Vol. 35 (2015)
No. 6

数 学 杂 志
J. of Math. (PRC)

ON RELATIVE MCCOY PROPERTIES WITH A RING ENDOMORPHISM

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Abstract: In this paper, we introduce the notions of an α-McCoy ring and weak α-McCoy rings to study McCoy properties and weak McCoy properties relative to an endomorphism α of a ring R. By using various ring extensions, we prove that a ring R is a right α-McCoy ring if and only if $R[x]$ is a right α-McCoy ring, and the direct limit of a direct system of right weak α-McCoy rings is investigated in the last section. It is shown that if R is a right weak α-McCoy ring. Some well-known results on McCoy rings are generalized.

Keywords: McCoy rings; α-McCoy rings; weak α-McCoy rings

2010 MR Subject Classification: 16N40; 16S50

Document code: A **Article ID:** 0255-7797(2015)06-1287-10

1 Introduction

Throughout this note, R denotes an associative ring with identity and α denotes a nonzero endomorphism, unless specified otherwise. For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R and $T_n(R)$ the n-by-n upper triangular matrix ring over R. In [8], Nielsen introduced the notion of a McCoy ring. A ring R is said to be right McCoy (resp., left McCoy) if for each pair of nonzero polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ with $f(x)r = 0$ (resp., $rg(x) = 0$). A ring R is McCoy if it is both left and right McCoy. The name of the ring was given due to N. H. McCoy who proved in [7] that commutative rings always satisfy this condition. A ring R is called weak McCoy if for each pair of nonzero polynomials $f(x) = \sum\limits_{i=0}^{m} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ such that $a_i r \in nil(R)$ (resp., $rb_j \in nil(R)$). A ring R is called weak McCoy if it is both right and left weak McCoy. Due to Rege and Chhawchharia [9], a ring R is called Armendariz if for given $f(x) = \sum\limits_{i=0}^{m} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for

* **Received date:** 2013-04-15 **Accepted date:** 2013-06-06

Foundation item: Supported by the Provincial Natural Science Research Program of Higher Education Institution of Anhui Province of China (KJ2012Z028); Partially Supported by the Foundation for Young Talents in College of Anhui Province of China (2012SQRL039); the National Natural Science Foundation of China (11226056).

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each i, j (the converse is obviously true). It is well-known that every reduced ring (i.e., rings without nonzero nilpotent elements in R) is an Armendariz ring and every Armendariz ring is McCoy. Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum\limits_{i=0}^{m} a_i x^i \to \sum\limits_{i=0}^{m} \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α. We shall also denote the extended map $R[x] \to R[x]$ by α and the image of $f(x) \in R[x]$ by $\alpha(f(x))$. For basic and other results on McCoy rings, see, e.g., [3, 8, 10, 11].

We consider the McCoy properties related to an endomorphism α of a ring R and call them α-McCoy rings. It is clear that every McCoy ring is an α-McCoy ring, but we shall give an example to show that there exists an α-McCoy ring which is not McCoy. A number of properties of this version are established. It is proved that a ring R is a right α-McCoy ring if and only if $R[x]$ is right α-McCoy. Moreover, we show that a ring R is right α-McCoy if and only if $R[x]/(x^n)$ is right α-McCoy. For a right Ore ring R, if α is an endomorphism of R with $Q(R)$ the classical right quotient ring of R. It is proved that R is right α-McCoy if and only if $Q(R)$ is right α-McCoy. Moreover, a weak form of α-McCoy rings is investigated in the last section. We show that in general weak α-McCoy rings need not be α-McCoy. It is proved that if R is a right weak α-McCoy ring, then the n-by-n upper triangular matrix ring $T_n(R)$ is a right weak α-McCoy ring. And hence some results on McCoy rings are generalized.

2 α-McCoy Rings and Examples

In this section, we relate the problem on the various McCoy properties of a ring R to an endomorphism α of R. We begin with the following definition.

Definition 2.1 An endomorphism α of a ring R is called right (resp., left) McCoy, if for each pair of nonzero polynomials $f(x) = \sum\limits_{i=0}^{m} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$ with $\alpha(f(x))g(x) = 0$ (resp., $f(x)\alpha(g(x)) = 0$), there exists a nonzero element $r \in R$ such that $\alpha(f(x))r = 0$ (resp., $r\alpha(g(x)) = 0$). A ring R is called right (resp., left) α-McCoy if there exists a right (resp., left) McCoy endomorphism α of R. R is an α-McCoy ring if it is both right and left α-McCoy.

It is clear that every right McCoy ring is right α-McCoy. However, we can give the following example to show that there exists a McCoy endomorphism α of a ring S such that S is not a McCoy ring.

Example 2.2 Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z} \right\}.$$

Let $\alpha : S \to S$ be an endomorphism defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. If $f(x) = \sum\limits_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ 0 & c_i \end{pmatrix} x^i$ and $g(x) = \sum\limits_{j=0}^{m} \begin{pmatrix} d_j & e_j \\ 0 & f_j \end{pmatrix} x^j$ are nonzero polynomials in $S[x]$ such that

$\alpha(f(x))g(x) = 0$. Then we have

$$\begin{aligned} \alpha\left(f\left(x\right)\right)g\left(x\right) &= \sum_{k=0}^{m+n}\left(\sum_{i+j=k}\begin{pmatrix} a_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_j & e_j \\ 0 & f_j \end{pmatrix}\right)x^k \\ &= \sum_{k=0}^{m+n}\left(\sum_{i+j=k}\begin{pmatrix} a_id_j & a_ie_j \\ 0 & 0 \end{pmatrix}\right)x^k = 0. \end{aligned}$$

This implies that

$$\sum_{k=0}^{n+m}(\sum_{i+j=k} a_id_j)x^k = 0, \sum_{k=0}^{n+m}(\sum_{i+j=k} a_ie_j)x^k = 0.$$

Let $f_1(x) = \sum\limits_{i=0}^{n} a_ix^i$, $g_1(x) = \sum\limits_{j=0}^{m} d_jx^j$ and $g_2(x) = \sum\limits_{j=0}^{m} e_jx^j$. Then we have $f_1(x)g_1(x) = f_1(x)g_2(x) = 0$. Since every reduced ring is an Armendariz ring, it follows that $a_id_j = a_ie_j = 0$ for each i, j. If $a_i = 0$, then we are done. If $a_i \neq 0$, then we have $d_j = e_j = 0$. Now if we let

$$r = \begin{pmatrix} 0 & 0 \\ 0 & f_j \end{pmatrix}$$

for some $f_j \neq 0$, then $r \neq 0$ and $\alpha(f(x))r = 0$. This shows that the endomorphism α of S is right McCoy. Similarly, we can prove that the endomorphism α of S is left McCoy. But S is neither left nor right McCoy by [10, Theorem 2.1].

According to [1], an endomorphism α of a ring R is called right (resp., left) reversible if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$). A ring R is called right (resp., left) α-reversible if there exists a right (resp., left) reversible endomorphism α of R. R is α-reversible if it is both left and right α-reversible.

Note 2.3 It is well-known that every reversible ring is a McCoy ring. Based on this fact, one may suspect that every left (resp., right) α-reversible ring is McCoy. But this is not true by Example 2.2 and [1, Example 2.2]. In general, we do not know if every α-reversible ring is α-McCoy. In fact, Example 2.2 shows that a right α-reversible ring can be α-McCoy.

The next proposition gives more examples of right α-McCoy rings.

Proposition 2.4 Let R be a ring and α an endomorphism of R. Then R is a right α-McCoy ring if and only if $R[x]$ is a right α-McCoy ring.

Proof Assume that R is a right α-McCoy ring. Let $p(y) = f_0 + f_1y + \cdots + f_my^m$, $q(y) = g_0 + g_1y + \cdots + g_ny^n$ be in $R[x][y]$ with $\alpha(p(y))q(y) = 0$. We also let

$$f_i = a_{i_0} + a_{i_1}x + \cdots + a_{w_i}x^{w_i}, g_j = b_{j_0} + b_{j_1}x + \cdots + b_{v_j}x^{v_j}$$

for each $0 \leq i \leq m$ and $0 \leq j \leq n$, where $a_{i_0}, a_{i_1}, \cdots, a_{w_i}, b_{j_0}, b_{j_1}, \cdots, b_{v_j} \in R$. We claim that $R[x]$ is right α-McCoy. Take a positive integer k such that $k > \max\{\deg(f_i), \deg(g_j)\}$ for any $0 \leq i \leq m$ and $0 \leq j \leq n$, where the degree is as polynomials in $R[x]$ and the degree of zero polynomial is take to be zero. Then

$$p(x^k) = f_0 + f_1x^k + \cdots + f_mx^{mk}, q(x^k) = g_0 + g_1x^k + \cdots + g_nx^{nk} \in R[x],$$

and hence the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of $p(x^k)$ (resp., $q(x^k)$). Since $\alpha(p(y))q(y)=0$, we have $\alpha(p(x^k))q(x^k)=0$. It follows that there exists $0\neq r\in R\subseteq R[x]$ such that $\alpha(p(x^k))r=0$. This implies that $\alpha(p(y))r=0$, and so $R[x]$ is right α-McCoy. Conversely, suppose that $f(y)=\sum\limits_{i=0}^{m}a_iy^i$, $g(y)=\sum\limits_{j=0}^{n}b_jy^j\in R[y]\backslash\{0\}$ such that $\alpha(f(y))g(y)=0$. Since $R[x]$ is right α-McCoy, there exists $0\neq r(x)\in R[x]$ such that $\alpha(f(y))r(x)=0$. This shows that $\alpha(a_i)r(x)=0$ for each i. It follows from $0\neq r(x)$ that there exists $0\neq r_j\in R$ such that $\alpha(a_i)r_j=0$ for each i. Therefore, $\alpha(f(y))r_j=0$ and so R is right α-McCoy.

Corollary 2.5 A ring R is a right McCoy ring if and only if $R[x]$ is right McCoy.

Let R be a ring and $\triangle$ a multiplicative monoid in R consisting of central regular elements, and let $\triangle^{-1}R=\{u^{-1}a|u\in\triangle,a\in R\}$, then $\triangle^{-1}R$ is a ring. For an endomorphism α of R with $\alpha(\Delta)\subseteq\Delta$, the induced map $\bar{\alpha}:\triangle^{-1}R\to\triangle^{-1}R$ defined by $\bar{\alpha}(u^{-1}a)=\alpha(u)^{-1}\alpha(a)$ is also an endomorphism. We have the following result for the right α-McCoy property.

Proposition 2.6 Let R be a ring with an endomorphism α. If R is right α-McCoy, then $\triangle^{-1}R$ is right α-McCoy.

Proof Assume that R is right α-McCoy and let

$$f(x)=\sum_{i=0}^{m}u_i^{-1}a_ix^i,g(x)=\sum_{j=0}^{n}v_j^{-1}b_jx^j\in\triangle^{-1}R[x]$$

with $\alpha(f(x))g(x)=0$. Then we have

$$F(x)=(u_mu_{m-1}\cdots u_0)f(x),G(x)=(v_nv_{n-1}\cdots v_0)g(x)\in R[x].$$

Since R is right α-McCoy and $\alpha(F(x))G(x)=0$, this implies that there exists a nonzero $r\in R$ such that $\alpha(u_mu_{m-1}\cdots u_0u_i^{-1}a_i)r=0$ for all i,j, and so $\alpha(a_i)r=0$ since $\triangle$ is a multiplicative monoid in R consisting of central regular elements and $u_i,v_j\in\triangle$ for all i,j. It follows that $\alpha(u_i^{-1}a_i)r=\alpha(u_i)^{-1}\alpha(a_i)r=0$ for all i,j. This shows that $\triangle^{-1}R$ is right α-McCoy.

The ring of Laurent polynomials in x, with coefficients in a ring R, consists of all formal sum $\sum\limits_{i=k}^{n}m_ix^i$ with obvious addition and multiplication, where $m_i\in R$ and k,n are (possibly negative) integers. We denote this ring by $R[x;x^{-1}]$. For an endomorphism α of a ring R, the map $\bar{\alpha}:R[x;x^{-1}]\to R[x;x^{-1}]$ defined by $\bar{\alpha}(\sum\limits_{i=k}^{n}a_ix^i)=\sum\limits_{i=k}^{n}\alpha(a_i)x^i$ extends α and is also an endomorphism of $R[x;x^{-1}]$.

Corollary 2.7 Let R be a ring. If R is a right α-McCoy ring, then $R[x;x^{-1}]$ is right α-McCoy.

Proof Let $\triangle=\{1,x,x^2,\cdots\}$, then clearly $\triangle$ is a multipicatively closed subset of $R[x]$. Since $R[x;x^{-1}]\cong\triangle^{-1}R[x]$, it follows directly from Proposition 2.6 that $R[x;x^{-1}]$ is right α-McCoy.

According to [2], an endomorphism α of a ring R is called semicommutative if $ab=0$ implies that $aR\alpha(b)=0$ for all $a,b\in R$. A ring R is called α-semicommutative if there exists a semicommutative endomorphism α of R. Recall from [3] that a ring R is said to be right linearly McCoy if given nonzero linear polynomials $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$, there exists a nonzero element $r \in R$ with $f(x)r = 0$. We can define linearly α-McCoy rings similarly. It was proved in [3, Proposition 5.3] that every semicommutative ring is right linearly McCoy. The next example gives an example of right linearly α-McCoy rings which is not α-semicommutative.

Example 2.8 Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers modulo *2*. Then R is a right linearly α-McCoy ring since R is a commutative reduced ring. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. For $(1, 0), (0, 1) \in R$, we have $(1, 0)(0, 1) = 0$ but $(1, 0)(1, 1)\alpha(0, 1) \neq 0$. It follows that R is not α-semicommutative.

Let $A(R, \alpha)$ be the subset $\{x^{-i}rx^i | r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, where $\alpha : R \to R$ is an injective ring endomorphism of a ring R (see [5] for more details). Elements of $R[x, x^{-1}; \alpha]$ are finite sums of elements of the form $x^{-i}rx^i$ where $r \in R$ and i is a non-negative integer. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$. Note that for each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set $A(R, \alpha)$ of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with

$$x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)},$$
$$(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}(\alpha^j(r)\alpha^i(s))x^{(i+j)}$$

for $r, s \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$.

Proposition 2.9 If R is an α-rigid ring, then $A(R, \alpha)$ is right α-McCoy.

Proof It follows directly from the fact that $A(R, \alpha)$ is an α-rigid ring by [4] and that every α-rigid ring is right α-McCoy.

Proposition 2.10 Let R be a ring and α an endomorphism of R. Then R is a right α-McCoy ring if and only if $R[x]/(x^n)$ is a right α-McCoy ring, where (x^n) is the ideal generated by x^n.

Proof Assume that R is right α-McCoy and we denote the element $\bar{x}$ in $R[x]/(x^n)$ by u. Then

$$R[x]/(x^n) = R[u] = R + Ru + \cdots + Ru^{n-1},$$

where u commutes with elements of R and $u^n = 0$. Let $f(y) = \sum_{i=0}^{p} f_i y^i$ and $g(y) = \sum_{j=0}^{q} g_j y^j$ be nonzero polynomials in $R[u][y]$ with $\alpha(f(y))g(y) = 0$, where

$$f_i = \sum_{s=0}^{n-1} a_{is}u^s, \;\; g_j = \sum_{t=0}^{n-1} b_{jt}u^t.$$

Moreover, if we let $k_s(y)=\sum\limits_{i=0}^{p}a_{is}y^i$, $h_t(y)=\sum\limits_{j=0}^{q}b_{jt}y^j$. Then we have

$$\begin{aligned}
0 &= \alpha(f(y))g(y)=(\sum_{i=0}^{p}\alpha(f_i)y^i)(\sum_{j=0}^{q}g_jy^j)\\
&= (\sum_{i=0}^{p}\sum_{s=0}^{n-1}\alpha(a_{is})u^sy^i)(\sum_{j=0}^{q}\sum_{t=0}^{n-1}b_{jt}u^ty^j)\\
&= \sum_{s=0}^{n-1}(\sum_{i=0}^{p}\alpha(a_{is})y^i)\sum_{t=0}^{n-1}(\sum_{j=0}^{q}b_{jt}y^j)u^{s+t}=(\sum_{s=0}^{n-1}\alpha(k_s(y)\sum_{t=0}^{n-1}h_t(y))u^{s+t}.
\end{aligned}$$

It follows that $\sum\limits_{s+t=k}\alpha(k_s(y))h_t(y)=0$, where $k=0,1,\cdots,n-1$. If $\alpha(k_0(y))=0$, take $r=u^{n-1}$. Then we have $0\neq r\in R[u]$, and so

$$\alpha(f(y))r=(\sum_{s=0}^{n-1}(\sum_{i=0}^{p}\alpha(a_{is})y^i)u^s)u^{n-1}=(\sum_{i=0}^{p}\alpha(a_{i0})y^i)u^{n-1}=\alpha(k_0(y))u^{n-1}=0.$$

If $\alpha(k_0(y))\neq 0$, it follows from $g(y)\neq 0$ that there is a minimal $k\in\{0,1,\cdots n-1\}$ such that $h_k(y)\neq 0$ and $\alpha(k_0(y))h_k(y)=0$. Since R is right α-McCoy, there exists a nonzero element $c\in R$ such that $\alpha(k_0(y))c=0$. Let $r'=cu^{n-1}$. Then we have $0\neq r'\in R[u]$ and

$$\alpha(f(y))r'=(\sum_{s=0}^{n-1}(\sum_{i=0}^{p}\alpha(a_{is})y^i)u^s)cu^{n-1}=(\sum_{i=0}^{p}\alpha(a_{i0})y^i)cu^{n-1}=0.$$

Conversely, suppose that

$$f(y)=\sum_{i=0}^{p}a_iy^i,g(y)=\sum_{j=0}^{q}b_jy^j\in R[y]\backslash\{0\}$$

such that $\alpha(f(y))g(y)=0$. Since $f(y)$ and $g(y)$ are nonzero polynomials of $R[x]/(x^n)[y]$ and $R[x]/(x^n)$ is right α-McCoy, it follows that there exists $0\neq r_1(x)=\sum\limits_{k=0}^{n-1}c_kx^k\in R[x]/(x^n)$ such that $\alpha(f(y))r_1(x)=0$. Let $c_{k_0}\neq 0$ with k_0 minimal. Then we obtain $\alpha(f(y))c_{k_0}=0$ and so R is right α-McCoy.

Corollary 2.11 Let R be a ring and n any positive integer. Then R is right McCoy if and only if $R[x]/(x^n)$ is right McCoy.

A ring R is called right Ore if given $a,b\in R$ with b regular, there exist $a_1,b_1\in R$ with b_1 regular such that $ab_1=ba_1$. It is well-known that R is a right Ore ring if and only if the classical right quotient ring $Q(R)$ of R exists. Suppose that the classical right quotient ring $Q(R)$ of R exists. Then for an endomorphism α of R and any $ab^{-1}\in Q(R)$ where $a,b\in R$ with b regular, the induced map $\bar{\alpha}:Q(R)\to Q(R)$ defined by $\bar{\alpha}(ab^{-1})=\alpha(a)\alpha(b)^{-1}$ is also an endomorphism.

Proposition 2.12 Let R be a right Ore ring with $Q(R)$ the classical right quotient ring of R. If α is an endomorphism of R, then R is right α-McCoy if and only if $Q(R)$ is right α-McCoy.

Proof Let $F(x) = \sum\limits_{i=0}^{m} \delta_i x^i, G(x) = \sum\limits_{j=0}^{n} \beta_j x^j$ be nonzero polynomials in $Q[x]$ with $\alpha(F(x))G(x) = 0$. By [6, Proposition 2.1.16], we may assume that $\delta_i = a_i u^{-1}$ and $\beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for each i, j and regular elements $u, v \in R$. Moreover, for each j, there exists $c_j \in R$ and a regular element $w \in R$ such that $\alpha(u)^{-1} b_j = c_j w^{-1}$ also by [6, Proposition 2.1.16]. Let $f(x) = \sum\limits_{i=0}^{m} a_i x^i, g(x) = \sum\limits_{j=0}^{n} c_j x^j$. Then we have

$$\begin{aligned} 0 &= \alpha(F(x))G(x) = (\sum_{i=0}^{m} \alpha(\delta_i)x^i)(\sum_{j=0}^{n} \beta_j x^j) \\ &= \sum_{k=0}^{m+n} (\sum_{i+j=k} \alpha(a_i)c_j(vw)^{-1})x^k = \alpha(f(x))g(x)(vw)^{-1}. \end{aligned}$$

This implies that $\alpha(f(x))g(x) = 0$. Then there exists a nonzero $r \in R$ such that $\alpha(f(x))r = 0$ since R is a right α-McCoy ring. Then $\alpha(a_i)r = 0$ for each i, and hence $\alpha(\delta_i)(\alpha(u)r) = 0$. Now Q being right α-McCoy follows from the fact that $\alpha(F(x))(\alpha(u)r) = 0$ since $\alpha(u)r$ is a nonzero element of Q. On the other hand, note that if

$$m(x) = \sum_{i=0}^{m} a_i x^i, n(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$$

such that $\alpha(m(x))n(x) = 0$. Then there exists a nonzero element γ in Q such that $\alpha(m(x))\gamma = 0$ since Q is right α-McCoy. We may assume $\gamma = d\kappa^{-1}$ with d a nonzero element in R and κ a regular element. So we obtain $\alpha(m(x))d\kappa^{-1} = 0$, and hence $\alpha(m(x))d = 0$. This implies that R is right α-McCoy. This completes the proof.

Corollary 2.13 Let R be a right Ore ring and $Q(R)$ be the classical right quotient ring of R. Then R is right McCoy if and only if $Q(R)$ is right McCoy.

3 Weak α-McCoy Rings and its Properties

Comparing with the definition of a weak McCoy ring, we give the following definition of weak α-McCoy rings accordingly.

Definition 3.1 An endomorphism α of a ring R is called right (resp., left) weak McCoy, if for each pair of nonzero polynomials $f(x) = \sum\limits_{i=0}^{m} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{n} b_j x^j \in R[x]$ with $\alpha(f(x))g(x) = 0$ (resp., $f(x)\alpha(g(x)) = 0$), there exists a nonzero element $r \in R$ such that $\alpha(a_i)r \in nil(R)$ (resp., $r\alpha(b_j) \in nil(R)$). A ring R is called right (resp., left) weak α-McCoy if there exists a right (resp., left) weak McCoy endomorphism α of R. R is a weak α-McCoy ring if it is both right and left α-McCoy.

It is clear that every right α-McCoy ring is right weak α-McCoy. It was shown in [10, Theorem 2.1] that $T_n(R)$ is not McCoy for $n \geq 2$. The following example shows that there exists a weak α-McCoy endomorphism α of a ring S such that S is not an α-McCoy ring.

Example 3.2 Let R be a reduced ring. Consider the ring

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in R \right\}.$$

Let $\alpha : S \to S$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

On the other hand, let

$$f(x) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x, \ \ g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x$$

be elements in $S[x]$. It is straightforward to check that $\alpha(f(x))g(x) = 0$, and we can not find a nonzero element $r \in S$ such that $\alpha(f(x)r = 0$. This implies that S is not an α-McCoy ring. However, S is a right weak α-McCoy ring by the following Proposition 3.3.

Proposition 3.3 Let R be a right weak α-McCoy ring and α an endomorphism of R. Then $T_n(R)$ is a right weak α-McCoy ring.

Proof Let $f(x) = \sum\limits_{i=0}^{m} A_i x^i, g(x) = \sum\limits_{j=0}^{n} B_j x^j$ be nonzero polynomials in $T_n(R)[x]$ with $\alpha(f(x))g(x) = 0$, where $A_i, B_j \in T_n(R)$ for all i, j. If we denote by E_{ij} the usual matrix unit with 1 in the (i, j)-coordinate and zero elsewhere, then for each $\alpha(A_i)$ there exists a nonzero element $C = rE_{1n}$ such that $\alpha(A_i)C \in nil(T_n(R))$, where $0 \neq r \in R$.

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrix $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$, $m \in M$ and the usual matrix operations are used. For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R, $\overline{\alpha}$: $T(R, R) \to T(R, R)$ defined by

$$\overline{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$$

is an endomorphism of $T(R, R)$. Since $T(R, 0)$ is isomorphic to R, we can identify the restriction of $\overline{\alpha}$ by $T(R, 0)$ to α.

Corollary 3.4 If R is right weak α-McCoy, then the trivial extension $T(R, R)$ is a right weak α-McCoy ring.

Based on Proposition 3.3, one may suspect that if R is a weak α-McCoy ring, then the n-by-n full matrix ring $M_n(R)$ is weak α-McCoy with $n \geq 2$. But the following example erases the possibility.

Example 3.5 Let R be a reduced ring. Then R is a weak α-McCoy ring. Put $S = M_n(R)$ and let α be an endomorphism of S defined by

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

We also let

$$f(x) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x, \ \ g(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

be polynomials in $S[x]$. Then we have $\alpha(f(x))g(x) = 0$, and it is easy to check that S is not a weak α-McCoy ring.

Now we consider the case of direct limits of direct systems of right weak α-McCoy rings.

Proposition 3.6 The direct limit of a direct system of right weak α-McCoy rings is also right weak α-McCoy.

Proof Let $D = \{R_i, \phi_{ij}\}$ be a direct system of right weak α-McCoy rings R_i for $i \in I$ and ring homomorphisms $\phi_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\phi_{ij}(1) = 1$, where I is a direct partially ordered set. Let $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \phi_{ij} = \iota_i$. We shall prove that R is a right weak α-McCoy ring. Let α be an endomorphism of R and take $x, y \in R$. It follows that $x = \iota_i(x_i), y = \iota_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Now define $x + y = \iota_k(\phi_{ik}(x_i) + \phi_{jk}(y_j))$ and $xy = \iota_k(\phi_{ik}(x_i)\phi_{jk}(y_j))$, where $\phi_{ik}(x_i)$ and $\phi_{jk}(y_j)$ are in R_k. It is easy to see that R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$. Let $\alpha(f(x))g(x) = 0$ with $f(x) = \sum\limits_{s=0}^{m} a_s x^s$ and $g(x) = \sum\limits_{t=0}^{n} b_t x^t$ in $R[x]$. Then there are $i_s, j_t, k \in I$ such that $\alpha(a_s) = \iota_{i_s}(a_{i_s}), b_t = \iota_{j_t}(b_{j_t}), i_s \leq k, j_t \leq k$. So we have

$$\alpha(a_s)b_t = \iota_k(\phi_{i_s k}(a_{i_s})\phi_{j_t k}(b_{j_t})),$$

and hence

$$\begin{aligned} \alpha(f(x))g(x) &= (\sum_{s=0}^{m} \iota_k(\phi_{i_s k}(a_{i_s}))x^s)(\sum_{t=0}^{n} \iota_k(\phi_{j_t k}(b_{j_t}))x^t) \\ &= \sum_{d=0}^{m+n}(\sum_{s+t=d} \iota_k(\phi_{i_s k}(a_{i_s})\phi_{j_t k}(b_{j_t})))x^d = 0 \end{aligned}$$

in $R_k[x]$ since $\alpha(f(x))g(x) = 0$. On the other hand, since R_k is right weak α-McCoy, there exists $s_k \in R_k \backslash \{0\}$ such that $\iota_k(\phi_{i_s k}(a_{i_s}))s_k \in nil(R_k)$ for all $0 \leq i \leq m$. Let $s = \iota_k(s_k)$. Then we have $\alpha(a_s)s \in nil(R)$ and R is right weak α-McCoy.

Corollary 3.7 The direct limit of a direct system of right weak McCoy rings is right weak McCoy.

Proposition 3.8 Let R be a ring and I an ideal of R such that R/I is right weak α-McCoy. If $I \subseteq nil(R)$, then R is a right weak α-McCoy ring.

Proof Let $f(x)=\sum\limits_{i=0}^{m} a_i x^i$ and $g(x)=\sum\limits_{j=0}^{n} b_j x^j$ be polynomials in $R[x]$ with $\alpha(f(x))g(x)=0$. Then we have $\sum\limits_{i=0}^{m}\alpha(\bar{a}_i)x^i)(\sum\limits_{j=0}^{n}\bar{b}_j x^j)=0$ in R/I. Since R/I is right weak α-McCoy, there exists $n_i\in\mathbb{N}$ and $s\notin I$ such that $(\alpha(\bar{a}_i)\bar{s})^{n_i}=0$. It follows that $(\alpha(a_i)s)^{n_i}\in nil(R)$ since $I\subseteq nil(R)$. This completes the proof.

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关于环自同态的相对McCoy性质

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摘要: 本文引入了α-McCoy环和弱α-McCoy环的概念分别研究了一个环R关于其自同态α的McCoy性质和弱McCoy性质. 利用各种环扩张, 证明了一个环R是α-McCoy环当且仅当$R[x]$是α-McCoy环, 得到了正向系上弱α-McCoy 环的正向极限是弱α-McCoy环, 推广和改进了McCoy环在矩阵环和多项式上的相关结论.

关键词: McCoy 环; α-McCoy 环; 弱α-McCoy 环

MR(2010) **主题分类号**: 16N40; 16S50 **中图分类号**: O153.3