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A MASCHKE-TYPE THEOREM FOR SMASH PRODUCTS AND A MORITA CONTEXT OVER HOPF ALGEBROIDS

WANG Yong

(*College of Mathematics and Information Technology, Nanjing Xiaozhuang University, Nanjing 211171, China*)

Abstract: In this paper, we consider the smash product algebras over Hopf algebroids. By integral theory for Hopf algebroids, we obtain a Maschke-type theorem for smash products and construct a Morita context over Hopf algebroids, which generalizes the corresponding result given by Cohen and Fishman in [1]. As an application, we obtain the Maschke-type Theorem for comodule algebras over Hopf algebroids.

Keywords: Hopf algebroid; Smash product; Maschke-type theorem; Morita context

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1 Introduction

Hopf algebroids, as a generalization of Hopf algebras to non-commutative base algebras, were introduced by Böhm in [3] (see also her joint work with Szlachányi in [4]) and studied further by Böhm in [5]. A Hopf algebroid consists of introducing two compatible bialgebroid structures, called left and right bialgebroids (see [2, 6]), on a given algebra, which are related with the antipode. More precisely, the best known examples of Hopf algebroids are Hopf algebras and weak Hopf algebras (see [7]), and some examples with commutative underlying algebra structure can be found in [8]. A survey of Hopf algebroids and their applications can be found in [9, 10]. As the study of Hopf algebroids has a quite short past, there are many aspects of Hopf algebras that have not yet been investigated how to extend to Hopf algebroids.

As we know, integrals in Hopf algebras are an essential tool in studying Hopf algebras and their action on algebras. Making use of integral theory, Cohen and Fishman presented a Maschke-type theorem and constructed a Morita context connecting $A\#H$ and A^H: $[A^H, {}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, A\#H]$ (see [1]). For further research and some applications, we refer to [11–15].

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Biography: Wang Yong(1982–), male, born at Xuzhou, Jiangsu, lecturer, major in Hopf algebras.

The purpose of the present paper is to investigate the above results in the case of Hopf algebroids.

This paper is organized as follows. In Section 1, we recall basic definitions and give a summary of the fundamental properties concerning Hopf algebroids. In Section 2, using integral theory for Hopf algebroids (see [5]), we mainly investigate a Maschke-type theorem for $A\#H$ over Hopf algebroids, which generalizes the corresponding result given by Cohen and Fishman in [1]. As an application, we obtain the Maschke-type theorem for comodule algebras over Hopf algebroids, which extends the corresponding result given by Năstăsescu et al. in [25]. In Section 3, we mainly claim that $A\#H$ and $A^{\mathcal{H}_L}$ the invariant subalgebra of $\mathcal{H}_L$ on A are connected via a Morita context over Hopf algebroids, using A as the connecting module.

In what follows, we recall some concepts and results used in this paper.

Throughout the paper, we always work over a commutative ring k and follow [16, 17] for terminologies on algebras, coalgebras, rings and corings. By an algebra R we mean an associative unital k-algebra. We denote by ${}_R\mathcal{M}, \mathcal{M}_R$ and ${}_R\mathcal{M}_R$ the categories of left, right and bimodules for R, respectively.

For an algebra R over k, an R-ring is a triple (A, μ, η). Here A is an R-bimodule, $\mu : A \otimes_R A \to A$ and $\eta : R \to A$ are R-bimodule maps, satisfying the associativity and unit conditions

$$\mu(\mu \otimes_R id) = \mu(id \otimes_R \mu), \quad \mu(\eta \otimes_R id) = id = \mu(id \otimes_R \eta).$$

An R-ring A is equivalent to a k-algebra A and a k-algebra map $\eta : R \to A$ (see [18]).

For an algebra R over k, an R-coring introduced in [19] is a triple (C, Δ, ε). Here C is an R-bimodule, $\Delta : C \to C \otimes_R C$ and $\varepsilon : C \to R$ are R-bimodule maps, satisfying the coassociativity and counit conditions

$$(\Delta \otimes_R id)\Delta = (id \otimes_R \Delta)\Delta, \quad (\varepsilon \otimes_R id)\Delta = id = (id \otimes_R \varepsilon)\Delta.$$

A left module for an R-ring (A, μ, η) is a pair (M, φ), where M is a left R-module and $\varphi : A \otimes_R M \to M$ is a morphism in ${}_R\mathcal{M}$, such that

$$\varphi(\mu \otimes_R id) = \varphi(id \otimes_R \varphi), \quad \varphi(\eta \otimes_R id) = id.$$

For an R-ring A, a left A-module morphism $f : M \to N$ is a left R-module map $f : M \to N$, satisfying $f\varphi_M = \varphi_N(id \otimes_R f)$. The category of left A-modules is denoted by ${}_A\mathcal{M}$. The category $\mathcal{M}_A$ of right A-modules is defined symmetrically (see [8]).

Note that a k-module M is a (left or right) module of an R-ring (A, μ, η) if and only if it is a (left or right) module of the corresponding k-algebra A. Furthermore, a k-module map $f : M \to N$ is a morphism of (left or right) modules of an R-ring (A, μ, η) if and only if it is a morphism of (left or right) modules of the corresponding k-algebra A (see [8]).

Definition 1.1 [2] A left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$ consists of two algebras H and L over k, which are called the total and base algebras, respectively. H is an $L \otimes_k L^{op}$-ring via the algebra homomorphisms $s_L : L \to H$ and $t_L : L^{op} \to H$, called the source and target maps, respectively (this means that the ranges of s_L and t_L are commuting subalgebras in H). In terms of s_L and t_L, we equip H with an L-L-bimodule structure as

$$l \cdot h \cdot l' = s_L(l)t_L(l')h, \quad l, l' \in L, h \in H. \tag{1.1}$$

The triple $(H, \Delta_L, \varepsilon_L)$ is an L-coring. Introducing Sweedler's convention $\Delta_L(h) = \Sigma h_{(1)} \otimes_L h_{(2)}$ for $h \in H$, the axioms

$$\Sigma h_{(1)}t_L(l) \otimes_L h_{(2)} = \Sigma h_{(1)} \otimes_L h_{(2)}s_L(l), \tag{1.2}$$
$$\Delta_L(1_H) = 1_H \otimes_L 1_H, \tag{1.3}$$
$$\Delta_L(hg) = \Delta_L(h)\Delta_L(g), \tag{1.4}$$
$$\varepsilon_L(1_H) = 1_L, \tag{1.5}$$
$$\varepsilon_L(hs_L\varepsilon_L(g)) = \varepsilon_L(hg) = \varepsilon_L(ht_L\varepsilon_L(g)) \tag{1.6}$$

are required for any $l \in L, h, g \in H$.

Symmetrically, a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \Delta_R, \varepsilon_R)$ consists of two algebras H and R over commutative ring k, which are called the total and base algebras, respectively. H is an $R \otimes_k R^{op}$-ring via the algebra homomorphisms $s_R : R \to H$ and $t_R : R^{op} \to H$, called the source and target maps, respectively (this means that the ranges of s_R and t_R are commuting subalgebras in H). In terms of s_R and t_R, we equip H with an R-R-bimodule structure as

$$r \cdot h \cdot r' = hs_R(r')t_R(r), \quad r, r' \in R, h \in H. \tag{1.7}$$

The triple $(H, \Delta_R, \varepsilon_R)$ is an R-coring. Introducing Sweedler's convention $\Delta_R(h) = \Sigma h^{(1)} \otimes_R h^{(2)}$ for $h \in H$, the axioms

$$\Sigma s_R(r)h^{(1)} \otimes_R h^{(2)} = \Sigma h^{(1)} \otimes_R t_R(r)h^{(2)}, \tag{1.8}$$
$$\Delta_R(1_H) = 1_H \otimes_R 1_H, \tag{1.9}$$
$$\Delta_R(hg) = \Delta_R(h)\Delta_R(g), \tag{1.10}$$
$$\varepsilon_R(1_H) = 1_R, \tag{1.11}$$
$$\varepsilon_R(s_R\varepsilon_R(h)g) = \varepsilon_R(hg) = \varepsilon_R(t_R\varepsilon_R(h)g) \tag{1.12}$$

are required for any $r \in R, h, g \in H$.

Definition 1.2 [5] A Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ consists of a left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$, a right bialgebroid $\mathcal{H}_R = (H, R, s_R, t_R, \Delta_R, \varepsilon_R)$ on the same total algebra H, and a k-module map $S : H \to H$, called the antipode, such that the following axioms hold:

$$s_L\varepsilon_L t_R = t_R, \quad t_L\varepsilon_L s_R = s_R, \quad s_R\varepsilon_R t_L = t_L, \quad t_R\varepsilon_R s_L = s_L, \tag{1.13}$$
$$(\Delta_L \otimes_R id)\Delta_R = (id \otimes_L \Delta_R)\Delta_L, \quad (\Delta_R \otimes_L id)\Delta_L = (id \otimes_R \Delta_L)\Delta_R, \tag{1.14}$$
$$S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l), \tag{1.15}$$
$$\Sigma S(h_{(1)})h_{(2)} = s_R\varepsilon_R(h), \quad \Sigma h^{(1)}S(h^{(2)}) = s_L\varepsilon_L(h), \tag{1.16}$$

for any $l \in L, r \in R, h \in H$.

By [20], (1.13) implies that Δ_L is R-R-bilinear and Δ_R is L-L-bilinear, that is

$$\Delta_L(ht_R(r)) = \Sigma h_{(1)}t_R(r) \otimes_L h_{(2)}, \quad \Delta_L(hs_R(r)) = \Sigma h_{(1)} \otimes_L h_{(2)}s_R(r), \tag{1.17}$$

$$\Delta_R(s_L(l)h) = \Sigma s_L(l)h^{(1)} \otimes_R h^{(2)}, \quad \Delta_R(t_L(l)h) = \Sigma h^{(1)} \otimes_R t_L(l)h^{(2)} \tag{1.18}$$

for any $l \in L, r \in R, h \in H$.

Similarly to the case of Hopf algebras, by [5], the antipode S of a Hopf algebroid $\mathcal{H}$ is an anti-algebra map on the total algebra H. That is, for any $h, g \in H$,

$$S(1_H) = 1_H, \quad S(hg) = S(g)S(h). \tag{1.19}$$

Moreover, the antipode S is an anti-coring map $\mathcal{H}_L \to \mathcal{H}_R$ and $\mathcal{H}_R \to \mathcal{H}_L$. That is, for any $h, g \in H$,

$$\varepsilon_R S = \varepsilon_R s_L \varepsilon_L, \quad \Sigma S(h)^{(1)} \otimes_R S(h)^{(2)} = \Sigma S(h_{(2)}) \otimes_R S(h_{(1)}), \tag{1.20}$$

$$\varepsilon_L S = \varepsilon_L s_R \varepsilon_R, \quad \Sigma S(h)_{(1)} \otimes_L S(h)_{(2)} = \Sigma S(h^{(2)}) \otimes_L S(h^{(1)}). \tag{1.21}$$

Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S. By [4, 8], S^{-1} is also both an anti-algebra map on the total algebra H and an anti-coring map $\mathcal{H}_L \to \mathcal{H}_R$ and $\mathcal{H}_R \to \mathcal{H}_L$. Moreover, the following identities hold.

For any $h \in H$,

$$\Sigma S^{-1}(h_{(2)})h_{(1)} = t_R \varepsilon_R(h), \quad \Sigma h^{(2)} S^{-1}(h^{(1)}) = t_L \varepsilon_L(h). \tag{1.22}$$

$$t_R = S^{-1}s_R, \quad t_L \varepsilon_L t_R = S^{-1}t_R, \quad t_L = S^{-1}s_L, \quad t_R \varepsilon_R t_L = S^{-1}t_L. \tag{1.23}$$

$$\varepsilon_L t_R \varepsilon_R = \varepsilon_L S^{-1}, \quad \varepsilon_R t_L \varepsilon_L = \varepsilon_R S^{-1}. \tag{1.24}$$

Definition 1.3 [21] Let $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$ be a left bialgebroid. A left module of $\mathcal{H}_L$ means a left module of the $L \otimes_k L^{op}$-ring (H, μ, η).

By [21, 22], a left $\mathcal{H}_L$-module morphism means a morphism of left modules of the $L \otimes_k L^{op}$-ring (H, μ, η). The left $\mathcal{H}_L$-module category ${}_{\mathcal{H}_L}\mathcal{M}$ has objects the left $\mathcal{H}_L$-modules and arrows the left $\mathcal{H}_L$-module maps. The category $\mathcal{M}_{\mathcal{H}_R}$ of right $\mathcal{H}_R$-modules is defined symmetrically.

Note that, for a left bialgebroid $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$, H is an $L \otimes_k L^{op}$-ring with unit $s_L \otimes_k t_L : L \otimes_k L^{op} \to H$. This endows ${}_{\mathcal{H}_L}\mathcal{M}$ with an additional piece of structure, that is, a forgetful functor $F : {}_{\mathcal{H}_L}\mathcal{M} \to {}_{L \otimes_k L^{op}}\mathcal{M} \cong {}_L\mathcal{M}_L$: a left $\mathcal{H}_L$-module M carries an underlying L-L-bimodule structure by $l \cdot m \cdot l' = s_L(l)t_L(l') \cdot m$ for any $l, l' \in L, m \in M$.

For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, a left $\mathcal{H}$-module is just a left $\mathcal{H}_L$-module and a right $\mathcal{H}$-module is just a right $\mathcal{H}_R$-module.

Definition 1.4 [2] Let $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$ be a left bialgebroid. A is called a left $\mathcal{H}_L$-module algebra if A is a left H-module as well as an L-ring, where A is viewed as an L-L-bimodule via

$$l \cdot a \cdot l' = (l \cdot 1_H \cdot l') \triangleright a = s_L(l)t_L(l') \triangleright a, \tag{1.26}$$

such that

$$h \triangleright (ab) = \Sigma(h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \tag{1.27}$$

$$h \triangleright 1_A = s_L\varepsilon_L(h) \triangleright 1_A \equiv t_L\varepsilon_L(h) \triangleright 1_A. \tag{1.28}$$

Note that the unit of L-ring A is the map $\eta_A : L \to A, l \mapsto l \cdot 1_A \equiv 1_A \cdot l$.

Definition 1.5 [2] Let $\mathcal{H}_L = (H, L, s_L, t_L, \Delta_L, \varepsilon_L)$ be a left bialgebroid and A a left $\mathcal{H}_L$-module algebra. The smash product algebra $A\#H$ is defined as the k-module $A \otimes_L H$ with product

$$(a\#h)(b\#g) = \Sigma a(h_{(1)} \triangleright b)\#h_{(2)}g, \tag{1.29}$$

here A is a right L-module via

$$a \cdot l = t_L(l) \triangleright a = a(s_L(l) \triangleright 1_A), \tag{1.30}$$

and H is a left L-module as in (1.1).

2 A Maschke-Type Theorem for Smash Products

In this section, we assume that $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a Hopf algebroid and A is a left $\mathcal{H}_L$-module algebra via the action "$\triangleright$". We mainly present a Maschke-type theorem for smash products over Hopf algebroids.

Recall that the left (or right) integrals in a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ are the left (or right) integrals in $\mathcal{H}_L$ (or $\mathcal{H}_R$), that is, the elements of

$$\mathcal{L}(\mathcal{H}) = \{x \in H | hx = s_L\varepsilon_L(h)x, h \in H\},$$
$$(\mathcal{R}(\mathcal{H}) = \{y \in H | yh = ys_R\varepsilon_R(h), h \in H\}).$$

By [5], a left integral $x \in \mathcal{L}(\mathcal{H})$ is normalized if $\varepsilon_L(x) = 1_L$, and, similarly, a right integral $y \in \mathcal{R}(\mathcal{H})$ is normalized if $\varepsilon_R(y) = 1_R$. And we have $S(\mathcal{L}(\mathcal{H})) \subseteq \mathcal{R}(\mathcal{H}), S(\mathcal{R}(\mathcal{H})) \subseteq \mathcal{L}(\mathcal{H})$.

Lemma 2.1 [5] Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid. The following properties of an element $y \in H$ are also equivalent.

(1) $y \in \mathcal{R}(\mathcal{H})$.

(2) $\Sigma y_{(1)} \otimes_L y_{(2)}S(h) = \Sigma y_{(1)}h \otimes_L y_{(2)}$ for any $h \in H$.

(3) $\Sigma S(y_{(1)}) \otimes_L y_{(2)}h = \Sigma hS(y_{(1)}) \otimes_L y_{(2)}$ for any $h \in H$.

Lemma 2.2 [5] Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid. The following assertions are equivalent.

(1) The L-ring (H, s_L) underlying $\mathcal{H}_L$ is left semisimple.

(2) The R-ring (H, s_R) underlying $\mathcal{H}_R$ is right semisimple.

(3) There exists a normalized left integral in $\mathcal{H}_L$.

(4) There exists a normalized right integral in $\mathcal{H}_R$.

Lemma 2.3 [23] Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid. For any $a, b \in A, h, g \in H$, we have

(1) $(1_A\#h)(1_A\#g) = 1_A\#hg$;

(2) $(1_A\#h)(a\#1_H)=\Sigma h_{(1)}\triangleright a\#h_{(2)}$;

(3) $(a\#1_H)(1_A\#h)=a\#h$;

(4) $(a\#1_H)(b\#1_H)=ab\#1_H$.

Proposition 2.4 Let $\mathcal{H}=(\mathcal{H}_L,\mathcal{H}_R,S)$ be a Hopf algebroid. Assume that the L-ring (H,s_L) underlying $\mathcal{H}_L$ is left semisimple. Consider V as a left $A\#H$-module and W as an $A\#H$-submodule of V. If W is an A-direct summand of V, then W is an $A\#H$-direct summand of V.

Proof Suppose that $\lambda:V\to W$ is a left A-module projection, and e is a normalized right integral in $\mathcal{H}_R$. Define a map

$$\widetilde{\lambda}:V\to W,v\mapsto\Sigma(1_A\#S(e_{(1)}))\cdot\lambda((1_A\#e_{(2)})\cdot v).$$

By Theorem 3.7 in [23], we know that $\Sigma(1_A\#S(e_{(1)}))\otimes_A(1_A\#e_{(2)})$ is a separable idempotent of $A\#H$. Hence it is easy to show that $\widetilde{\lambda}$ is both left $A\#H$-linear and a projection.

The proof is completed.

By the above conclusions, we obtain the Maschke-type theorem for smash products over Hopf algebroids.

Theorem 2.5 Let $\mathcal{H}=(\mathcal{H}_L,\mathcal{H}_R,S)$ be a Hopf algebroid. Assume that the L-ring (H,s_L) underlying $\mathcal{H}_L$ is left semisimple.

(1) Let V be an $A\#H$-module. If V is completely reducible as an A-module, then V is completely reducible as an $A\#H$-module.

(2) If the L-ring (A,η_A) is semisimple Artinian, then the L-ring $(A\#H,1_A\#s_L)$ is also semisimple Artinian.

Proof (1) is immediate from Proposition 2.4.

(2) follows from (1), using the fact that a ring is semisimple if and only if every module is completely reducible.

Remark (1) Let $\mathcal{H}=(\mathcal{H}_L,\mathcal{H}_R,S)$ be a Hopf algebroid and $U=s_L(L)s_R(R)$. Then, by (1.13), (1.15) and (1.18), we can deduce that $\mathcal{U}_L=(U,L,s_L,t_L,\Delta_L,\varepsilon_L)$ has a structure of a left bialgebroid, $\mathcal{U}_R=(U,R,s_R,t_R,\Delta_R,\varepsilon_R)$ has a structure of a right bialgebroid and $\mathcal{U}=(\mathcal{U}_L,\mathcal{U}_R,S)$ has a structure of a Hopf algebroid, that is, $\mathcal{U}$ is a sub-Hopf algebroid of $\mathcal{H}$. And it is obvious that $s_L(L)$ is a left $\mathcal{U}_L$-module algebra via the action $u\cdot s_L(l)=s_L\varepsilon_L(us_L(l))$. Assume that the L-ring (U,s_L) underlying $\mathcal{U}_L$ is left semisimple. Hence, by Theorem 2.5, L-ring $(s_L(L)\#U,1_H\#s_L)$ is also semisimple. And for any $s_L(L)\#U$-module V, if V is completely reducible as a $s_L(L)$-module, then V is completely reducible as a $s_L(L)\#U$-module.

(2) Let (H,Δ,ε,S) be a weak Hopf algebra with bijective antipode S (see [7]). Define the maps $\sqcap^L,\sqcap^R:H\longrightarrow H$ by the formulas

$$\sqcap^L(h)=\varepsilon(1_1h)1_2;\sqcap^R(h)=1_1\varepsilon(h1_2),$$

where $\Delta(1_H)=1_1\otimes 1_2$. Denote by H^L the image $\sqcap^L(H)$ and H^R the image $\sqcap^R(H)$ (see [7]). By [8], we know that $\mathcal{H}_L=(H,H^L,id,S^{-1}|_{H^L},p_L\circ\Delta,\sqcap^L)$ has a structure of a left bialgebroid, $\mathcal{H}_R = (H, H^R, id, S^{-1}|_{H^R}, p_R \circ \Delta, \sqcap^R)$ has a structure of a right bialgebroid and $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ has a structure of a Hopf algebroid, where p_L and p_R are the canonical projections $p_L : H \otimes_k H \to H \otimes_{H^L} H$ and $p_R : H \otimes_k H \to H \otimes_{H^R} H$, respectively.

Let A be a left H-module algebra, then it is easy to show that A is also a left $\mathcal{H}_L$-module algebra with an H^L-H^L-bimodule structure $x \cdot h \cdot x' = S^{-1}(x')xh$ for any $h \in H, x, x' \in H^L$. Assume that H is semisimple, then it is obvious that H^L-ring (H, id) is also semisimple underlying $\mathcal{H}_L$. Therefore, by Theorem 2.5, we obtain the Maschke-type theorem for smash products over weak Hopf algebras, which was given by Zhang in [13].

(a) Let V be an $A\#H$-module. If V is completely reducible as an A-module, then V is completely reducible as an $A\#H$-module.

(b) If A is semisimple Artinian, then $A\#H$ is also semisimple Artinian.

(3) Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid, B a right $\mathcal{H}$-comodule algebra with $\mathcal{H}_R$-coinvariant subalgebra $B^{co\mathcal{H}_R} = \{b \in B | \rho_R(b) = b \otimes_R 1\}$ (see [20]). Assume that the L-ring (H, s_L) underlying $\mathcal{H}_L$ is left semisimple and there exists a right $\mathcal{H}$-comodule algebra map $\phi : H \to B$. By Lemma 3.1 in [24], we know that $B^{co\mathcal{H}_R}$ is a left $\mathcal{H}_L$-module algebra via $h \cdot b = h^{(1)} b \phi(S(h^{(2)}))$ for any $b \in B^{co\mathcal{H}_R}$. By Theorem 3.3 in [24], $B^{co\mathcal{H}_R}\#H \cong B$ as right $\mathcal{H}$-comodule algebras, hence, according to Theorem 2.5, we obtain the Maschke-type theorem for comodule algebras over Hopf algebroids, which generalizes the corresponding result given by Năstăsescu et al. in [25].

(a) For a B-module V, if V is completely reducible as a $B^{co\mathcal{H}_R}$-module, then V is completely reducible as a B-module.

(b) If $B^{co\mathcal{H}_R}$ is semisimple Artinian, then B is also semisimple Artinian.

3 A Morita Context Connecting $A\#H$ and $A^{\mathcal{H}_L}$

In this section, we assume that $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a Hopf algebroid, A is a left $\mathcal{H}_L$-module algebra, and

$$\begin{aligned} A^{\mathcal{H}_L} &= \{a \in A | h \triangleright a = s_L \varepsilon_L(h) \triangleright a, h \in H\} \\ &= \{a \in A | h \triangleright a = t_L \varepsilon_L(h) \triangleright a, h \in H\}, \end{aligned} \tag{3.1}$$

the invariant subalgebra of $\mathcal{H}_L$ on A (see [2]). Compared with the corresponding result in [23], we mainly construct a Morita context connecting $A\#H$ and $A^{\mathcal{H}_L}$ under a different condition.

By [2], we know that for any $a \in A, b \in A^{\mathcal{H}_L}, h \in H$,

$$h \triangleright (ab) = (h \triangleright a)b, \quad h \triangleright (ba) = b(h \triangleright a). \tag{3.2}$$

Lemma 3.1 Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S. Then A is a left and right $A\#H$-module via

$$(a\#h) \rightharpoonup b = a(h \triangleright b), \; b \leftharpoonup (a\#h) = S^{-1}(h) \triangleright (ba) \tag{3.3}$$

for any $a\#h \in A\#H, b \in A$.

Proof By (2.4) in [23], A is a left $A\#H$-module via "$\rightharpoonup$". In what follows, we check that A is a right $A\#H$-module via the action "$\leftharpoonup$".

For any $a, b \in A, l \in L, h \in H$,

$$\begin{aligned} b \leftharpoonup (a\#l \cdot h) &= S^{-1}(s_L(l)h) \triangleright (ba) = S^{-1}(h)t_L(l) \triangleright (ba) \\ &= S^{-1}(h) \triangleright (b(t_L(l) \triangleright a)) = S^{-1}(h) \triangleright (b(a \cdot l)) \\ &= b \leftharpoonup (a \cdot l\#h), \end{aligned}$$

where the second equality follows by (1.23) and the fact that S^{-1} is an anti-algebra map on the total algebra H, and the third one by (1.27) and the fact that Δ_L is right L-linear. Hence the action "$\leftharpoonup$" is well defined.

We now compute for any $a\#h, b\#g \in A\#H, c \in A$,

$$\begin{aligned} c \leftharpoonup [(a\#h)(b\#g)] &= \Sigma c \leftharpoonup (a(h_{(1)} \triangleright b)\#h_{(2)}g) = \Sigma S^{-1}(h_{(2)}g) \triangleright [ca(h_{(1)} \triangleright b)] \\ &= \Sigma S^{-1}(g)S^{-1}(h_{(2)}) \triangleright [ca(h_{(1)} \triangleright b)] \\ &= \Sigma S^{-1}(g) \triangleright (S^{-1}(h_{(2)})_{(1)} \triangleright (ca))(S^{-1}(h_{(2)})_{(2)}h_{(1)} \triangleright b) \\ &= \Sigma S^{-1}(g) \triangleright (S^{-1}(h_{(2)}{}^{(2)}) \triangleright (ca))(S^{-1}(h_{(2)}{}^{(1)})h_{(1)} \triangleright b) \\ &\overset{(1.14)}{=} \Sigma S^{-1}(g) \triangleright (S^{-1}(h^{(2)}) \triangleright (ca))(S^{-1}(h^{(1)}{}_{(2)})h^{(1)}{}_{(1)} \triangleright b) \\ &\overset{(1.22)}{=} \Sigma S^{-1}(g) \triangleright (S^{-1}(h^{(2)}) \triangleright (ca))(t_R\varepsilon_R(h^{(1)}) \triangleright b) \\ &= \Sigma S^{-1}(g) \triangleright (1_{(1)}S^{-1}(h^{(2)}) \triangleright (ca))(1_{(2)}t_R\varepsilon_R(h^{(1)}) \triangleright b) \\ &\overset{(1.13)}{=} \Sigma S^{-1}(g) \triangleright (1_{(1)}S^{-1}(h^{(2)}) \triangleright (ca))(1_{(2)}s_L\varepsilon_L t_R\varepsilon_R(h^{(1)}) \triangleright b) \\ &\overset{(1.2)}{=} \Sigma S^{-1}(g) \triangleright (1_{(1)}t_L\varepsilon_L t_R\varepsilon_R(h^{(1)})S^{-1}(h^{(2)}) \triangleright (ca))(1_{(2)} \triangleright b) \\ &= \Sigma S^{-1}(g) \triangleright [(S^{-1}(h^{(2)}t_R\varepsilon_R(h^{(1)})) \triangleright (ca))b] \\ &= S^{-1}(g) \triangleright [(S^{-1}(h) \triangleright (ca))b] = (c \leftharpoonup (a\#h)) \leftharpoonup (b\#g), \end{aligned}$$

where the fifth equality follows by the fact that S^{-1} is an anti-coring map, and the tenth one by (1.23) and the fact that S^{-1} is an anti-algebra map on the total algebra H. Hence A is a right $A\#H$-module.

The proof is completed.

Remark The action "$\rightharpoonup$" determines a right $A^{\mathcal{H}_L}$-module map

$$\pi : A\#H \to End_{A^{\mathcal{H}_L}}(A), \pi(a\#h)(b) = (a\#h) \rightharpoonup b,$$

where A is a right $A^{\mathcal{H}_L}$-module via its multiplication. Moreover π is an algebra map.

If the antipode S is bijective, then the action "$\leftharpoonup$" determines a left $A^{\mathcal{H}_L}$-module map

$$\pi' : A\#H \to {}_{A^{\mathcal{H}_L}}End(A), \pi'(a\#h)(b) = b \leftharpoonup (a\#h).$$

Further π' is an anti-algebra map.

Corollary 3.2 Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid. Consider A as a left and right $A\#H$-module as in (3.3). Then

(1) $A^{\mathcal{H}_L} \cong {}_{A\#H}End(A)^{op}$ as algebras.

(2) If the antipode S is bijective, then $A^{\mathcal{H}_L} \cong End_{A\#H}(A)$ as algebras.

Proof (2) Let $f_a : A \to A$ denote left multiplication by a, that is, $f_a(a') = aa'$ for any $a, a' \in A$. Define a map

$$\phi : A^{\mathcal{H}_L} \to End_{A\#H}(A), b \mapsto f_b.$$

ϕ is well-defined, since for any $b \in A^{\mathcal{H}_L}, a \in A, c\#h \in A\#H$,

$$\begin{aligned} f_b(a \leftharpoonup (c\#h)) &= b(S^{-1}(h) \triangleright (ac)) \overset{(3.2)}{=} S^{-1}(h) \triangleright (bac) \\ &= (ba) \leftharpoonup (c\#h) = f_b(a) \leftharpoonup (c\#h), \end{aligned}$$

that is, f_b is right $A\#H$-linear.

It is obvious that ϕ is an algebra map, and is also injective. In what follows, we check that ϕ is surjective.

Choose $\beta \in End_{A\#H}(A)$. For any $a \in A$, $\beta(a) = \beta(1_A \leftharpoonup (a\#1_H)) = \beta(1_A)a$, so $\beta = f_{\beta(1_A)}$. Moreover, $\beta(1_A) \in A^{\mathcal{H}_L}$, since for any $h \in H$,

$$\begin{aligned} h \triangleright \beta(1_A) &= \beta(1_A) \leftharpoonup (1_A\#S(h)) = \beta(1_A \leftharpoonup (1_A\#S(h))) \\ &= \beta(h \triangleright 1_A) = \beta(t_L\varepsilon_L(h) \triangleright 1_A) = \beta(1_A \leftharpoonup (1_A\#s_L\varepsilon_L(h))) \\ &= \beta(1_A) \leftharpoonup (1_A\#s_L\varepsilon_L(h)) = t_L\varepsilon_L(h) \triangleright \beta(1_A), \end{aligned}$$

where the fifth and seventh equalities follow by (1.23) and (3.3). Hence ϕ is surjective.

Similarly, we can check that (1) holds.

The proof is completed.

Since $A^{\mathcal{H}_L}$ is a subalgebra of A, A is a left and right $A^{\mathcal{H}_L}$-module via its multiplication. We may consider the bimodules: ${}_{A^{\mathcal{H}_L}}A_{A\#H}$ and ${}_{A\#H}A_{A^{\mathcal{H}_L}}$ where A is a left and right $A\#H$-module via (3.3).

This is true since for any $b \in A^{\mathcal{H}_L}, a, c \in A, h \in H$,

$$\begin{aligned} (b \cdot a) \leftharpoonup (c\#h) &= (ba) \leftharpoonup (c\#h) = S^{-1}(h) \triangleright (bac) \\ &\overset{(3.2)}{=} b(S^{-1}(h) \triangleright (ac)) = b \cdot (a \leftharpoonup (c\#h)). \end{aligned}$$

In a similar way, we can prove that $(c\#h) \rightharpoonup (a \cdot b) = ((c\#h) \rightharpoonup a) \cdot b$.

Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S, and x a nonzero left integral in $\mathcal{H}_L$. Define two maps

$$\begin{aligned} &[,] : A \otimes_{A^{\mathcal{H}_L}} A \to A\#H, a \otimes_{A^{\mathcal{H}_L}} b \mapsto (a\#x)(b\#1_H), \\ &(,) : A \otimes_{A\#H} A \to A^{\mathcal{H}_L}, a \otimes_{A\#H} b \mapsto x \triangleright (ab). \end{aligned}$$

We now obtain the main result of this section, which generalizes corresponding result given by Cohen and Fishman in [1].

Theorem 3.3 Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S, and x an S-fixed left integral in $\mathcal{H}_L$, i.e. $S(x) = x$. Then $[A^{\mathcal{H}_L}, {}_{A^{\mathcal{H}_L}}A_{A\#H}, {}_{A\#H}A_{A^{\mathcal{H}_L}}, A\#H]$ forms a Morita context.

Proof To satisfy the conditions for a Morita context given in [26], we must check that $[,]$ is an $A\#H$-bimodule map which is middle $A^{\mathcal{H}_L}$-linear, and that $(,)$ is an $A^{\mathcal{H}_L}$-bimodule map which is middle $A\#H$-linear, and that the "associativity" is satisfied.

(1) $[,]$ is an $A\#H$-bimodule map which is middle $A^{\mathcal{H}_L}$-linear.

$[,]$ is a left $A\#H$-module map, since for any $a,b\in A, c\#h\in A\#H$,

$$\begin{aligned}(c\#h)[a,b] &= (c\#h)(a\#x)(b\#1_H)=\Sigma(c(h_{(1)}\triangleright a)\#h_{(2)}x)(b\#1_H)\\ &=\Sigma(c(h_{(1)}\triangleright a)\#s_L\varepsilon_L(h_{(2)})x)(b\#1_H)\\ &=\Sigma(c(h_{(1)}\triangleright a)\#\varepsilon_L(h_{(2)})\cdot x)(b\#1_H)\\ &=\Sigma(c(h_{(1)}\triangleright a)(s_L\varepsilon_L(h_{(2)})\triangleright 1_A)\#x)(b\#1_H)\\ &=\Sigma(c(h_{(1)}\triangleright a)(h_{(2)}\triangleright 1_A)\#x)(b\#1_H)\\ &=(c(h\triangleright a)\#x)(b\#1_H)=(c(h\triangleright a)\#x)(b\#1_H)\\ &=((c\#h)\rightharpoonup a\#x)(b\#1_H)=[(c\#h)\rightharpoonup a,b].\end{aligned}$$

And it is a right $A\#H$-module map, because

$$\begin{aligned}[a,b](c\#h) &= (a\#x)(b\#1_H)(c\#h)=(a\#S(x))(bc\#h)\\ &=a(S(x)_{(1)}\triangleright(bc))\#S(x)_{(2)}h\\ &=a(S(x)_{(1)}S^{-1}(h)\triangleright(bc))\#S(x)_{(2)}\\ &=(a\#S(x))(S^{-1}(h)\triangleright(bc)\#1_H)\\ &=(a\#x)(S^{-1}(h)\triangleright(bc)\#1_H)\\ &=[a,S^{-1}(h)\triangleright(bc)]=[a,b\leftharpoonup(c\#h)],\end{aligned}$$

where the second and the sixth equalities follow by the fact that x is an S-fixed left integral, and the fourth one by $S(\mathcal{L}(\mathcal{H}))\subseteq\mathcal{R}(\mathcal{H})$ and Lemma 2.1.

Obviously, $[,]$ is middle $A^{\mathcal{H}_L}$-linear in the light of the fact that for any $a\in A^{\mathcal{H}_L}, h\in H$,

$$\begin{aligned}(1_A\#h)(a\#1_H) &= \Sigma h_{(1)}\triangleright a\#h_{(2)}\overset{(3.1)}{=}\Sigma t_L\varepsilon_L(h_{(1)})\triangleright a\#h_{(2)}\\ &=\Sigma a\cdot\varepsilon_L(h_{(1)})\#h_{(2)}=\Sigma a\#\varepsilon_L(h_{(1)})\cdot h_{(2)}\\ &=a\#h=(a\#1_H)(1_A\#h).\end{aligned}$$

(2) $(,)$ is an $A^{\mathcal{H}_L}$-bimodule map which is middle $A\#H$-linear.

By (3.1), it is easy to prove that $(,)$ is an $A^{\mathcal{H}_L}$-bimodule map. We check that $(,)$ is middle $A\#H$-linear. For any $a,b\in A, c\#h\in A\#H$,

$$\begin{aligned}(a\leftharpoonup(c\#h),b) &= (S^{-1}(h)\triangleright(ac),b)=x\triangleright((S^{-1}(h)\triangleright(ac))b)\\ &=S(x)\triangleright((S^{-1}(h)\triangleright(ac))b)\\ &=\Sigma(S(x)_{(1)}S^{-1}(h)\triangleright(ac))(S(x)_{(2)}\triangleright b)\\ &=\Sigma(S(x)_{(1)}\triangleright(ac))(S(x)_{(2)}h\triangleright b)\\ &=S(x)\triangleright(ac(h\triangleright b))=x\triangleright(ac(h\triangleright b))\\ &=(a,(c\#h)\rightharpoonup b).\end{aligned}$$

(3) The "associativity" holds: for any $a,b,c\in A$, $(a,b)c=a\leftharpoonup[b,c]$ and $[a,b]\rightharpoonup c=a(b,c)$. By (2.4) and (3.2), it is straightforward.

The proof is completed.

Remark (1) According to remark in Section 2 and Theorem 3.3, we can obtain the Morita context over weak Hopf algebras, which was given by Zhang in [13].

(2) Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S, and let x be an S-fixed and normalized left integral in $\mathcal{H}_L$. Assume that L-ring $(A\#H, 1_A\#s_L)$ is simple. It is obvious that $[A, A]$ is an idea of $A\#H$, hence $[A, A] = A\#H$, that is, $[,]$ is surjective. And $(,)$ is also surjective since x is normalized. Hence $A\#H$ is Morita equivalent to $A^{\mathcal{H}_L}$.

(3) Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid with bijective antipode S, and let x be an S-fixed and normalized left integral in $\mathcal{H}_L$. Assume that A is semisimple Artinian, which is left (or right) $A\#H$-faithful. By Theorem 2.5, we know that $A\#H$ is semisimple Artinian, and it is not difficult to show that $[A, A]$ is an essential ideal of A, since A is left (or right) $A\#H$-faithful. However, semisimple Artinian rings have no nontrivial essential ideals, so, $[,]$ is surjective. Since $(,)$ is also surjective, $A\#H$ is Morita equivalent to $A^{\mathcal{H}_L}$.

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Hopf代数胚上的Smash积的Maschke型定理和Morita关系

王 勇

(南京晓庄学院数学与信息技术学院, 江苏 南京 211171)

摘 要: 本文研究了Hopf代数胚上的Smash积代数. 利用Hopf代数胚的积分理论, 获得了Hopf代数胚上的Smash的Maschke型定理并构造了一个Morita关系, 推广了Cohen和Fishman在文献[1]中的相应结果. 作为应用, 获得了Hopf代数胚上的余模代数的Maschke型定理.

关 键 词: Hopf代数胚; Smash积; Maschke型定理; Morita关系

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