Vol. 35 (2015)
No. 6

数 学 杂 志
J. of Math. (PRC)

INVARIANT EINSTEIN METRICS ON FULL FLAG MANIFOLD OF $SO(8)/T$

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Abstract: We consider the problem of the invariant Einstein metrics on the full flag manifolds $M = SO(8)/T$ with twelve isotropy summands. With the help of the computer we obtain there are one hundred and sixty invariant Einstein metrics (up to a scale) on the full flag manifold of $SO(8)/T$, of which one is Kähler Einstein metric (up to isometry) and four are non-Kähler Einstein metrics (up to isometry). We promote the original methods which are applied to the full flag manifolds with not more than six isotropy summands.

Keywords: full flag manifold; Einstein metric; Ricc tensor; isotropy representation

2010 MR Subject Classification: 53C30

Document code: A **Article ID:** 0255-7797(2015)06-1319-10

1 Introduction

An important class of homogeneous manifolds are the orbits of the adjoint action of a semisimple compact Lie Group, called generalized flag manifolds. Such manifolds can be described by a quotient $M = G/C(T)$, where $C(T)$ is the centralizer of a torus T of the Lie group G. If $C(T) = T$ then $M = G/T$ is called full flag manifold.

Non-Kähler Einstein metrics on full flag manifolds corresponding to classical Lie group were studied by several authors [1–4]. But when the isotropy representation of the full flag manifolds increases, it is very difficult to find all the non-Kähler Einstein metrics (up to isometry). It is well known that there are only some results for the G-invariant Einstein metrics on the full flag manifold with no more than six isotropy summands. In this paper we study the classification problem of homogeneous Einstein metrics on the full flag manifold $SO(8)/T$. It is the first known example for the full flag manifold of a classical Lie group with twelve isotropy summands which admits four non-Kähler Einstein metrics.

* **Received date:** 2014-10-24 **Accepted date:** 2014-12-22

Foundation item: Supported by Sichuan University of Science and Engineering grant (2012PY17); Sichuan Province University Key Laboratory of Bridge Non-Destruction Detecting and Engineering Computing grant (2014QZJ03); Partially Supported by Scientific Research Fund of Sichuan Province Education Department Grant (15SB0122).

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This paper is organized as follows: In Section 2 we recall the Lie theoretic description of a generalized flag manifold G/K of a compact and connected semisimple Lie group G. In Section 3 we consider the classical full flag manifold $SO(8)/T$ and isometric problem, then we prove $SO(8)/T$ admits five (up to isometry) $SO(8)$-invariant Einstein metrics.

2 Generalized Flag Manifold

Let G be a compact connected simple Lie group and g be the corresponding Lie algebra. We denote by $g^{\mathbb{C}}$ the complexification of g and $Ad: G \to Aut(g)$ be the adjoint representation of G. Let G/K be generalized flag manifold and $\mathfrak{k}$ be the Lie algebra of K. We denote by $o = eK$ the origin of the flag manifold (the identity coset of G/K). Since the Lie group G is simple and compact, the Cartan-Killing form $< \cdot, \cdot >$ is non-degenerated and negative definite. Thus $Q(\cdot, \cdot) = - < \cdot, \cdot >$ is an inner product. Let $\mathfrak{m} = \mathfrak{k}^{\perp}$ be the orthogonal complement of $\mathfrak{k}$ with respect to Q. Then the decomposition $g = \mathfrak{m} \oplus \mathfrak{k}$ is reductive, that is, $Ad(K)\mathfrak{m} \subset \mathfrak{m}$ and the tangent space at the origin $T_o(G/K)$ is identified with $\mathfrak{m}$.

We denote by $j: K \to Aut(\mathfrak{m})$ the isotropy representation of K on $\mathfrak{m}$. For a generalized flag manifold it is well known that the isotropy representation is completely reducible, that is,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s, \tag{1}$$

where each $\mathfrak{m}_i$ is an irreducible inequivalent component of the isotropy representation.

Let T be a maximal torus of G, and η be the Lie algebra of T. The complexification $\eta^{\mathbb{C}}$ is a Cartan subalgebra of $g^{\mathbb{C}}$. Let $\mathcal{R}$ be a root system of $(g^{\mathbb{C}}, \eta^{\mathbb{C}})$ and consider the root space decomposition

$$g^{\mathbb{C}} = \eta^{\mathbb{C}} \oplus \sum_{\alpha \in \mathcal{R}} g^{\mathbb{C}}_{\alpha}, \tag{2}$$

where $g^{\mathbb{C}}_{\alpha}$ denote the complex 1-dimensional root space.

Let $\mathcal{R}^+$ be a choice of positive roots and Π be corresponding set of simple roots. We fix once and for all a Weyl basis of $g^{\mathbb{C}}$ which amounts to take $E_\alpha \in g^{\mathbb{C}}_{\alpha}$ such that $Q(E_\alpha, E_{-\alpha}) = -1$, and $[E_\alpha, E_{-\alpha}] = -H_\alpha$, where $H_\alpha \in \eta^{\mathbb{C}}$ is determined by the equation $Q(H, H_\alpha) = \alpha(H)$, for all $H \in \eta^{\mathbb{C}}$. The vectors E_α satisfy the relation $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ with $N_{\alpha,\beta} \in \mathbb{R}, N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ and $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \mathcal{R}$.

Let $A_\alpha = E_\alpha - E_{-\alpha}$ and $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. The vectors

$$A_\alpha, B_\alpha, \sqrt{-1}H_\beta, \quad (\alpha \in \mathcal{R}^+ \text{ and } \beta \in \Pi) \tag{3}$$

form a basis of g (compact real form of the Lie algebra $g^{\mathbb{C}}$).

For $\alpha \in \mathcal{R}^+$, let

$$\mathfrak{m}_\alpha = span_{\mathbb{R}}\{A_\alpha, B_\alpha\} \tag{4}$$

be the real root space.

We have the following decomposition

$$g = \eta \oplus \sum_{\alpha \in \mathcal{R}^+} \mathfrak{m}_\alpha. \tag{5}$$

The next lemma gives us information about the Lie algebra structure of g.

Lemma 2.1 The Lie bracket between the elements of (3) of g are given by

$$\begin{aligned}
&[\sqrt{-1}H_\alpha, A_\beta] = \beta(H_\alpha)B_\beta, \quad [A_\alpha, A_\beta] = N_{\alpha,\beta}A_{\alpha+\beta} + N_{-\alpha,\beta}A_{\alpha-\beta},\\
&[\sqrt{-1}H_\alpha, B_\beta] = -\beta(H_\alpha)A_\beta, \quad [B_\alpha, B_\beta] = -N_{\alpha,\beta}A_{\alpha+\beta} - N_{\alpha,-\beta}A_{\alpha-\beta},\\
&[A_\alpha, B_\alpha] = 2\sqrt{-1}H_\alpha, \quad [A_\alpha, B_\beta] = N_{\alpha,\beta}B_{\alpha+\beta} + N_{\alpha,-\beta}B_{\alpha-\beta}.
\end{aligned} \tag{6}$$

Since $\eta^{\mathbb{C}}$ is also a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ (complexification of the Lie algebra of K), let $\mathcal{R}_K$ be the root system for $(\mathfrak{k}^{\mathbb{C}}, \eta^{\mathbb{C}})$ and let $\mathcal{R}_M = \mathcal{R}\backslash\mathcal{R}_K$. In a similar way, let $\mathcal{R}_K^+$ be a choice of positive roots and Π_K the corresponding set of simple roots for $\mathfrak{k}^{\mathbb{C}}$ and define $\mathcal{R}_M = \mathcal{R}\setminus\mathcal{R}_K$ and $\Pi_M = \Pi\setminus\Pi_K$ be the set of positive and simple complementary roots.

For convenience, we fix a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r, \phi_1, \cdots, \phi_k\}$ of $\mathcal{R}$, so that $\Pi_K = \{\phi_1, \cdots, \phi_k\}$ is a basis of the root system $\mathcal{R}_K$ and $\Pi_M = \Pi\backslash\Pi_K = \{\alpha_1, \cdots, \alpha_r\}(r+k = l)$. We consider the decomposition $\mathcal{R} = \mathcal{R}_K \cup \mathcal{R}_M$, and we define the set

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{k}^{\mathbb{C}}) \cap i\eta = \{X \in \eta : \phi(X) = 0, \text{for all } \phi \in \mathcal{R}_K\}, \tag{7}$$

where η is the real ad-diagonal subalgebra $\eta = \eta^{\mathbb{C}} \cap i\mathfrak{k}$, $\mathfrak{z}$ presents the center of $\mathfrak{k}^{\mathbb{C}}$. Consider the linear restriction map $\kappa : \eta^* \to \mathfrak{t}^*$ defined by $\kappa(\alpha) = \alpha\mid_{\mathfrak{t}}$, and set $\mathcal{R}_{\mathfrak{t}} = \kappa(\mathcal{R}) = \kappa(\mathcal{R}_M)$. Note that $\kappa(\mathcal{R}_K) = 0$ and $\kappa(0) = 0$. The elements of $\mathcal{R}_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots. A $\mathfrak{t}$-root is called simple if it is not a sum of two positive $\mathfrak{t}$-root.

Proposition 2.2 (see [5, Proposition 4.1]) There is one-to-one correspondence between $\mathfrak{t}$-roots and complex irreducible ad$(\mathfrak{k}^{\mathbb{C}})$-submodules $\mathfrak{m}_\xi$ of $\mathfrak{m}^{\mathbb{C}}$. This correspondence is given by

$$\mathcal{R}_{\mathfrak{t}} \ni \xi \leftrightarrow \mathfrak{m}_\xi = \sum_{\alpha\in\mathcal{R}_M:\kappa(\alpha)=\xi} \mathbb{C}E_\alpha.$$

Thus $\mathfrak{m}^{\mathbb{C}} = \sum\limits_{\xi\in\mathcal{R}_{\mathfrak{t}}} \mathfrak{m}_\xi$. Moreover, these submodules are inequivalent as ad$(\mathfrak{k}^{\mathbb{C}})$-modules.

Since the complex conjugation $\tau : g^{\mathbb{C}} \to g^{\mathbb{C}}$, $X + iY \mapsto X - iY$ $(X, Y \in g)$ of $g^{\mathbb{C}}$ with respect to the compact real form g interchanges the root spaces, i.e. $\tau(E_\alpha) = E_{-\alpha}$ and $\tau(E_{-\alpha}) = E_\alpha$, a decomposition of the real ad$(\mathfrak{k})$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbb{C}})^\tau$ into real irreducible ad$(\mathfrak{k})$-submodule is given by

$$\mathfrak{m} = \sum_{\xi\in\mathcal{R}^+=\kappa(\mathcal{R}_M^+)} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau, \tag{8}$$

where $\mathfrak{n}^\tau$ denotes the set of fixed points of the complex conjugation τ in a vector subspace $\mathfrak{n} \subset g^{\mathbb{C}}$. If, for simplicity, we set $\mathcal{R}_{\mathfrak{t}}^+ = \{\xi_1, \cdots, \xi_s\}$, then according to (8) each real irreducible ad$(\mathfrak{k})$-submodules $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau$ $(1 \le i \le s)$ corresponding to the positive $\mathfrak{t}$ -roots ξ_i, is given by

$$\mathfrak{m}_i = \mathbb{R}A_\alpha + \mathbb{R}B_\alpha, \tag{9}$$

where $\alpha \in \mathcal{R}^+$.

Proposition 2.3 (see [6]) Let $\Pi_M = \Pi \setminus \Pi_K = \{\alpha_1, \cdots, \alpha_r\}$. Then the set $\{\bar{\alpha}_i = \alpha_i \mid_{\mathfrak{t}}: \alpha_i \in \Pi_M\}$ is a $\mathfrak{t}$-base of $\mathfrak{t}^*$.

The space of G-invariant Riemannian metric $\mathfrak{g} = - < \cdot, \cdot >$ on M is given by

$$\{x_1 \cdot Q(\cdot, \cdot) \mid_{\mathfrak{m}_1} + \cdots + x_s \cdot Q(\cdot, \cdot) \mid_{\mathfrak{m}_s}: x_1 > 0, \cdots, x_s > 0\}. \tag{10}$$

Then the Ricci tensor $\mathrm{Ric}_{\mathfrak{g}}$ of G/K, as a G-invariant symmetric covariant 2-tensor on G/K, is identified with an $Ad(K)$-invariant symmetric bilinear form on $\mathfrak{m}$ is given by

$$\mathrm{Ric}_{\mathfrak{g}} = \gamma_1 x_1 (Q(\cdot, \cdot)) \mid_{\mathfrak{m}_1} + \cdots + \gamma_s x_s (Q(\cdot, \cdot)) \mid_{\mathfrak{m}_s}, \tag{11}$$

here $\gamma_1, \cdots, \gamma_s$ are the components of the Ricci tensor on each $\mathfrak{m}_i$.

Proposition 2.4 (see [7]) Let $\mathfrak{g} = - < \cdot, \cdot >$ be a G-invariant metric given by (10), and J be a G-invariant complex structure induced by an invariant ordering $\mathcal{R}_M^+$. Then, $\mathfrak{g}$ is a Kähler metric with respect to the complex structure J, if and only if the positive real numbers x_ξ satisfy $x_{\xi+\zeta} = x_\xi + x_\zeta$ for any $\xi, \zeta, \xi + \zeta \in \mathcal{R}_{\mathfrak{t}}^+ = \kappa(\mathcal{R}_M^+)$. Equivalently, $\mathfrak{g}$ is Kähler, if and only if $x_{\alpha+\beta} = x_\alpha + x_\beta$, where $\alpha, \beta, \alpha + \beta \in \mathcal{R}_M^+$ are such that $\kappa(\alpha) = \xi$ and $\kappa(\beta) = \zeta$.

Let$\{e_\alpha\}$ be a orthogonal basis with respect to $Q(\cdot, \cdot)$ adapted to the decomposition of $\mathfrak{m}$: $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$ with $i < j$ then $\alpha < \beta$. Following [8] we set $A_{\alpha,\beta}^\gamma := Q([e_\alpha, e_\beta], e_\gamma)$, thus $[e_\alpha, e_\beta]_{\mathfrak{m}} = \sum_\gamma A_{\alpha,\beta}^\gamma e_\gamma$. Consider

$$c_{ij}^k := \sum (A_{\alpha,\beta}^\gamma)^2, \tag{12}$$

where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i, e_\beta \in \mathfrak{m}_j, e_\gamma \in \mathfrak{m}_k$, and $i, j, k \in \{1, \cdots, s\}$.

Definition 2.5 A symmetric $\mathfrak{t}$-triple in $\mathfrak{t}^*$ is a triple $\Omega = (\xi_i, \xi_j, \xi_k)$ of $\mathfrak{t}$-roots $\xi_i, \xi_j, \xi_k \in \mathcal{R}_{\mathfrak{t}}$ such that $\xi_i + \xi_j + \xi_k = 0$.

Lemma 2.6 (see [9]) Let (ξ_i, ξ_j, ξ_k) be symmetric $\mathfrak{t}$-triples. Then there exist roots $\alpha, \beta, \gamma \in \mathcal{R}_M$ with $\kappa(\alpha) = \xi_i, \kappa(\beta) = \xi_j, \kappa(\gamma) = \xi_k$, such that $\alpha + \beta + \gamma = 0$.

Lemma 2.7 (see [7, Corollary 1.9]) Let G/K be a generalized flag manifold of a compact simple Lie group G and $\mathcal{R}_{\mathfrak{t}}$ be the associated $\mathfrak{t}$-root system. Assume that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ is a $Q(\cdot, \cdot)$-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $Ad(\mathfrak{k})$-module, and let $\xi_i, \xi_j, \xi_k \in \mathcal{R}_{\mathfrak{t}}$ be the $\mathfrak{t}$-root associated to the components $\mathfrak{m}_i, \mathfrak{m}_j$ and $\mathfrak{m}_k$ respectively, then, $c_{ij}^k \neq 0$, if and only if (ξ_i, ξ_j, ξ_k) is a symmetric $\mathfrak{t}$-triples, i.e. $\xi_i + \xi_j + \xi_k = 0$.

3 Invariant Einstein Metrics on $SO(8)/T$

Let $M = G/T$ be a full flag manifold and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be $Q(\cdot, \cdot)$-orthogonal decomposition of $\mathfrak{m}$. Then the set

$$\{X_\alpha = \frac{A_\alpha}{\sqrt{2}} = \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}, Y_\alpha = \frac{B_\alpha}{\sqrt{2}} = \sqrt{-1}\frac{E_\alpha + E_{-\alpha}}{\sqrt{2}} : \alpha \in R^+, \kappa(\alpha) = \xi_i \in R_{\mathfrak{t}}^+\} \tag{13}$$

is a $Q(\cdot, \cdot)$-orthogonal basis of $\mathfrak{m}_i$.

Theorem 3.1 (see [10]) For a full flag manifold G/T the non-zero structure constants c_{ij}^k is given by

$$c_{ij}^k = (A_{\alpha,\beta}^{\alpha+\beta})^2 = 2N_{\alpha,\beta}^2, \tag{14}$$

where $\alpha, \beta \in R^+$ with $\kappa(\alpha) = \xi_i, \kappa(\beta) = \xi_j, \kappa(\alpha+\beta) = \xi_k$.

Lemma 3.2 (see [11]) Let $M = G/K$ be a reductive homogeneous space of a compact semisimple Lie group G and let $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ be a decomposition of $\mathfrak{m}$ into mutually inequivalent irreducible $Ad(K)$-submodules. Then the components $\gamma_1, \cdots, \gamma_s$ of the Ricci tensor of a G-invariant metric (10) on M are given by

$$\gamma_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{i,j}\frac{x_k}{x_i x_j}c_{ij}^k - \frac{1}{2d_k}\sum_{i,j}\frac{x_j}{x_k x_i}c_{ki}^j \ (k = 1, \cdots, s). \tag{15}$$

Next we talk about the isometric problem about a flag manifold, in general, this is not a trivial problem.

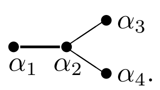
Let G/K be a generalized flag manifold with isotropy decomposition (1), and $d = \sum_{i=1}^{s} d_i = \dim M$. For any G-invariant Einstein metrics $\mathfrak{g} = (x_1, \cdots, x_s)$ on M, we determine a scale invariant given by $H_{\mathfrak{g}} = V^{\frac{1}{d}} S_{\mathfrak{g}}$, where $S_{\mathfrak{g}}$ is the scalar curvature of $\mathfrak{g}$, and $V = V_{\mathfrak{g}}/V_B$ is the quotient of the volumes $V_{\mathfrak{g}} = \prod_{i=1}^{s} x_i^{d_i}$ of the given metric $\mathfrak{g}$, and V_B the volume of the normal metric induced by the negative of the Killing form of G. We normalize $V_B = 1$, so $H_{\mathfrak{g}} = V_{\mathfrak{g}}^{\frac{1}{d}} S_{\mathfrak{g}}$. The scalar curvature $S_{\mathfrak{g}}$ of a G-invariant metric $\mathfrak{g}$ on M is given by the following well-known formula [8]:

$$S_{\mathfrak{g}} = \sum_{i=1}^{s} d_i \gamma_i = \frac{1}{2}\sum_{i=1}^{s}\frac{d_i}{x_i} - \frac{1}{4}\sum_{1\le i,j,k\le s} c_{ij}^k \frac{x_k}{x_i x_j}, \tag{16}$$

where the components γ_i of the Ricc tensor are given by (15). The scalar curvature is a homogeneous polynomial of degree -1 on the variables $x_i (i = 1, \cdots, s)$. The volume $V_{\mathfrak{g}}$ is a monomial of degree d, so $H_{\mathfrak{g}} = V_{\mathfrak{g}}^{\frac{1}{d}} S_{\mathfrak{g}}$ is a homogeneous polynomial of degree 0. Therefore, $H_{\mathfrak{g}}$ is invariant under a common scaling of the variables x_i.

If two metrics are isometric then they have the same scale invariant, so if the scale invariant $H_{\mathfrak{g}}$ and $H_{\mathfrak{g}'}$ are different, then the metrics $\mathfrak{g}$ and $\mathfrak{g}'$ cant not be isometric. However, if $H_{\mathfrak{g}} = H_{\mathfrak{g}'}$ we can not immediately conclude if the metrics $\mathfrak{g}$ and $\mathfrak{g}'$ are isometric or not. For such a case we have to look at the group of automorphisms of G and check if there is an automorphism which permutes the isotopy summands and takes one metric to another. This usually arises for the Kähler-Einstein metrics.

Now we consider the full flag manifold of $SO(8)/T$ with the painted Dynkin graph



Here $\Pi_M = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, let $\bar\alpha_1 = \kappa(\alpha_1), \bar\alpha_2 = \kappa(\alpha_2)$, $\bar\alpha_3 = \kappa(\alpha_3)$ and $\bar\alpha_4 = \kappa(\alpha_4)$, we have $R_{\mathfrak{t}}^+ = \kappa(R_M^+) = \{\bar\alpha_1, \bar\alpha_2, \bar\alpha_3, \bar\alpha_4, \bar\alpha_1 + \bar\alpha_2, \bar\alpha_2 + \bar\alpha_3, \bar\alpha_2 + \bar\alpha_4, \bar\alpha_1 + \bar\alpha_2 + \bar\alpha_3, \bar\alpha_1 + \bar\alpha_2 + \bar\alpha_4, \bar\alpha_2 +$

$\bar{\alpha}_3+\bar{\alpha}_4, \bar{\alpha}_1+\bar{\alpha}_2+\bar{\alpha}_3+\bar{\alpha}_4, \bar{\alpha}_1+2\bar{\alpha}_2+\bar{\alpha}_3+\bar{\alpha}_4\}$, thus we conclude that isotropy representation $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3\oplus\mathfrak{m}_4\oplus\mathfrak{m}_5\oplus\mathfrak{m}_6\oplus\mathfrak{m}_7\oplus\mathfrak{m}_8\oplus\mathfrak{m}_9\oplus\mathfrak{m}_{10}\oplus\mathfrak{m}_{11}\oplus\mathfrak{m}_{12}$.

By Lemma 2.7 we obtain that the non-zero structure constants are

$$c_{1,2}^5, c_{1,6}^8, c_{1,7}^9, c_{1,10}^{11}, c_{2,3}^6, c_{2,4}^7, c_{2,11}^{12}, c_{3,5}^8, c_{3,7}^{10}, c_{3,9}^{11}, c_{4,5}^9, c_{4,6}^{10}, c_{4,8}^{11}, c_{5,10}^{12}, c_{6,9}^{12}, c_{7,8}^{12}.$$

Lemma 3.3 The non-zero structure constants of generalized flag manifold $SO(8)/T$ are given by $c_{1,2}^5=c_{1,6}^8=c_{1,7}^9=c_{1,10}^{11}=c_{2,3}^6=c_{2,4}^7=c_{2,11}^{12}=c_{3,5}^8=c_{3,7}^{10}=c_{3,9}^{11}=c_{4,5}^9=c_{4,6}^{10}=c_{4,8}^{11}=c_{4,10}^{12}=c_{6,9}^{12}=c_{7,8}^{12}=\frac{1}{6}$.

Proof From the theorem of Lie algebra we can get $N_{\alpha,\beta}^2=\frac{q(p+1)}{2}(\alpha,\alpha)$, $(\alpha,\beta)=-\frac{q-p}{2}(\alpha,\alpha)$, where p,q are the largest nonnegative integers such that $\beta+k\alpha\in R$ with $-p\le k\le q$.

By Lemma 3.1 we can calculate the non-zero structure constants of M as follows:

$$\begin{aligned}
&c_{1,2}^5=2N_{\alpha_1,\alpha_2}^2=(\alpha_1,\alpha_1),\ \ c_{1,6}^8=2N_{\alpha_1,\alpha_2+\alpha_3}^2=(\alpha_1,\alpha_1),\ \ c_{1,7}^9=2N_{\alpha_1,\alpha_2+\alpha_4}^2=(\alpha_1,\alpha_1),\\
&c_{1,10}^{11}=2N_{\alpha_1,\alpha_2+\alpha_3+\alpha_4}^2=(\alpha_1,\alpha_1),\ \ c_{2,3}^6=2N_{\alpha_2,\alpha_3}^2=(\alpha_2,\alpha_2),\ \ c_{2,4}^7=2N_{\alpha_2,\alpha_4}^2=(\alpha_2,\alpha_2),\\
&c_{2,11}^{12}=2N_{\alpha_2,\alpha_1+\alpha_2+\alpha_3+\alpha_4}^2=(\alpha_2,\alpha_2),\ \ c_{3,5}^8=2N_{\alpha_3,\alpha_1+\alpha_2}^2=(\alpha_3,\alpha_3),\ \ c_{3,7}^{10}=2N_{\alpha_3,\alpha_2+\alpha_4}^2=(\alpha_3,\alpha_3),\\
&c_{3,9}^{11}=2N_{\alpha_3,\alpha_1+\alpha_2+\alpha_4}^2=(\alpha_3,\alpha_3),\ \ c_{4,5}^9=2N_{\alpha_4,\alpha_1+\alpha_2}^2=(\alpha_4,\alpha_4),\ \ c_{4,6}^{10}=2N_{\alpha_4,\alpha_2+\alpha_3}^2=(\alpha_4,\alpha_4),\\
&c_{4,8}^{11}=2N_{\alpha_4,\alpha_1+\alpha_2+\alpha_3}^2=(\alpha_4,\alpha_4),\ \ c_{5,10}^{12}=2N_{\alpha_1+\alpha_2,\alpha_2+\alpha_3+\alpha_4}^2=(\alpha_1,\alpha_1),\\
&c_{6,9}^{12}=2N_{\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_4}^2=(\alpha_2,\alpha_2),\ \ c_{7,8}^{12}=2N_{\alpha_2+\alpha_4,\alpha_1+\alpha_2+\alpha_3}^2=(\alpha_2,\alpha_2).
\end{aligned}$$

As $(\alpha_1,\alpha_1)=(\alpha_2,\alpha_2)=(\alpha_3,\alpha_3)=(\alpha_4,\alpha_4)=\frac{1}{5}$ we obtain that $c_{12}^5=c_{16}^8=c_{19}^{10}=c_{23}^6=c_{27}^9=c_{34}^7=c_{35}^8=c_{46}^9=c_{48}^{10}=c_{57}^{10}=\frac{1}{6}$.

Lemma 3.4 The components $\gamma_i(i=1,\cdots,12)$ of Ricci tensor associated to the $SO(8)$-invariant Riemmanian metric $\mathfrak{g}$ are the following:

$$\begin{cases}
\gamma_1=\frac{1}{2x_1}+\frac{x_1^2-x_2^2-x_5^2}{24x_1x_2x_5}+\frac{x_1^2-x_6^2-x_8^2}{24x_1x_6x_8}+\frac{x_1^2-x_7^2-x_9^2}{24x_1x_7x_9}+\frac{x_1^2-x_{10}^2-x_{11}^2}{24x_1x_{10}x_{11}},\\
\gamma_2=\frac{1}{2x_2}+\frac{x_2^2-x_1^2-x_5^2}{24x_1x_2x_5}+\frac{x_2^2-x_3^2-x_6^2}{24x_2x_3x_6}+\frac{x_2^2-x_4^2-x_7^2}{24x_2x_4x_7}+\frac{x_2^2-x_{11}^2-x_{12}^2}{24x_2x_{11}x_{12}},\\
\gamma_3=\frac{1}{2x_3}+\frac{x_3^2-x_2^2-x_6^2}{24x_2x_3x_6}+\frac{x_3^2-x_7^2-x_{10}^2}{24x_3x_7x_{10}}+\frac{x_3^2-x_5^2-x_8^2}{24x_3x_5x_8}+\frac{x_3^2-x_9^2-x_{11}^2}{24x_3x_9x_{11}},\\
\gamma_4=\frac{1}{2x_4}+\frac{x_4^2-x_2^2-x_7^2}{24x_2x_4x_7}+\frac{x_4^2-x_5^2-x_9^2}{24x_4x_5x_9}+\frac{x_4^2-x_6^2-x_{10}^2}{24x_4x_6x_{10}}+\frac{x_4^2-x_8^2-x_{11}^2}{24x_4x_8x_{11}},\\
\gamma_5=\frac{1}{2x_5}+\frac{x_5^2-x_1^2-x_2^2}{24x_1x_2x_5}+\frac{x_5^2-x_3^2-x_8^2}{24x_3x_5x_8}+\frac{x_5^2-x_4^2-x_9^2}{24x_4x_5x_9}+\frac{x_5^2-x_{10}^2-x_{12}^2}{24x_5x_{10}x_{12}},\\
\gamma_6=\frac{1}{2x_6}+\frac{x_6^2-x_1^2-x_8^2}{24x_1x_6x_8}+\frac{x_6^2-x_2^2-x_3^2}{24x_2x_3x_6}+\frac{x_6^2-x_4^2-x_{10}^2}{24x_4x_6x_{10}}+\frac{x_6^2-x_9^2-x_{12}^2}{24x_6x_9x_{12}},\\
\gamma_7=\frac{1}{2x_7}+\frac{x_7^2-x_1^2-x_9^2}{24x_1x_7x_9}+\frac{x_7^2-x_2^2-x_4^2}{24x_2x_4x_7}+\frac{x_7^2-x_3^2-x_{10}^2}{24x_3x_7x_{10}}+\frac{x_7^2-x_8^2-x_{12}^2}{24x_7x_8x_{12}},\\
\gamma_8=\frac{1}{2x_8}+\frac{x_8^2-x_1^2-x_6^2}{24x_1x_6x_8}+\frac{x_8^2-x_3^2-x_5^2}{24x_3x_5x_8}+\frac{x_8^2-x_4^2-x_{11}^2}{24x_4x_8x_{11}}+\frac{x_8^2-x_7^2-x_{12}^2}{24x_7x_8x_{12}},\\
\gamma_9=\frac{1}{2x_9}+\frac{x_9^2-x_1^2-x_7^2}{24x_1x_7x_9}+\frac{x_9^2-x_3^2-x_{11}^2}{24x_3x_9x_{11}}+\frac{x_9^2-x_6^2-x_{12}^2}{24x_6x_9x_{12}}+\frac{x_9^2-x_4^2-x_5^2}{24x_4x_5x_9},\\
\gamma_{10}=\frac{1}{2x_{10}}+\frac{x_{10}^2-x_1^2-x_{11}^2}{24x_1x_{10}x_{11}}+\frac{x_{10}^2-x_3^2-x_7^2}{24x_3x_7x_{10}}+\frac{x_{10}^2-x_4^2-x_6^2}{24x_4x_6x_{10}}+\frac{x_{10}^2-x_5^2-x_{12}^2}{24x_5x_{10}x_{12}},\\
\gamma_{11}=\frac{1}{2x_{11}}+\frac{x_{11}^2-x_1^2-x_{10}^2}{24x_1x_{10}x_{11}}+\frac{x_{11}^2-x_2^2-x_{12}^2}{24x_2x_{11}x_{12}}+\frac{x_{11}^2-x_3^2-x_9^2}{24x_3x_9x_{11}}+\frac{x_{11}^2-x_4^2-x_8^2}{24x_4x_8x_{11}},\\
\gamma_{12}=\frac{1}{2x_{12}}+\frac{x_{12}^2-x_2^2-x_{11}^2}{24x_2x_{11}x_{12}}+\frac{x_{12}^2-x_5^2-x_{10}^2}{24x_5x_{10}x_{12}}+\frac{x_{12}^2-x_6^2-x_9^2}{24x_6x_9x_{12}}+\frac{x_{12}^2-x_7^2-x_8^2}{24x_7x_8x_{12}}.
\end{cases}$$

From (10) and (11) we get that a G-invariant Riemmanian metric $\mathfrak{g}$ on $M=SO(8)/T$ is Einstein, if and only if, there is a positive constant e such that $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma_5=$

$\gamma_6 = \gamma_7 = \gamma_8 = \gamma_9 = \gamma_{10} = \gamma_{11} = \gamma_{12} = e$, or equivalently,

$$
\begin{aligned}
&\gamma_1 - \gamma_2 = 0, \gamma_2 - \gamma_3 = 0, \gamma_3 - \gamma_4 = 0, \gamma_4 - \gamma_5 = 0, \gamma_5 - \gamma_6 = 0, \gamma_6 - \gamma_7 = 0, \gamma_7 - \gamma_8 = 0, \\
&\gamma_8 - \gamma_9 = 0, \gamma_9 - \gamma_{10} = 0, \gamma_{10} - \gamma_{11} = 0, \gamma_{11} - \gamma_{12} = 0.
\end{aligned} \tag{17}
$$

By Lemma 3.4 and system (17) we obtain the following polynomial system (we apply the normalization $x_1 = 1$).

$$
\begin{cases}
12x_2x_5x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} + 2x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} - 2x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12}x_2^2 + x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12} \\
-x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12}x_6^2 - x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12}x_8^2 + x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12} - x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12}x_7^2 \\
-x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12}x_9^2 + x_2x_5x_6x_8x_7x_9x_3x_4x_{12} - x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{10}^2 - x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{11}^2 \\
+x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12}^2 + x_5x_6x_8x_7x_9x_{10}x_3x_4x_{11}^2 - x_5x_6x_8x_7x_9x_{10}x_3x_4x_2^2 - 12x_5x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} \\
-x_5x_8x_7x_9x_{10}x_{11}x_4x_{12}x_2^2 + x_5x_8x_7x_9x_{10}x_{11}x_4x_{12}x_3^2 + x_5x_8x_7x_9x_{10}x_{11}x_4x_{12}x_6^2 - x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_2^2 \\
+x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_4^2 + x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_7^2 = 0; \\
x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} + x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12}x_2^2 - x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12}x_5^2 - x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12}^2 \\
-x_5x_6x_8x_7x_9x_{10}x_3x_4x_{11}^2 + x_5x_6x_8x_7x_9x_{10}x_3x_4x_2^2 - x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_3^2 + x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_7^2 \\
+x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_{10}^2 - x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_3^2 + x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_9^2 + x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_{11}^2 \\
+12x_5x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} + 2x_5x_8x_7x_9x_{10}x_{11}x_4x_{12}x_2^2 - 2x_5x_8x_7x_9x_{10}x_{11}x_4x_{12}x_3^2 + x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_2^2 \\
-x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_4^2 - x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_7^2 - 12x_2x_5x_6x_8x_7x_9x_{10}x_{11}x_4x_{12} - x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_3^2 \\
+x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_5^2 + x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_8^2 = 0; \\
12x_2x_6x_7x_{10}x_5x_8x_9x_{11}x_4 - 12x_3x_2x_6x_7x_{10}x_5x_8x_9x_{11} + x_7x_{10}x_5x_8x_9x_{11}x_4x_3^2 - x_7x_{10}x_5x_8x_9x_{11}x_4x_2^2 \\
-x_7x_{10}x_5x_8x_9x_{11}x_4x_6^2 + x_2x_6x_5x_8x_9x_{11}x_4x_3^2 - x_2x_6x_5x_8x_9x_{11}x_4x_7^2 - x_2x_6x_5x_8x_9x_{11}x_4x_{10}^2 \\
+x_2x_6x_7x_{10}x_9x_{11}x_4x_3^2 - x_2x_6x_7x_{10}x_9x_{11}x_4x_5^2 - x_2x_6x_7x_{10}x_9x_{11}x_4x_8^2 + x_2x_6x_7x_{10}x_5x_8x_4x_3^2 \\
-x_2x_6x_7x_{10}x_5x_8x_4x_9^2 - x_2x_6x_7x_{10}x_5x_8x_4x_{11}^2 - x_3x_6x_{10}x_5x_8x_9x_{11}x_4^2 + x_3x_6x_{10}x_5x_8x_9x_{11}x_2^2 \\
+x_3x_6x_{10}x_5x_8x_9x_{11}x_7^2 - x_3x_2x_6x_7x_{10}x_8x_{11}x_4^2 + x_3x_2x_6x_7x_{10}x_8x_{11}x_5^2 + x_3x_2x_6x_7x_{10}x_8x_{11}x_9^2 \\
-x_3x_2x_7x_5x_8x_9x_{11}x_4^2 + x_3x_2x_7x_5x_8x_9x_{11}x_6^2 + x_3x_2x_7x_5x_8x_9x_{11}x_{10}^2 - x_3x_2x_6x_7x_{10}x_5x_9x_4^2 \\
+x_3x_2x_6x_7x_{10}x_5x_9x_8^2 + x_3x_2x_6x_7x_{10}x_5x_9x_{11}^2 = 0; \\
x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} + x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12}x_2^2 - x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12}x_5^2 + x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_4^2 \\
-x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_{11}^2 - x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_6^2 - x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_{10}^2 + x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_4^2 \\
-x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_8^2 - x_2x_6x_8x_7x_9x_{11}x_3x_4x_5^2 + x_2x_6x_8x_7x_9x_{11}x_3x_4x_{10}^2 + x_2x_6x_8x_7x_9x_{11}x_3x_4x_{12}^2 \\
-x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_2^2 + x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_4^2 - x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_7^2 + x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_3^2 \\
-x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_5^2 + x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_8^2 + 12x_2x_5x_6x_8x_7x_9x_{10}x_{11}x_3x_{12} - 12x_2x_6x_8x_7x_9x_{10}x_{11}x_3x_4x_{12} \\
+2x_2x_6x_8x_7x_{10}x_{11}x_3x_{12}x_4^2 - 2x_2x_6x_8x_7x_{10}x_{11}x_3x_{12}x_5^2 = 0; \\
-x_3x_8x_4x_9x_{10}x_{12}x_6 + x_3x_8x_4x_9x_{10}x_{12}x_6x_5^2 - x_3x_8x_4x_9x_{10}x_{12}x_6x_2^2 + x_5x_2x_3x_4x_9x_{10}x_{12} - x_5x_2x_3x_4x_9x_{10}x_{12}x_6^2 \\
+x_5x_2x_3x_4x_9x_{10}x_{12}x_8^2 + 12x_2x_3x_8x_4x_9x_{10}x_{12}x_6 - 12x_5x_2x_3x_8x_4x_9x_{10}x_{12} + x_2x_4x_9x_{10}x_{12}x_6x_5^2 - x_2x_4x_9x_{10}x_{12}x_6x_3^2 \\
-x_2x_4x_9x_{10}x_{12}x_6x_8^2 + x_2x_3x_8x_{10}x_{12}x_6x_5^2 - x_2x_3x_8x_{10}x_{12}x_6x_4^2 - x_2x_3x_8x_{10}x_{12}x_6x_9^2 + x_2x_3x_8x_4x_9x_6x_5^2 \\
-x_2x_3x_8x_4x_9x_6x_{10}^2 - x_2x_3x_8x_4x_9x_6x_{12}^2 - x_5x_8x_4x_9x_{10}x_{12}x_6^2 + x_5x_8x_4x_9x_{10}x_{12}x_2^2 + x_5x_8x_4x_9x_{10}x_{12}x_3^2 \\
-x_5x_2x_3x_8x_9x_{12}x_6^2 + x_5x_2x_3x_8x_9x_{12}x_4^2 + x_5x_2x_3x_8x_9x_{12}x_{10}^2 - x_5x_2x_3x_8x_4x_{10}x_6^2 + x_5x_2x_3x_8x_4x_{10}x_9^2 \\
+x_5x_2x_3x_8x_4x_{10}x_{12}^2 = 0; \\
-x_2x_3x_4x_{10}x_9x_{12}x_7 + x_2x_3x_4x_{10}x_9x_{12}x_7x_6^2 - x_2x_3x_4x_{10}x_9x_{12}x_7x_8^2 + x_6x_8x_2x_3x_4x_{10}x_{12} - x_6x_8x_2x_3x_4x_{10}x_{12}x_7^2 \\
+x_6x_8x_2x_3x_4x_{10}x_{12}x_9^2 - 12x_2x_3x_8x_4x_9x_{10}x_{12}x_6 + 12x_8x_2x_3x_4x_{10}x_9x_{12}x_7 + x_8x_4x_{10}x_9x_{12}x_7x_6^2 - x_8x_4x_{10}x_9x_{12}x_7x_2^2 \\
-x_8x_4x_{10}x_9x_{12}x_7x_3^2 + x_8x_2x_3x_9x_{12}x_7x_6^2 - x_8x_2x_3x_9x_{12}x_7x_4^2 - x_8x_2x_3x_9x_{12}x_7x_{10}^2 + x_8x_2x_3x_4x_{10}x_7x_6^2 \\
-x_8x_2x_3x_4x_{10}x_7x_9^2 - x_8x_2x_3x_4x_{10}x_7x_{12}^2 - x_6x_8x_3x_{10}x_9x_{12}x_7^2 + x_6x_8x_3x_{10}x_9x_{12}x_2^2 + x_6x_8x_3x_{10}x_9x_{12}x_4^2 \\
-x_6x_8x_2x_4x_9x_{12}x_7^2 + x_6x_8x_2x_4x_9x_{12}x_3^2 + x_6x_8x_2x_4x_9x_{12}x_{10}^2 - x_6x_2x_3x_4x_{10}x_9x_7^2 + x_6x_2x_3x_4x_{10}x_9x_8^2 \\
+x_6x_2x_3x_4x_{10}x_9x_{12}^2 = 0;
\end{cases} \tag{18}
$$

$$
\left\{\begin{array}{l}
x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12} + x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12}x_6^2 - x_2x_5x_7x_9x_{10}x_{11}x_3x_4x_{12}x_8^2 - x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12} \\
+x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12}x_7^2 - x_2x_5x_6x_8x_{10}x_{11}x_3x_4x_{12}x_9^2 - x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_3^2 + x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_7^2 \\
-x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_{10}^2 + x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_{11}^2 + x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_4^2 - x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_8^2 \\
-x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_2^2 - x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_4^2 + x_5x_6x_8x_9x_{10}x_{11}x_3x_{12}x_7^2 + x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_3^2 \\
+x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_5^2 - x_2x_6x_7x_9x_{10}x_{11}x_4x_{12}x_8^2 + 12x_2x_5x_6x_8x_9x_{10}x_{11}x_3x_4x_{12} - 12x_2x_5x_6x_7x_9x_{10}x_{11}x_3x_4x_{12} \\
+2x_2x_5x_6x_9x_{10}x_{11}x_3x_4x_7^2 - 2x_2x_5x_6x_9x_{10}x_{11}x_3x_4x_8^2 = 0; \\
x_6x_4x_{11}x_7x_{12}x_9x_8^2 - x_6x_4x_{11}x_7x_{12}x_9x_5^2 - x_6x_4x_{11}x_7x_{12}x_9x_3^2 + 12x_6x_3x_5x_4x_{11}x_7x_{12}x_9 + x_8x_6x_5x_4x_7x_{12}x_3^2 \\
-x_8x_6x_5x_4x_7x_{12}x_9^2 - x_6x_3x_5x_4x_{11}x_9x_{12}^2 - x_6x_3x_5x_4x_{11}x_9x_7^2 + x_6x_3x_5x_4x_{11}x_9x_8^2 - x_6x_3x_5x_7x_{12}x_9x_{11}^2 \\
-x_6x_3x_5x_7x_{12}x_9x_4^2 + x_6x_3x_5x_7x_{12}x_9x_8^2 + x_8x_6x_3x_{11}x_7x_{12}x_5^2 - 12x_8x_6x_3x_5x_4x_{11}x_7x_{12} + x_8x_6x_3x_{11}x_7x_{12}x_4^2 \\
-x_8x_6x_3x_{11}x_7x_{12}x_9^2 + x_8x_6x_5x_4x_7x_{12}x_{11}^2 - x_3x_5x_4x_{11}x_7x_{12}x_9 + x_8x_3x_5x_4x_{11}x_7x_{12}^2 + x_8x_3x_5x_4x_{11}x_7x_6^2 \\
-x_8x_3x_5x_4x_{11}x_7x_9^2 + x_3x_5x_4x_{11}x_7x_{12}x_9x_8^2 - x_3x_5x_4x_{11}x_7x_{12}x_9x_6^2 + x_8x_6x_3x_5x_4x_{11}x_{12} + x_8x_6x_3x_5x_4x_{11}x_{12}x_7^2 \\
-x_8x_6x_3x_5x_4x_{11}x_{12}x_9^2 = 0; \\
-12x_6x_3x_5x_4x_{11}x_7x_{12}x_9 + 12x_7x_3x_{11}x_4x_5x_6x_{12}x_{10} + x_3x_{11}x_4x_5x_6x_{12}x_{10}x_9^2 - x_3x_{11}x_4x_5x_6x_{12}x_{10} \\
+x_9x_7x_3x_4x_5x_6x_{12} - x_3x_{11}x_4x_5x_6x_{12}x_{10}x_7^2 + x_7x_4x_5x_6x_{12}x_{10}x_9^2 - x_7x_3x_{11}x_4x_5x_{10}x_{12}^2 - x_7x_3x_{11}x_4x_5x_{10}x_6^2 \\
+x_7x_3x_{11}x_4x_5x_{10}x_9^2 - x_7x_3x_{11}x_6x_{12}x_{10}x_5^2 - x_7x_3x_{11}x_6x_{12}x_{10}x_4^2 - x_9x_7x_3x_4x_5x_6x_{12}x_{10}^2 + x_7x_3x_{11}x_6x_{12}x_{10}x_9^2 \\
-x_7x_4x_5x_6x_{12}x_{10}x_{11}^2 - x_7x_4x_5x_6x_{12}x_{10}x_3^2 - x_9x_7x_3x_{11}x_5x_{12}x_{10}^2 + x_9x_{11}x_4x_5x_6x_{12}x_7^2 + x_9x_{11}x_4x_5x_6x_{12}x_3^2 \\
-x_9x_{11}x_4x_5x_6x_{12}x_{10}^2 + x_9x_7x_3x_4x_5x_6x_{12}x_{11}^2 + x_9x_7x_3x_{11}x_5x_{12}x_6^2 + x_9x_7x_3x_{11}x_5x_{12}x_4^2 - x_9x_7x_3x_{11}x_4x_6x_{10}^2 \\
+x_9x_7x_3x_{11}x_4x_6x_{12}^2 + x_9x_7x_3x_{11}x_4x_6x_5^2 = 0; \\
-x_5x_6x_8x_7x_9x_{10}x_3x_4x_{11}^2 - 12x_2x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12} + 12x_2x_5x_6x_8x_7x_9x_{11}x_3x_4x_{12} + x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12}^2 \\
+x_5x_6x_8x_7x_9x_{10}x_3x_4x_2^2 + 2x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{10}^2 - x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_6^2 - x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_4^2 \\
+x_2x_5x_8x_7x_9x_{11}x_3x_{12}x_{10}^2 - 2x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{11}^2 + x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_3^2 - x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_{11}^2 \\
+x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_9^2 - x_2x_6x_8x_7x_9x_{11}x_3x_4x_5^2 + x_2x_6x_8x_7x_9x_{11}x_3x_4x_{10}^2 - x_2x_6x_8x_7x_9x_{11}x_3x_4x_{12}^2 \\
-x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_3^2 - x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_7^2 + x_2x_5x_6x_8x_9x_{11}x_4x_{12}x_{10}^2 - x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_{11}^2 \\
+x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_4^2 + x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_8^2 = 0; \\
2x_5x_6x_8x_7x_9x_{10}x_3x_4x_{11}^2 - x_2x_5x_6x_9x_{10}x_{11}x_3x_4x_{12}^2 + 12x_2x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12} - 12x_2x_5x_6x_8x_7x_9x_{10}x_{11}x_3x_4 \\
-2x_5x_6x_8x_7x_9x_{10}x_3x_4x_{12}^2 - x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{10}^2 - x_2x_5x_6x_8x_7x_9x_3x_4x_{12} + x_2x_5x_6x_8x_7x_9x_3x_4x_{12}x_{11}^2 \\
-x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_3^2 + x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_{11}^2 - x_2x_5x_6x_8x_7x_{10}x_4x_{12}x_9^2 + x_2x_6x_8x_7x_9x_{11}x_3x_4x_5^2 \\
+x_2x_6x_8x_7x_9x_{11}x_3x_4x_{10}^2 - x_2x_6x_8x_7x_9x_{11}x_3x_4x_{12}^2 - x_2x_5x_8x_7x_{10}x_{11}x_3x_4x_{12}^2 + x_2x_5x_8x_7x_{10}x_{11}x_3x_4x_6^2 \\
+x_2x_5x_8x_7x_{10}x_{11}x_3x_4x_9^2 + x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_{11}^2 - x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_4^2 - x_2x_5x_6x_7x_9x_{10}x_3x_{12}x_8^2 \\
+x_2x_5x_6x_9x_{10}x_{11}x_3x_4x_7^2 + x_2x_5x_6x_9x_{10}x_{11}x_3x_4x_8^2 = 0.
\end{array}\right. \tag{19}
$$

Any positive real solution

$$x_2>0, x_3>0, x_4>0, x_5>0, x_6>0, x_7>0, x_8>0, x_9>0, x_{10}>0, x_{11}>0, x_{12}>0$$

of the system above determines a $SO(8)$-invariant Einstein metric

$$(1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \in \mathbb{R}_+^{12}$$

on $M = G/T$. With the help of computer we obtain that there are one hundred and sixty invariant Einstein metrics on $SO(8)/T$, of which one hundred and twelve are Kähler Einstein metrics (up to a scale).

Theorem 3.5 The full flag manifold $M = SO(8)/T$ admits five (up to isometry) $SO(8)$-invariant Einstein metrics. There is a unique Kähler Einstein metric (up to a scale) given by

$$\mathfrak{g} = (1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 5)$$

and the other four are non-Kähler (up to a scale) approximately given by as follows:

(a) (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),

(b) 1, 0.5, 1, 1, 1, 1, 1, 0.5, 0.5, 0.5, 1, 1),

(c) (1, 1.4, 1, 1.96, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1.4, 1),

(d) (1, 1.5230, 1, 1.9907, 1.5230, 1.5230, 1.2011, 1.5230, 1.2011, 1.2011, 1.2011, 0.8669).

Proof We compute $H_{\mathfrak{g}}$ of all the one hundred and sixty positive (real) solutions by formula (16) and obtain five non equal values, the five values are

$$8.0356, 7.9370, 8.0000, 7.9975, 7.9959.$$

Thus there are at least five non-isometry Einstein metrics. When $H_{\mathfrak{g}} = H_{\mathfrak{g}'}$ it is easy to check that there is an element of Weyl group of G which permutes the isotopy summands and takes one metric to another. Thus there are five $SO(8)$- invariant non-isometric Einstein metrics.

Acknowledge The Authors declared that they have no conflict to any work.

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满旗流形$SO(8)/T$上不变爱因斯坦度量

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摘要: 本文研究了迷向表示分为12个不可约子空间的满旗流形$SO(8)/T$上不变爱因斯坦度量的问题. 利用计算机计算满旗流形$SO(8)/T$爱因斯坦方程组的方法, 得到了满旗流形$SO(8)/T$ 上有160 个不变爱因斯坦度量(up to a scale)的结果, 在等距情况下考虑这160个不变爱因斯坦度量, 其中1个是凯莱爱因斯坦度量, 4 个是非凯莱爱因斯坦度量. 推广了只对迷向表示分为小于等于6个不可约子空间的满旗流形上不变爱因斯坦度量的研究.

关键词: 满旗流形; 爱因斯坦度量; Ricci 张量; 迷向表示

MR(2010)**主题分类号**: 53C30 **中图分类号**: O186.14