|}{(1+c\|x(\tau)\|^k)^{\frac{2}{k}}}\frac{(1+c\|x\|^k)^{\frac{2}{k}}}{\|x\|}.$$

Thus we obtain

$$\|f(x)\| \geq \frac{\|x\|}{(1+c\|x\|^k)^{\frac{2}{k}}}.$$

Also from the right of (6) we can obtain

$$\|f(x)\| \leq \frac{\|x\|}{(1-c\|x\|^k)^{\frac{2}{k}}},$$

this completes the proof.

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强β型螺形映照的增长和掩盖定理

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摘 要: 本文研究了复Banach空间单位球上的强β型螺形映照. 利用强β 型螺形映照的定义及其几何特征, 获得了复Banach空间单位球上强β 型螺形映照的增长和掩盖定理, 并结合k阶零点得到强β 型螺形映照相应的增长和掩盖定理, 推广了螺型映照的相应结论.

关键词: 强β型螺形映照; k阶零点; 增长和掩盖定理

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