Vol. 35 (2015)
No. 5

数 学 杂 志
J. of Math. (PRC)

THE RIGIDITY OF TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

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Abstract: In this paper, we investigate totally real submanifolds in a complex projective space. By using moving-frame method and the DDVV inequality, we obtain two rigidity theorems and an integral inequality, improve the related results.

Keywords: complex projective space; pseudo-umbilical submanifolds; sectional curvature

2010 MR Subject Classification: 53C20; 53C40

Document code: A **Article ID:** 0255-7797(2015)05-1139-09

1 Introduction

Let CP^{n+p} be a $2(n+p)$-dimensional complex projective space endowed with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let M^n be an n-dimensional submanifold in CP^{n+p}. M^n is called totally real if each tangent space of M^n is mapped into the normal space by the complex structure J of CP^{n+p}. It plays an important role in geometry of submanifolds to investigate rigidity of totally real submanifolds in complex projective space. Totally real submanifolds in complex projective space were extensively studied and many rigidity theorems were proved, see, for example [1–6], etc.

Recently, Cao [7] and Gu, Xu [8] proved the following rigidity theorems, respectively, by using the DDVV inequality verified by Ge and Tang [9], Lu [10].

Theorem A (see [7]) Let M^n be an n-dimensional oriented closed minimal submanifold in an n-dimensional simply connected and locally symmetric Riemannian manifold N^{n+p}. Suppose the sectional curvature K_N of N satisfies $\delta \leq K_N \leq 1$. If the sectional curvature K_M of M^n satisfies

$$K_M \geq \frac{4}{3n(p+1)}(n-1)^{\frac{1}{2}}(p-1)(p+2)(1-\delta) + (\frac{p+2}{2(p+1)} - \frac{\delta}{p+1})\mathrm{sgn}(p-1),$$

then either M is totally geodesic, or $N^{n+p} = S^{n+p}$ and M is isometric to the standard immersion of the product of two spheres or the Veronese surface in S^4.

* **Received date:** 2013-08-28 **Accepted date:** 2014-03-12

Foundation item: Supported by National Natural Science Foundation of China (11101352); Natural Science Foundation of Anhui Educational Committee (KJ2013Z263); Provincial Natural Science Research Programme Foundation of Anhui's Universities (2014KJ002).

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Theorem B (see [8]) Let M^n be an n-dimensional oriented compact submanifold with parallel mean curvature $H \neq 0$ in $F^{n+p}(c)$. If $c+H^2>0$ and

$$K_M \geq \frac{\mathrm{sgn}(p-2)(p-1)}{2p}(c+H^2),$$

then M is either a totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$ in $F^{n+p}(c)$, the standard immersion of the product of two spheres or the Veronese surface in $S^4(\frac{1}{\sqrt{c+H^2}})$.

In this paper, we study totally real submanifolds in CP^{n+p}, obtain two rigidity theorems and an integral inequality, by using moving-frame method and the DDVV inequality.

Theorem 1.1 Let M^n be an $n(n\geq 2)$-dimensional compact totally real submanifolds with parallel mean curvature vector $\xi \neq 0$ in CP^{n+p} $(p \geq 1)$. If the sectional curvature R_M of M^n satisfies

$$R_M \geq \frac{n+2p-1}{2(n+2p)}(1+H^2),$$

then M^n is a totally umbilical sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, where H is the mean curvature of M^n.

Theorem 1.2 Let M^n be an $n(n \geq 2)$-dimensional complete totally real pseudo-umbilical submanifold in CP^{n+p} $(p \geq 1)$. If $J\xi$ is normal to M^n, then either M^n is totally umbilical or $\inf \rho \leq n(1+H^2)(n-\frac{5}{3})$, where ξ, ρ, H are the mean curvature vector, the scalar curvature, the mean curvature of M^n.

Compared with the result in [5], we do not need the submanifolds to have parallel mean curvature vector condition in Theorem 1.2.

Theorem 1.3 Let M^n be an $n(n \geq 2)$-dimensional compact totally real pseudo-umbilical submanifold in CP^{n+p} $(p>0)$ .If $J\xi$ is tangent to M^n , then

$$\int_{M^n}[2(1+4H^2)nS-3S^2-5n^2H^4-4n^2H^2+2nH^2]dV \leq 0,$$

where S, H are the length square of the second fundamental form, the mean curvature of M^n.

2 Basic Formulas

Let M^n be an $n(n\geq 2)$-dimensional totally real submanifold in CP^{n+p}. Choose a local field of orthonormal frames

$$e_1,\cdots,e_n,e_{n+1},\cdots,e_{n+p},e_{1^*}=Je_1,\cdots,e_{n^*}=Je_n,e_{(n+1)^*}=Je_{n+1},\cdots,e_{(n+p)^*}=Je_{n+p}$$

in CP^{n+p}, in such a way that, restricted to M^n, $e_1,\cdots,e_n$ are tangent to M^n and

$$e_{n+1},\cdots,e_{n+p},e_{1^*},\cdots,e_{n^*},e_{(n+1)^*},\cdots,e_{(n+p)^*}$$

are normal to M^n. We shall make use of the following convention on the range of indices:

$$A,B,C,\cdots=1,\cdots,n+p,1^*,\cdots,n+p^*;$$

$$i,j,k,\cdots=1,\cdots,n;\ \alpha,\beta,\gamma,\cdots=n+1,\cdots,n+p,1^*,\cdots,n+p^*.$$

Let ω^A and ω^A_B be the dual frame field and the connection 1-forms of CP^{n+p}, respectively, then the stucture equations of CP^{n+p} are given by

$$d\omega_A = -\sum_B \omega_{AB}\wedge\omega_B, \omega_{AB}+\omega_{BA}=0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC}\wedge\omega_{CB}+\frac{1}{2}\sum_{C,D}K_{ABCD}\omega_C\wedge\omega_D, \tag{2.2}$$

where

$$K_{ABCD}=\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}+J_{AC}J_{BD}-J_{AD}J_{BC}+2J_{AB}J_{CD}. \tag{2.3}$$

Restricting these forms to M^n, we have

$$\omega_\alpha = 0,\quad \omega_{\alpha i}=\sum_j h^\alpha_{ij}\omega_j,\quad h=\sum_{ij\alpha}h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha,$$

$$h^{i^*}_{jk} = h^{j^*}_{ik}=h^{k^*}_{ij},\quad \xi=\frac{1}{n}\sum_\alpha(\sum_i h^\alpha_{ii})e_\alpha, \tag{2.4}$$

$$R_{ijkl} = K_{ijkl}+\sum_\alpha(h^\alpha_{ik}h^\alpha_{jl}-h^\alpha_{il}h^\alpha_{jk}), \tag{2.5}$$

$$h^\alpha_{ijk}-h^\alpha_{ikj} = -K_{\alpha ijk}, \tag{2.6}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}+\frac{1}{2}\sum_{kl}R_{\alpha\beta kl}\omega_k\wedge\omega_l, \tag{2.7}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl}+\sum_m(h^\alpha_{km}h^\beta_{ml}-h^\alpha_{lm}h^\beta_{km}), \tag{2.8}$$

where h, ξ, R_{ijkl}, $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector,the curvature tensor, the normal curvature tensor of M^n and h^α_{ijk} is the covariant of h^α_{ij}. We define

$$S=|h|^2, H=|\xi|, H_\alpha=(h^\alpha_{ij})_{n\times n}. \tag{2.9}$$

The scalar curvature ρ of M^n is given by

$$\rho=n(n-1)+n^2H^2-S. \tag{2.10}$$

Denoting the first and second covariant derivatives of h^α_{ij} by h^α_{ijk} and h^α_{ijkl}, respectively, we have

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij}-\sum_k h^\alpha_{kj}\omega_{ki}-\sum_k h^\alpha_{ik}\omega_{kj}-\sum_\beta h^\beta_{ij}\omega_{\beta\alpha},$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk}-\sum_l h^\alpha_{ljk}\omega_{li}-\sum_l h^\alpha_{ilk}\omega_{lj}-\sum_l h^\alpha_{ijl}\omega_{lk}+\sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}.$$

Then the Laplacian of h^{α}_{ij} is

$$\triangle h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk} = \sum_k h^{\alpha}_{kkij} + \sum_{km}(h^{\alpha}_{km}R_{mijk} + h^{\alpha}_{mi}R_{mkjk}) - \sum_{\beta k} h^{\beta}_{ki}R_{\alpha\beta jk}. \tag{2.11}$$

Lemma 2.1 (see [9,10]) Let $B_1, \ldots, B_m$ be symmetric $(n\times n)$-matrices, then

$$\sum_{r,s=1}^{m} \|[B_r, B_s]\|^2 \leq (\sum_{r=1}^{m} \|B_r\|^2)^2,$$

where the equality holds if and only if under rotation all B_r's are zero except two matrices which can be written as

$$\tilde{B_r} = P\begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B_s} = P\begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

where P is an orthogonal $(n\times n)$-matrix, $[B_r, B_s] = B_rB_s - B_sB_r$ is the commutator of the matrices B_r, B_s.

Lemma 2.2 (see [11]) Let $A_1, A_2, \cdots, A_m$ $(m \geq 2)$ be symmetric $(n\times n)$-matrices. Then

$$-2\sum_{\alpha\beta=1}^{m}[\mathrm{tr}(A^2_{\alpha}B^2_{\beta}) - \mathrm{tr}(A_{\alpha}A_{\beta})^2] - \sum_{\alpha\beta=1}^{m}[\mathrm{tr}(A_{\alpha}A_{\beta})]^2 \geq -\frac{3}{2}(\sum_{\alpha=1}^{m}\mathrm{tr}(A^2_{\alpha}))^2.$$

3 Proof of Main Theorems

Proof of Theorem 1.1 M^n is a submanifold with parallel mean curvature vector ξ. Choose e_{n+1} such that it is parallel to ξ, and

$$\mathrm{tr}H_{n+1} = nH, \quad \mathrm{tr}H_{\alpha} = 0, \quad \alpha \neq n+1. \tag{3.1}$$

The mean curvature vector ξ is parallel, so we have

$$D^{\perp}\xi = dHe_{n+1} + HD^{\perp}e_{n+1} = dHe_{n+1} + H\sum_{\beta}\omega_{n+1\beta}e_{\beta} = 0. \tag{3.2}$$

(3.2) and (2.7) imply

$$d\omega_{n+1\beta} = -\sum_{\gamma}\omega_{n+1\gamma}\wedge\omega_{\gamma\beta} + \frac{1}{2}\sum_{kl}R_{n+1\beta kl}\omega_k\wedge\omega_l = \frac{1}{2}\sum_{kl}R_{n+1\beta kl}\omega_k\wedge\omega_l = 0. \tag{3.3}$$

From (3.3), we know $R_{n+1\beta kl} = 0$. Set $S_H = \mathrm{tr}H^2_{n+1}$, $\tau = S - \mathrm{tr}H^2_{n+1}$. From (2.11), noting that M^n has parallel mean curvature vector and $\sum\limits_k h^{\alpha}_{kkij} = 0$, one gets

$$\begin{aligned} \frac{1}{2}\triangle S_H &= \sum_{ijk}(h^{n+1}_{ijk})^2 + \sum_{ij}h^{n+1}_{ij}\triangle h^{n+1}_{ij} \\ &= \sum_{ijk}(h^{n+1}_{ijk})^2 + \sum_{ijkm}h^{n+1}_{ij}(h^{n+1}_{km}R_{mijk} + h^{n+1}_{mi}R_{mkjk}). \end{aligned} \tag{3.4}$$

Denote $R_M(p,\pi)$ the sectional curvature of M^n for 2-plane $\pi \subset T_pM$ at point $p \in M^n$. Set $R_{\min}(p) = \min_{\pi\subset T_pM} R_M(p,\pi)$. We choose the orthonormal fields $\{e_i\}$ such that $h_{ij}^{n+1} = \lambda_i\delta_{ij}$, hence, we get

$$\sum_{ijkm} h_{ij}^{n+1}(h_{km}^{n+1}R_{mijk} + h_{mi}^{n+1}R_{mkjk}) = \frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2R_{ijij} \geq \frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2R_{\min}. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\frac{1}{2}\triangle S_H \geq \sum_{ijk}(h_{ijk}^{n+1})^2 + \frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2R_{\min}.$$

It follows from $R_M \geq \frac{n+2p-1}{2(n+2p)}(1+H^2)$ and lemma of Hopf that S_H is a constant, and

$$\frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2R_{\min} = 0. \quad (3.6)$$

(3.6) implies that $\lambda_i = \lambda_j$, $\forall i,j$ and M^n is pseudo-umbilical. From (2.11), nothing that M^n has parallel mean curvature vector and $\sum\limits_k h_{kkij}^\alpha = 0$, one gets

$$\begin{aligned}\frac{1}{2}\triangle\tau &= \sum_{\alpha\neq n+1}\sum_{ijk}(h_{ijk}^\alpha)^2 + \sum_{\alpha\neq n+1}\sum_{ijkm}h_{ij}^\alpha(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk}) \\ &\quad -\sum_{\alpha\neq n+1}\sum_{\beta ijk}h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk}. \end{aligned} \quad (3.7)$$

By using (2.3), (2.5), (3.1) and the fact that M^n is pseudo-umbilical, we can get

$$\begin{aligned}&\sum_{\alpha\neq n+1}\sum_{ijkm}h_{ij}^\alpha(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk}) \\ =\ & n(1+H^2)\tau + \sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\alpha^2H_\beta^2)] - \sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)]^2. \end{aligned} \quad (3.8)$$

Combining (2.3), (2.8) and (3.1), we obtain

$$\sum_{\alpha\neq n+1}\sum_{\beta ijk}h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} = -\sum_i \mathrm{tr}H_{i^*}^2 - \sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\alpha^2H_\beta^2)]. \quad (3.9)$$

Substituting (3.8) and (3.9) into (3.7), for any real number a, we have

$$\begin{aligned}\frac{1}{2}\triangle\tau &= \sum_{\alpha\neq n+1}\sum_{ijk}(h_{ijk}^\alpha)^2 + \sum_i trH_{i^*}^2 - an(1+H^2)\tau \\ &\quad +(1+a)\sum_{\alpha\neq n+1}\sum_{ijkm}h_{ij}^\alpha(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk}) + a\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)]^2 \\ &\quad +(1-a)\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\alpha^2H_\beta^2)]. \end{aligned} \quad (3.10)$$

For fixed α, we choose the orthonormal frame field $\{e_i\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$. From (3.1), we get

$$\begin{aligned}\sum_{ijkm} h^\alpha_{ij}(h^\alpha_{km}R_{mijk} + h^\alpha_{mi}R_{mkjk}) &= \frac{1}{2}\sum_{ij}(\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \geq \frac{1}{2}\sum_{ij}(\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{\min} \\ &= n\mathrm{tr}H^2_\alpha R_{\min}. \qquad (3.11)\end{aligned}$$

(3.11) implies

$$\sum_{\alpha\neq n+1}\sum_{ijkm} h^\alpha_{ij}(h^\alpha_{km}R_{mijk} + h^\alpha_{mi}R_{mkjk}) \geq n\tau R_{\min}. \qquad (3.12)$$

By a direct computation and the DDVV inequality, we obtain

$$\begin{aligned}\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H^2_\alpha H^2_\beta) - \mathrm{tr}(H_\alpha H_\beta)^2] &= \frac{1}{2}\sum_{\alpha\beta\neq n+1}\mathrm{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 \\ &\leq \frac{1}{2}(\sum_{\alpha\neq n+1}\mathrm{tr}H^2_\alpha)^2 = \frac{1}{2}\tau^2. \qquad (3.13)\end{aligned}$$

We also have

$$\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)]^2 \geq \frac{1}{n+2p-1}\tau^2. \qquad (3.14)$$

Taking $a = \frac{n+2p-1}{n+2p+1}$ in (3.10), it follows from (3.12), (3.13) and (3.14) that

$$\frac{1}{2}\triangle\tau \geq [-\frac{n+2p-1}{n+2p+1}(1+H^2) + \frac{2n+4p}{n+2p+1}R_{\min}]n\tau. \qquad (3.15)$$

Hence, if $R_M \geq \frac{n+2p-1}{2(n+2p)}(1+H^2)$, then $\frac{1}{2}\triangle\tau \geq 0$. Thus, by a well-known lemma of Hopf, we have $\frac{1}{2}\triangle\tau = 0$, consequently we have either

(i) $\tau = 0$, or (ii) $R_M = \frac{n+2p-1}{2(n+2p)}(1+H^2)$.

(i) If $\tau = 0$, then M^n is totally umbilical. From (2.3) and (2.5), we obtain

$$R_{ijij} = 1 + H^2,$$

therefore, M^n is a totally umbilical sphere $S^n(\frac{1}{\sqrt{1+H^2}})$.

(ii) If $R_M = \frac{n+2p-1}{2n+4p}(1+H^2)$, then inequality signs in (3.12), (3.13), (3.14) and (3.15) become equalities. Now, we will prove that case (ii) can not occur. The equality of (3.13) implies that either all H_α's are zero or two of the H_α's are nonzero ($\alpha \neq n+1$). When inequality signs in (3.14) and (3.15) become equality, respectively, we get that

$$\mathrm{tr}H^2_\alpha = \mathrm{tr}H^2_\beta \ \ (\alpha, \beta \neq n+1),$$

and $\sum_i \mathrm{tr}H^2_{i^*} = 0$. Hence, we have that all H_α's are zero ($\alpha \neq n+1$). Thus, M^n is totally umbilical, $R_{ijij} = 1 + H^2$. This leads to a contradiction.

Proof of Theorem 1.2 $J\xi$ is normal to M^n. Without loss of generality, we can choose e_{n+1} such that it is parallel to ξ, and

$$\mathrm{tr}H_{n+1}=nH, \mathrm{tr}H_\alpha=0, \alpha\neq n+1. \tag{3.16}$$

From (2.11), we have

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum_{\alpha ijk}(h^\alpha_{ijk})^2+\sum_{\alpha ijk}h^\alpha_{ij}h^\alpha_{kkij}\\ &\quad+\sum_{\alpha ijkm}h^\alpha_{ij}(h^\alpha_{km}R_{mijk}+h^\alpha_{mi}R_{mkjk})-\sum_{\alpha\beta ijk}h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}.\end{aligned} \tag{3.17}$$

Combining (2.3), (2.5), (2.8), (3.16) and the fact that M^n is pseudo-umbilical, we can get

$$\begin{aligned}&\sum_{\alpha ijkm}h^\alpha_{ij}(h^\alpha_{km}R_{mijk}+h^\alpha_{mi}R_{mkjk})\\ &=n(1+H^2)S-n^2H^2+\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)]^2,\end{aligned} \tag{3.18}$$

$$\sum_{\alpha\beta ijk}h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}=-\sum_i\mathrm{tr}H^2_{i^*}-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]. \tag{3.19}$$

Using (3.16) and pseudo-umbilical condition $h^{n+1}_{ij}=H\delta_{ij}$, we have

$$\sum_{\alpha ijk}h^\alpha_{ij}h^\alpha_{kkij}=nH\triangle H, \tag{3.20}$$

$$\sum_{\alpha ijk}(h^\alpha_{ijk})^2\geq\sum_{ik}(h^{n+1}_{iik})^2=n\sum_i(\nabla_iH)^2, \tag{3.21}$$

$$\frac{1}{2}\triangle H^2=H\triangle H+\sum_i(\nabla_iH)^2. \tag{3.22}$$

By Lemma 2.2 and pseudo-umbilical condition $h^{n+1}_{ij}=H\delta_{ij}$, we have

$$\begin{aligned}&2\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)]^2\\ =\ &2\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]-\sum_{\alpha\beta\neq n+1}[\mathrm{tr}(H_\alpha H_\beta)]^2-(\mathrm{tr}H^2_{n+1})^2\\ \geq\ &-\frac{3}{2}\tau^2-n^2H^4=-\frac{3}{2}(S-nH^2)^2-n^2H^4.\end{aligned} \tag{3.23}$$

Substituting (3.18)–(3.23) into (3.17), we have

$$\begin{aligned}\frac{1}{2}\triangle S &\geq \frac{1}{2}n\triangle H^2+n(1+H^2)S-\frac{3}{2}(S-nH^2)^2-n^2H^4-n^2H^2\\ &= \frac{1}{2}n\triangle H^2+(S-nH^2)[n(1+H^2)-\frac{3}{2}(S-nH^2)]\\ &= \frac{1}{2}n\triangle H^2+\tau[n(1+H^2)-\frac{3}{2}\tau].\end{aligned} \tag{3.24}$$

By the same argument as in [5], we conclude that either M^n is totally umbilical or

$$\inf\rho \le n(1+H^2)(n-\frac{5}{3}).$$

Proof of Theorem 1.3 $J\xi$ is tangent to M^n. Without loss of generality, we can choose e_{1^*} such that it is parallel to ξ, and $\mathrm{tr}H_{1^*}=nH, \mathrm{tr}H_\alpha=0, \alpha\neq 1^*$. This, together with (2.3) and (2.8), implies

$$\begin{aligned}\sum_{\alpha\beta ijk} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\alpha\beta jk} &= n^2H^2-\sum_i \mathrm{tr}H_{i^*}^2-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]\\ &\le n^2H^2-\mathrm{tr}H_{1^*}^2-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]\\ &= n^2H^2-nH^2-\sum_{\alpha\beta}[\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\alpha^2H_\beta^2)]. \qquad (3.25)\end{aligned}$$

By the same argument as in Theorem 1.2, we conclude that

$$\frac{1}{2}\triangle S \ge \frac{1}{2}n\triangle H^2+n(1+H^2)S-\frac{3}{2}(S-nH^2)^2-n^2H^4-2n^2H^2+nH^2.$$

As this and M^n is compact, we obtain

$$\int_{M^n}[2(1+4H^2)nS-3S^2-5n^2H^4-4n^2H^2+2nH^2]dV\le 0.$$

Acknowledgments The author would like to thank the referee for his very valuable comments and suggestions to improve this article.

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复射影空间中全实子流形的刚性

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摘 要: 本文研究了复射影空间中的全实子流形. 通过使用活动标架的方法和DDVV不等式, 得到了两个刚性定理和一个积分不等式, 改进了相关的结果.

关 键 词: 复射影空间; 伪脐子流形; 截面曲率

MR(2010)**主 题 分 类 号**: 53C20; 53C40 **中 图 分 类 号**: O186.12