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一维半导体量子能量输运模型稳态解的唯一性

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摘 要: 本文研究了一维半导体稳态量子能量输运模型的古典解. 利用一些不等式技巧证明了当晶格温度充分大且电流密度较小时其解是唯一的, 这在文献 [3] 中没有得到.

关键词: 量子能量输运模型; 稳态解; 唯一性

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1 引言及结果

本文研究一个简化的量子能量输运模型 [1]:

$$n_t + \operatorname{div}\left[\frac{\varepsilon^2}{6} n \nabla \left(\frac{\triangle\sqrt{n}}{\sqrt{n}}\right) - \nabla(nT) + n\nabla V\right] = 0, \tag{1.1}$$

$$-\operatorname{div}(n\nabla T) = \frac{n}{\tau}(T_L(x) - T), \tag{1.2}$$

$$\lambda^2 \triangle V = n - C(x), \tag{1.3}$$

这里电子浓度 n, 电子温度 T 和电位势 V 为未知函数; 带电粒子杂质 $C(x)$ 和晶格温度 $T_L(x)$ 为给定函数; 普朗克常数 $\varepsilon > 0$, 能量松弛时间 $\tau > 0$ 和标度的德拜长度 $\lambda > 0$ 为物理参数. 模型 (1.1)–(1.3) 可以从量子流体动力学方程组中通过取大时间及小速度极限推导出 [1]. 文献 [1] 在周期边界条件下证明了 (1.1)–(1.3) 弱解的整体存在性, 文献 [2] 得到了其解的半古典极限.

最近文献 [3] 研究了 (1.1)–(1.3) 一维稳态模型古典解的存在性, 具体来说, 我们考虑了如下边值问题:

$$\frac{\varepsilon^2}{6} n \left(\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right)_x - (nT)_x + nV_x = J_0, \tag{1.4}$$

$$-(nT_x)_x = \frac{n}{\tau}(T_L(x) - T), \tag{1.5}$$

$$\lambda^2 V_{xx} = n - C(x) \quad \text{in } (0,1), \tag{1.6}$$

$$n(0) = n(1) = 1, \quad n_x(0) = n_x(1) = 0, \quad T(0) = T_0, \quad T_x(0) = T_x(1) = 0, \tag{1.7}$$

$$V(0) = V_0 = -\frac{\varepsilon^2}{6}(\sqrt{n})_{xx}(0) + T_0, \tag{1.8}$$

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其中 J_0 表示电流密度, 得到了如下结果:

定理 1.1 (见文献 [3]) 设 $C(x),\ T_L(x)\in L^\infty(0,1),\ C(x)>0,\ 0<m_L\le T_L(x)\le M_L$, $x\in\ (0,1)$, 则问题 (1.4)–(1.8) 存在古典解 (n,T,V) 使得 $n(x)\ge e^{-M}>0$, $x\in(0,1)$, 其中 M 是

$$M=\sqrt{\frac{e^{2M}}{\tau m_L^2}(M_L-m_L)M_L+\frac{2(e^{-1}+\parallel C(x)\log C(x)\parallel_{L^\infty(0,1)})}{\lambda^2 m_L}} \tag{1.9}$$

的解.

本文继续证明问题 (1.4)–(1.8) 解的唯一性, 我们的主要结果是

定理 1.2 设定理 1.1 中的条件成立, 再若 m_L 充分大且 $|J_0|$ 较小, 则问题 (1.4)–(1.8) 的解是唯一的.

注 1.1 模型 (1.1)–(1.3) 是一种简化的量子能量输运模型, 对于其他类型的量子能量输运模型, 我们可以参考文献 [4–6].

注 1.2 当晶格温度 $T_L(x)$ 和电子温度 T 都为常数时, 则模型 (1.1)–(1.3) 变成量子漂移-扩散模型, 此类模型的研究结果见文献 [7–19].

2 定理 1.2 的证明

文献 [3] 通过指数变换 $n=e^u$ 把问题 (1.4)–(1.8) 变成了如下与之等价的问题

$$\frac{\varepsilon^2}{12}\left(u_{xx}+\frac{u_x^2}{2}\right)_{xx}-T_{xx}-(u_xT)_x+\frac{e^u-C(x)}{\lambda^2}=J_0(e^{-u})_x, \tag{2.1}$$

$$-(e^uT_x)_x=\frac{e^u}{\tau}(T_L(x)-T), \tag{2.2}$$

$$V(x)=-\frac{\varepsilon^2}{12}\left(u_{xx}+\frac{u_x^2}{2}\right)(x)+T(x)+\int_0^x u_x(s)T(s)ds+J_0\int_0^x e^{-u(s)}ds, \tag{2.3}$$

$$u(0)=u(1)=0,\quad u_x(0)=u_x(1)=0,\quad T(0)=T_0,\quad T_x(0)=T_x(1)=0, \tag{2.4}$$

$$V(0)=V_0=-\frac{\varepsilon^2}{12}u_{xx}(0)+T_0, \tag{2.5}$$

并证明了此问题存在解 $(u,T,V)\in H^4(0,1)\times H^2(0,1)\times H^2(0,1)$, 所以为了得到定理 1.2, 只需证明问题 (2.1), (2.2), (2.4) 解的唯一性即可. 为此有如下定理

定理 2.1 设定理 1.1 中的条件成立, 再若 m_L 充分大且 $|J_0|$ 较小, 使得

$$m_L-\frac{\varepsilon M^2}{24}\sqrt{6m_L}-\frac{|J_0|e^M}{\sqrt{2}}-\frac{e^{2M}(\sqrt{2\varepsilon}+\sqrt[4]{6m_L}\cdot M)(\sqrt{2}e^{2M}+1)(M_L-m_L)}{2\sqrt{2\varepsilon}\cdot\tau}>0, \tag{2.6}$$

则问题 (2.1), (2.2), (2.4) 的解 $(u,T)\in H^4(0,1)\times H^2(0,1)$ 是唯一的.

注 2.1 由 (1.9) 式知, 当 m_L 较大时 M 会较小, 从而可以保证当 m_L 充分大且 $|J_0|$ 较小时 (2.6) 式成立.

在文献 [3] 的引理 2.1 中已经得到了如下估计:

$$
\begin{aligned}
&\frac{\varepsilon^2}{12}\parallel u_{xx}\parallel^2_{L^2(0,1)}+\frac{m_L}{2}\parallel u_x\parallel^2_{L^2(0,1)}\\
\le\ &\frac{e^{2M}}{2\tau m_L}(M_L-m_L)M_L+\lambda^{-2}(e^{-1}+\parallel C(x)\log C(x)\parallel_{L^\infty(0,1)}), \qquad (2.7)\\
&\parallel u\parallel_{L^\infty(0,1)}\le M, \qquad (2.8)\\
&0<m_L\le T\le M_L, \qquad (2.9)
\end{aligned}
$$

其中 (2.9) 式见文献 [3] 中引理 2.1 的证明. 为了证明定理 2.1, 还需要如下估计

引理 2.1 设 $(u,T)\in H^4(0,1)\times H^2(0,1)$ 是问题 (2.1), (2.2), (2.4) 的解, 则

$$\parallel u_x\parallel_{L^\infty(0,1)}\le\frac{\sqrt[4]{6m_L}\cdot M}{\sqrt{\varepsilon}}, \qquad (2.10)$$

$$\parallel T_x\parallel_{L^\infty(0,1)}\le\frac{e^{2M}}{\tau}(M_L-m_L). \qquad (2.11)$$

证 由均值不等式及 (2.7) 式, 得

$$
\begin{aligned}
&\frac{\sqrt{m_L}}{\sqrt{6}}\varepsilon\parallel u_{xx}\parallel_{L^2(0,1)}\parallel u_x\parallel_{L^2(0,1)}\le\frac{\varepsilon^2}{12}\parallel u_{xx}\parallel^2_{L^2(0,1)}+\frac{m_L}{2}\parallel u_x\parallel^2_{L^2(0,1)}\\
\le\ &\frac{e^{2M}}{2\tau m_L}(M_L-m_L)M_L+\lambda^{-2}(e^{-1}+\parallel C(x)\log C(x)\parallel_{L^\infty(0,1)}),
\end{aligned}
$$

所以再由 Hölder 不等式及 (1.9) 式, 得

$$
\begin{aligned}
&u_x^2(x)=2\int_0^x u_{xx}(s)u_x(s)ds\le 2\parallel u_{xx}\parallel_{L^2(0,1)}\parallel u_x\parallel_{L^2(0,1)}\\
\le\ &\frac{2\sqrt{6}}{\sqrt{m_L}\varepsilon}\left[\frac{e^{2M}}{2\tau m_L}(M_L-m_L)M_L+\lambda^{-2}(e^{-1}+\parallel C(x)\log C(x)\parallel_{L^\infty(0,1)})\right]\\
=\ &\frac{2\sqrt{6}}{\sqrt{m_L}\varepsilon}\cdot M^2\cdot\frac{m_L}{2}=\frac{\sqrt{6m_L}\cdot M^2}{\varepsilon},
\end{aligned}
$$

从而 (2.10) 式成立.

(2.2) 式两边在 $(0,x)$ 上积分, 得

$$T_x=-\frac{e^{-u}}{\tau}\int_0^x e^u(T_L(x)-T)dx,$$

所以由 (2.8) 及 (2.9) 式知 (2.11) 式成立.

定理 2.1 的证明 设 (u_1,T_1), $(u_2,T_2)\in H^4(0,1)\times H^2(0,1)$ 为问题 (2.1), (2.2), (2.4) 的两个解. 用 T_1-T_2 分别作为

$$-(e^{u_1}T_{1x})_x=\frac{e^{u_1}}{\tau}(T_L(x)-T_1)$$

和

$$-(e^{u_2}T_{2x})_x=\frac{e^{u_2}}{\tau}(T_L(x)-T_2)$$

的试验函数并两式相减, 得

$$\begin{aligned}
&\int_0^1 e^{u_1}(T_1-T_2)_x^2dx = -\int_0^1 T_{2x}(e^{u_1}-e^{u_2})(T_1-T_2)_x dx - \frac{1}{\tau}\int_0^1 e^{u_1}(T_1-T_2)^2dx \\
&+\frac{1}{\tau}\int_0^1 (T_L(x)-T_2)(e^{u_1}-e^{u_2})(T_1-T_2)dx \\
\leq\ & -\int_0^1 T_{2x}(e^{u_1}-e^{u_2})(T_1-T_2)_x dx \\
&+\frac{1}{\tau}\int_0^1 (T_L(x)-T_2)(e^{u_1}-e^{u_2})(T_1-T_2)dx. \qquad (2.12)
\end{aligned}$$

由 (2.8) 式知

$$\int_0^1 e^{u_1}(T_1-T_2)_x^2dx \geq e^{-M}\int_0^1 (T_1-T_2)_x^2dx. \qquad (2.13)$$

由拉格朗日中值定理及 (2.8) 式知

$$|e^{u_1}-e^{u_2}| \leq e^M|u_1-u_2|,$$

所以再由 (2.11) 式, Hölder 不等式及 Poincaré不等式, 得

$$\begin{aligned}
&-\int_0^1 T_{2x}(e^{u_1}-e^{u_2})(T_1-T_2)_x dx \leq \frac{e^{2M}}{\tau}(M_L-m_L)\int_0^1 e^M|u_1-u_2|\cdot|(T_1-T_2)_x|dx \\
\leq\ & \frac{e^{3M}}{\tau}(M_L-m_L)\left[\int_0^1 (u_1-u_2)^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}} \\
\leq\ & \frac{e^{3M}}{\sqrt{2}\tau}(M_L-m_L)\left[\int_0^1 (u_1-u_2)_x^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}}. \qquad (2.14)
\end{aligned}$$

利用 (2.9) 式, 类似上式估计, 可以得到

$$\begin{aligned}
&\frac{1}{\tau}\int_0^1 (T_L(x)-T_2)(e^{u_1}-e^{u_2})(T_1-T_2)dx \\
\leq\ & \frac{e^M}{2\tau}(M_L-m_L)\left[\int_0^1 (u_1-u_2)_x^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}}. \qquad (2.15)
\end{aligned}$$

由 (2.12)–(2.15) 式可得

$$\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}} \leq \frac{e^{2M}(\sqrt{2}e^{2M}+1)(M_L-m_L)}{2\tau}\left[\int_0^1 (u_1-u_2)_x^2dx\right]^{\frac{1}{2}}. \qquad (2.16)$$

用 u_1-u_2 分别作为

$$\frac{\varepsilon^2}{12}\left(u_{1xx}+\frac{u_{1x}^2}{2}\right)_{xx} - T_{1xx} - (u_{1x}T_1)_x + \frac{e^{u_1}-C(x)}{\lambda^2} = J_0(e^{-u_1})_x$$

和

$$\frac{\varepsilon^2}{12}\left(u_{2xx}+\frac{u_{2x}^2}{2}\right)_{xx} - T_{2xx} - (u_{2x}T_2)_x + \frac{e^{u_2}-C(x)}{\lambda^2} = J_0(e^{-u_2})_x$$

的试验函数并两式相减, 得

$$
\begin{aligned}
&\frac{\varepsilon^2}{12}\int_0^1 (u_1-u_2)_{xx}^2dx+\frac{\varepsilon^2}{24}\int_0^1 (u_{1x}^2-u_{2x}^2)(u_1-u_2)_{xx}dx+\int_0^1 (T_1-T_2)_x(u_1-u_2)_xdx\\
&+\int_0^1 T_1(u_1-u_2)_x^2dx+\int_0^1 u_{2x}(T_1-T_2)(u_1-u_2)_xdx+\frac{1}{\lambda^2}\int_0^1 (e^{u_1}-e^{u_2})(u_1-u_2)dx\\
=\ &-J_0\int_0^1 (e^{-u_1}-e^{-u_2})(u_1-u_2)_xdx. \qquad (2.17)
\end{aligned}
$$

由 (2.10) 式及 Young 不等式, 得

$$
\begin{aligned}
&\frac{\varepsilon^2}{24}\int_0^1 (u_{1x}^2-u_{2x}^2)(u_1-u_2)_{xx}dx=\frac{\varepsilon^2}{24}\int_0^1 (u_{1x}+u_{2x})(u_1-u_2)_x(u_1-u_2)_{xx}dx\\
\geq\ &-\frac{\varepsilon^{\frac{3}{2}}\cdot\sqrt[4]{6m_L}\cdot M}{12}\int_0^1 |(u_1-u_2)_x|\cdot|(u_1-u_2)_{xx}|dx\\
\geq\ &-\frac{\varepsilon^2}{24}\int_0^1 (u_1-u_2)_{xx}^2dx-\frac{\varepsilon M^2}{24}\sqrt{6m_L}\int_0^1 (u_1-u_2)_x^2dx. \qquad (2.18)
\end{aligned}
$$

由 Hölder 不等式及 (2.16) 式, 得

$$
\begin{aligned}
&\int_0^1 (T_1-T_2)_x(u_1-u_2)_xdx\geq-\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (u_1-u_2)_x^2dx\right]^{\frac{1}{2}}\\
\geq\ &-\frac{e^{2M}(\sqrt{2}e^{2M}+1)(M_L-m_L)}{2\tau}\int_0^1 (u_1-u_2)_x^2dx. \qquad (2.19)
\end{aligned}
$$

由 (2.9) 式知

$$
\int_0^1 T_1(u_1-u_2)_x^2dx\geq m_L\int_0^1 (u_1-u_2)_x^2dx. \qquad (2.20)
$$

由 (2.10) 式, Hölder 不等式, Poincaré不等式及 (2.16) 式, 得

$$
\begin{aligned}
&\int_0^1 u_{2x}(T_1-T_2)(u_1-u_2)_xdx\geq-\frac{\sqrt[4]{6m_L}\cdot M}{\sqrt{\varepsilon}}\int_0^1 |T_1-T_2|\cdot|(u_1-u_2)_x|dx\\
\geq\ &-\frac{\sqrt[4]{6m_L}\cdot M}{\sqrt{2\varepsilon}}\left[\int_0^1 (T_1-T_2)_x^2dx\right]^{\frac{1}{2}}\left[\int_0^1 (u_1-u_2)_x^2dx\right]^{\frac{1}{2}}\\
\geq\ &-\frac{\sqrt[4]{6m_L}\cdot M\cdot e^{2M}(\sqrt{2}e^{2M}+1)(M_L-m_L)}{2\sqrt{2\varepsilon}\cdot\tau}\int_0^1 (u_1-u_2)_x^2dx. \qquad (2.21)
\end{aligned}
$$

由函数 e^x 的单调递增性可知

$$
\frac{1}{\lambda^2}\int_0^1 (e^{u_1}-e^{u_2})(u_1-u_2)dx\geq 0. \qquad (2.22)
$$

由格朗日中值定理, Hölder 不等式及 Poincaré不等式, 得

$$
\begin{aligned}
&-J_0\int_0^1 (e^{-u_1}-e^{-u_2})(u_1-u_2)_xdx\leq|J_0|e^M\int_0^1 |u_1-u_2|\cdot|(u_1-u_2)_x|dx\\
\leq\ &\frac{|J_0|e^M}{\sqrt{2}}\int_0^1 (u_1-u_2)_x^2dx. \qquad (2.23)
\end{aligned}
$$

由 (2.17)–(2.23) 式可得

$$\frac{\varepsilon^2}{24}\int_0^1 (u_1-u_2)_{xx}^2 dx + C_0\int_0^1 (u_1-u_2)_x^2 dx \leq 0, \tag{2.24}$$

其中

$$C_0 = m_L - \frac{\varepsilon M^2}{24}\sqrt{6m_L} - \frac{|J_0|e^M}{\sqrt{2}} - \frac{e^{2M}(\sqrt{2\varepsilon}+\sqrt[4]{6m_L}\cdot M)(\sqrt{2}e^{2M}+1)(M_L-m_L)}{2\sqrt{2\varepsilon}\cdot\tau}.$$

由 (2.24) 式及条件 (2.6) 式可知 $u_1=u_2$, 再由 (2.16) 式得 $T_1=T_2$.

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UNIQUENESS OF STATIONARY SOLUTIONS TO ONE-DIMENSIONAL QUANTUM ENERGY-TRANSPORT MODEL FOR SEMICONDUCTORS

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Abstract: In this paper, we study the classical solutions to the stationary quantum energy-transport model for semiconductors in one space dimension. The uniqueness of the solutions is proved when the lattice temperature is sufficiently large and the current density is relatively small by using some inequality techniques, which is not obtained in [3].

Keywords: quantum energy-transport model; stationary solutions; uniqueness

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