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# COMPLEMENTS OF DISTANCE-REGULAR GRAPHS

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**Abstract:** In this paper, we study the complement of $\Gamma$ which is a distance-regular graph with diameter $d(\Gamma) \geq 2$. By using intersection numbers of $\Gamma$, we show that the complement of $\Gamma$ is strongly regular or generalized strongly regular as $d = 2$ or $d \geq 3$, respectively. We get the complements of Grassmann graph, dual polar graph and Hamming graph in [2], which are the generalized strongly regular.

**Keywords:** distance-regular graph; complement; strongly regular graph; generalized strongly regular graph

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## 1 Introduction

Let $\Gamma = (X, R)$ denote a finite undirected graph without loops or multiple edges, with vertex set $X$ and edge set $R$. Assume that $\Gamma$ is a connected regular graph. For vertices $u$ and $v$ in $X$, let $\partial_\Gamma(u, v)$ denote the distance between $u$ and $v$. The maximum value of the distance function in $\Gamma$ is called the diameter of $\Gamma$, denoted by $d(\Gamma)$. For all $u \in X$ and for all integers $i$ $(0 \leq i \leq d)$, set

$$\Gamma_i(u) := \{v \mid v \in X, \partial_\Gamma(u, v) = i\},\ \Gamma_1(u) := \Gamma(u),$$

$\Gamma$ is said to be distance-regular whenever for all integers $h, i, j$ $(0 \leq h, i, j \leq d(\Gamma))$ and for all $u, v \in X$ with $\partial_\Gamma(u, v) = h$, the number

$$p_{ij}^h := |\Gamma_i(u) \cap \Gamma_j(v)| \tag{1.1}$$

is independent of $u, v$. The constants $p_{ij}^h$ are known as the intersection numbers of $\Gamma$. For convenience, set

$$c_i := p_{i-1,1}^i\ (1 \leq i \leq d(\Gamma)),\ \ a_i := p_{i1}^i\ (0 \leq i \leq d(\Gamma)),$$
$$b_i := p_{i+1,1}^i\ (0 \leq i \leq d(\Gamma) - 1),\ \ k_i := p_{ii}^0\ (0 \leq i \leq d(\Gamma)),$$

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and put $c_0 := 0, b_d := 0, k := k_1$. Note that $c_1 = 1, a_0 = 0$, and

$$k_j = \sum_{i=0}^{d(\Gamma)} p^h_{ij} = \sum_{i=0}^{d(\Gamma)} p^h_{ji}, \quad |X| = 1 + k_1 + \cdots + k_{d(\Gamma)}. \tag{1.2}$$

The reader is referred to [1–3] for general theory of distance-regular graphs.

The complement $\bar{G}$ of a graph $G$ has the same vertex set as $G$, where vertices $x$ and $y$ are adjacent in $\bar{G}$ if and only if they are not adjacent in $G$.

A simple graph $G$ is called generalized strongly regular with parameters $(v, \lambda, a, b, c)$ if it consists of $v$ vertices such that for any $x, y \in G$,

$$|G(x) \cap G(y)| = \begin{cases} \lambda, & \text{if } x = y, \\ a \text{ or } b, & \text{if } x, y \text{ are adjacent}, \\ c, & \text{otherwise}, \end{cases}$$

where $a, b$ are integers such that $b \le a$. In particular, if $a = b$, then $G$ is called strongly regular with parameters $(v, k, a, c)$. Clearly, strongly regular graphs are generalized strongly regular graphs.

Let $\Gamma = (X, R)$ be the distance-regular graph and $\bar{\Gamma}$ be the complement of $\Gamma$. In this paper, we obtain the following result.

**Theorem 1.1** Let $\Gamma = (X, R)$ be the distance-regular graph with diameter $d(\Gamma) \ge 2$ and intersection numbers

$$p^h_{j\,t} \ (0 \le h, j, t \le d(\Gamma)).$$

Then the following hold.

(i) If $d(\Gamma) \ge 3$, then $\bar{\Gamma}$ is a generalized strongly regular graph with parameters

$$(|X|, |X| - k - 1, |X| - 2k + c_2 - 2, |X| - 2k - 2, |X| - 2k + a_1),$$

where $k, c_2$ and $a_1$ are parameters of $\Gamma$.

(ii) If $d(\Gamma) = 2$, then $\bar{\Gamma}$ is a strongly regular graph with parameters

$$(|X|, |X| - k - 1, |X| - 2k + c_2 - 2, |X| - 2k + a_1).$$

Moreover, $\bar{\Gamma}$ is connected if and only if $|X| - 2k + a_1 > 0$.

**Proof** For any $x, y \in X$ with $\partial_\Gamma(x, y) = l$, where $1 \le l \le d(\Gamma)$. By (1.1) and (1.2), the number of vertices $z \in X$ satisfying both $\partial_{\bar{\Gamma}}(x, z) = 1$ and $\partial_{\bar{\Gamma}}(y, z) = 1$ is

$$\begin{aligned}
\sum_{2 \le j \le d(\Gamma)} \sum_{2 \le t \le d(\Gamma)} p^l_{j\,t} &= \sum_{2 \le j \le d(\Gamma)} (k_j - p^l_{j\,1} - p^l_{j\,0}) \\
&= \sum_{2 \le j \le d(\Gamma)} k_j - \sum_{2 \le j \le d(\Gamma)} p^l_{j\,1} - \sum_{2 \le j \le d(\Gamma)} p^l_{j\,0} \\
&= (|X| - k - 1) - (k - p^l_{1\,1} - p^l_{0\,1}) - (1 - p^l_{1\,0} - p^l_{0\,0}) \\
&= |X| - 2k + p^l_{1\,1} + 2p^l_{0\,1} - 2 \\
&= \begin{cases} |X| - 2k + a_1, & \text{if } l = 1, \\ |X| - 2k + p^l_{1\,1} - 2, & \text{if } l \ne 1. \end{cases}
\end{aligned}$$

(i) Suppose that $x$ and $y$ are two distinct vertices of $\bar{\Gamma}$. If $\partial_{\bar{\Gamma}}(x,y)=1$, then there exists some $l\in\{2,\cdots,d(\Gamma)\}$ such that $\partial_{\Gamma}(x,y)=l$, which implies that the number of vertices $z\in X$ satisfying both $\partial_{\bar{\Gamma}}(x,z)=1$ and $\partial_{\bar{\Gamma}}(y,z)=1$ is

$$|X|-2k+c_2-2$$

or

$$|X|-2k-2,$$

according to $l=2$ or $l\neq 2$, respectively. If $\partial_{\bar{\Gamma}}(x,y)\neq 1$, then $\partial_{\Gamma}(x,y)=1$, which implies that the number of vertices $z\in X$ satisfying both $\partial_{\bar{\Gamma}}(x,z)=1$ and $\partial_{\bar{\Gamma}}(y,z)=1$ is $|X|-2k+a_1$. Therefore, the desired result follows.

(ii) Similar to the proof of (i), we have $\bar{\Gamma}$ is a strongly regular graph with parameters

$$(|X|,|X|-k-1,|X|-2k+c_2-2,|X|-2k+a_1).$$

Suppose that $\bar{\Gamma}$ is not connected and let $Z$ be a component of $\bar{\Gamma}$. Then a vertex in $Z$ has no common neighbours with a vertex not in $Z$, and so

$$|X|-2k+a_1=0.$$

If $|X|-2k+a_1=0$, then any two neighbours of a vertex $x\in\bar{\Gamma}$ must be adjacent, and so the component containing $x$ must be a complete graph, and hence $\bar{\Gamma}$ is a disjoint union of complete graphs.

## 2 Examples

Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a prime power. Let $\mathbb{F}_q^n$ be the $n$-dimensional vector space over the finite field $\mathbb{F}_q$. Let $1\leq m\leq n-1$. The Grassmann graph $\Gamma(m,q,n)$ is the graph the vertices of which are the $m$-dimensional subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if and only if they meet in a subspace of dimension $m-1$. It can shown (see [2, Theorem 9.3.3]) that $\Gamma(m,q,n)$ is a distance-regular graph of diameter $\min\{m,n-m\}$.

**Example 2.1** For $2\leq m\leq n-2$, let $\bar{\Gamma}(m,q,n)$ be the complement of $\Gamma(m,q,n)$ and

$$\beta=\begin{bmatrix} n\\ m\end{bmatrix}_q,\ \alpha=2q\begin{bmatrix} n-m\\ 1\end{bmatrix}_q\begin{bmatrix} m\\ 1\end{bmatrix}_q,\ \gamma=\frac{q^m+q^{n-m}-2q}{q-1}.$$

Then the following hold.

(i) If $\min\{m,n-m\}>2$, then $\bar{\Gamma}(m,q,n)$ is a generalized strongly regular graph with parameters

$$(\beta,\beta-\alpha-1,\beta-2\alpha+(q+1)^2-2,\beta-2\alpha-2,\beta-2\alpha+\gamma).$$

(ii) If $\min\{m,n-m\}=2$, then $\bar{\Gamma}(m,q,n)$ is a strongly regular graph with parameters

$$(\beta,\beta-\alpha-1,\beta-2\alpha+(q+1)^2-2,\beta-2\alpha+\gamma).$$

Let $q, r$ be prime powers. Let $V$ be one of the following spaces equipped with a specified form:

- $[C_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate symplectic form;
- $[B_d(q)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate quadratic form;
- $[D_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate quadratic form of Witt index $d$;
- $[{}^2D_{d+1}(q)] = \mathbb{F}_q^{2d+2}$ with a nondegenerate quadratic form of Witt index $d$;
- $[{}^2A_{2d}(r)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate Hermitean form $q = r^2$;
- $[{}^2A_{2d-1}(r)] = \mathbb{F}_q^{2d}$ with a nondegenerate Hermitean form $q = r^2$.

A subspace of $V$ is called isotropic whenever the form vanishes completely on this subspace. Maximal isotropic subspaces have dimension $d$. The dual polar graph $\Gamma$ (on $V$) has as vertices the maximal isotropic subspaces; two points $P, Q$ are adjacent if and only if $\dim(P \cap Q) = d-1$. It can shown (see [2, Theorem 9.4.3]) that $\Gamma$ is a distance-regular graph of diameter $d$.

**Example 2.2** Let $2 \leq d$, and let $e$ be $1, 1, 0, 2, 3/2, 1/2$ in the respective cases

$$[C_d(q)],\ [B_d(q)],\ [D_d(q)],\ [{}^2D_{d+1}(q)],\ [{}^2A_{2d}(r)],\ [{}^2A_{2d-1}(r)].$$

Let $\bar{\Gamma}$ be the complement of $\Gamma$ and

$$\beta = \prod_{i=0}^{d-1}(q^{d+e-i-1}+1),\ \alpha = q^e\frac{q^d-1}{q-1},\ \gamma = q^e - 1.$$

Then the following hold.

(i) If $d > 2$, then $\bar{\Gamma}$ is a generalized strongly regular graph with parameters

$$(\beta, \beta-\alpha-1, \beta-2\alpha+q-1, \beta-2\alpha-2, \beta-2\alpha+\gamma).$$

(ii) If $d = 2$, then $\bar{\Gamma}$ is a strongly regular graph with parameters

$$(\beta, \beta-\alpha-1, \beta-2\alpha+q-1, \beta-2\alpha+\gamma).$$

Let $Y$ be a finite set of cardinality $q \geq 2$. The Hamming graph $H(d, q)$ with diameter $d$ has vertex set $Y^d = \bigotimes_{i=1}^{d} Y$, the cartesian product of $d$ copies of $Y$; two points of $H(d, q)$ are adjacent whenever they differ in precisely one coordinate. It can show (see [2, Theorem 9.2.1]) that $H(d, q)$ is a distance-regular graph of diameter $d$.

**Example 2.3** Let $2 \leq d$ and $\bar{H}(d, q)$ be the complement of $H(d, q)$. Then the following hold.

(i) If $d > 2$, then $\bar{H}(d, q)$ is a generalized strongly regular graph with parameters

$$(q^d, q^d - d(q-1) - 1, q^d - 2d(q-1), q^d - 2d(q-1) - 2, q^d - 2d(q-1) + q - 2).$$

(ii) If $d = 2$, then $\bar{H}(d, q)$ is a strongly regular graph with parameters

$$(q^d, q^d - d(q-1) - 1, q^d - 2d(q-1) - 2, q^d - 2d(q-1) + q - 2).$$

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# 距离正则图的推广


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**摘要**: 本文研究了直径为$d(\Gamma) \geq 2$的距离正则图$\Gamma$的补图. 利用$\Gamma$的交叉数分别证明了当$d = 2$时, $\Gamma$的补图式强正则; 当$d \geq 3$时, $\Gamma$的补图是广义强正则. 将文献[2]中的距离正则图Grassmann图、对偶极图、Hamming图推广到它们的补图, 从而得到广义强正则图.



**关键词**: 距离正则图; 推广; 强正则图; 广义强正则图

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