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# SIGNED CLIQUE EDGE DOMINATION NUMBERS OF GRAPHS

AO Guo-yan[1,2,3], Jirimutu[2,3], ZHAO Ling-qi[2]

(*1.School of Mathematical Sciences, Hulunbuir College, Hailaer 021008, China*)

(*2.Institute of Discrete Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China*)

(*3.College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China*)

**Abstract:** In this paper, we study signed clique edge domination number of graph. By using pigeonhole principle, we obtain the signed clique edge domination numbers of graphs $K_n \vee P_m$ and $K_n \vee C_m$, which extend the known results.

**Keywords:** graphs; signed clique edge domination number; signed clique edge dominating function

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## 1 Introduction

In this paper, the graphs are undirected simple graphs and for other terminologies we follow [1]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Every maximal complete subgraph $K$ of graph $G$ is called a clique of $G$, the order of a largest complete subgraph is called the clique number of $G$, denoted by $\omega(G)$. A clique $K$ is called non-trivial if $K \neq K_1$. Let $G_1$ and $G_2$ be any two disjoint graphs. Then $G_1 \vee G_2$ denotes the join graphs of $G_1$ and $G_2$:

$$\begin{aligned}&V(G_1 \vee G_2) = V(G_1) \bigcup V(G_2),\\&E(G_1 \vee G_2) = E(G_1) \bigcup E(G_2) \bigcup \{uv : u \in V(G_1), v \in V(G_2)\}.\end{aligned}$$

Let $G = (V, E)$ be a graph. For a function $f : E \to \{+1, -1\}$ and a subset $S$ of $E(G)$, define $f(S) = \sum\limits_{e\in S} f(e)$. For convenience, for a given graph $G = (V, E)$, an edge $e \in E(G)$ is said to be a $+1$ edge of $G$ if $f(e) = +1$, analogously, an edge $e \in E(G)$ is said to be a -1 edge of $G$ if $f(e) = -1$. Write $E_1 = \{e \in E(G)|f(e) = +1\}$, $E_2 = \{e \in E(G)|f(e) = -1\}$.

**Definition 1.1** [2] Let $G = (V, E)$ be a simple graph. A function $f : E \to \{+1, -1\}$ is said to be a signed clique edge dominating function of $G$ if $\sum\limits_{e\in E(K)} f(e) \geq 1$ for every

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non-trivial clique $K$ in $G$. The signed clique edge domination number of $G$ is defined to be $\gamma'_{scl}(G) = \min\{\sum\limits_{e\in E(G)} f(e) : f$ is a signed clique edge dominating function of $G\}$. In particular, for empty graph $\overline{K_n}$, define $\gamma'_{scl}(\overline{K_n}) = 0$.

In recent years, domination number and its variations were studied extensively. The monographs [2] contain extensive reviews of topics. Signed edge domination was studied in [3, 4], signed clique edge domination was studied in [5], signed star domination in [6], signed cycle domination in [7], minus edge domination in [8], signed edge total domination in [9]. In this paper, we determine the signed clique edge domination numbers of graphs $K_n \vee P_m$ and $K_n \vee C_m$.

## 2 Main Result

**Theorem 2.1** For any positive integer $n \geq 3$ and $m \geq 3$,

$$\gamma'_{scl}(K_n \vee P_m) = \begin{cases} 2(6-n)\lfloor \frac{m}{2} \rfloor - (n+1)m + \frac{n(n-1)}{2} + 1, & \text{when } n = 3,4,5, \\ -(n+1)m + 2n + 3 + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor + 1} + 1}{2}, & \text{when } n \geq 6. \end{cases}$$

**Proof** Let $f$ be a signed clique edge dominating function of graph $G = K_n \vee P_m$ such that $\gamma'_{scl}(G) = f(E) = \sum\limits_{e\in E} f(e)$. The vertices of $K_n$ are $v_1, v_2, \cdots v_n$ in this order, and the vertices of $P_m$ are $u_1, u_2, \cdots u_m$ in this order. Then $|E(G)| = \frac{n(n-1)}{2} + (n+1)m - 1$. Let $A = \{v_i u_j | i = 1, 2, \cdots n, j = 1, 2, \cdots m\} \bigcup \{u_i u_{i+1} | i = 1, 2, \cdots, m-1\}$.

We first prove lower bound.

**Case 1** $n = 3, 4, 5$, then

$$\gamma'_{scl}(G) \geq 2(6-n)\lfloor \frac{m}{2} \rfloor - (n+1)m + \frac{n(n-1)}{2} + 1. \tag{2.1}$$

Let $s$ (respectively $t$) be the number of +1 (respectively -1) edges of $G$, thus $\frac{n(n-1)}{2} + (n+1)m - 1 = s + t$, $\gamma'_{scl}(G) = s - t$.

Suppose that (2.1) does not hold. Then $\gamma'_{scl}(G) < 2(6-n)\lfloor \frac{m}{2} \rfloor - (n+1)m + \frac{n(n-1)}{2} + 1$, Hence $t > (n+1)m - (6-n)\lfloor \frac{m}{2} \rfloor - 1$. Let the number of -1 edges in $A$ be $r$.

**Case 1.1** $m \equiv 0 \pmod 2$.

Suppose $3(n-2)\frac{m}{2} + (m-1) < r \leq (n+1)m - 1$. By the pigeonhole principle, there exists a clique $K_{n+2} \in G$, that the number of -1 edges is at least $3n-4$, such that $\sum\limits_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

If

$$(n+1)m - (6-n)\frac{m}{2} - \frac{n(n-1)}{2} \leq r \leq 3(n-2)\frac{m}{2} + (m-1),$$

then the number of -1 edges in $E(G) \setminus A$ is at least 1. By the pigeonhole principle, there exists a $K_{n+2} \in G$, such that $\sum_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

**Case 1.2** $m \equiv 1 \pmod 2$.

Suppose $n\lceil\frac{m}{2}\rceil+2(n-3)\lfloor\frac{m}{2}\rfloor+(m-1)<r\le(n+1)m-1$. By the pigeonhole principle, there exists a clique $K_{n+2}\in G$, that the number of -1 edges is at least $3n-4$, such that $\sum\limits_{e\in E(K_{n+2})}f(e)\le 0$. This is a contradiction.

If

$$(n+1)m-(6-n)\lfloor\frac{m}{2}\rfloor-\frac{n(n-1)}{2}\le r\le 3(n-2)\lfloor\frac{m}{2}\rfloor+n+(m-1),$$

then the number of -1 edges in $E(G)\setminus A$ is at least 1. By the pigeonhole principle, there exists a $K_{n+2}\in G$, such that $\sum\limits_{e\in E(K_{n+2})}f(e)\le 0$. This is a contradiction.

Hence $\gamma_{scl}^{'}(G)\ge 2(6-n)\lfloor\frac{m}{2}\rfloor-(n+1)m+\frac{n(n-1)}{2}+1$.

**Case 2** $n\ge 6$. Then

$$\gamma_{scl}^{'}(G)\ge-(n+1)m+2n+3+\frac{(-1)^{\lfloor\frac{n}{2}\rfloor+1}+1}{2}.$$

Let $f$ be a signed clique edge dominating function of G such that $\gamma_{scl}^{'}(G)=f(G)$, and $s$ the number of +1 edges of $G$. Then $\gamma_{scl}^{'}(G)=2s-|E(G)|$. And $\sum\limits_{e\in E(K_{n+2})}f(e)\ge 1$ for every non-trivial clique $K_{n+2}$ in $G$. Hence $s\ge s_0=|\{e\in E(K_{n+2})\mid f(e)=1\}|$.

Note that

$$|E(G)|=\frac{n(n-1)}{2}+(n+1)m-1,\ |E(K_{n+2})|=\frac{(n+2)(n+1)}{2}.$$

Since $f(K_{n+2})\ge 1$, $s_0\ge\lfloor\frac{(n+2)(n+1)}{4}\rfloor+1$. Then $s\ge\lfloor\frac{(n+2)(n+1)}{4}\rfloor+1$. Hence

$$\gamma_{scl}^{'}(G)=2s-|E(G)|\ge-(n+1)m+2n+3+\frac{(-1)^{\lfloor\frac{n}{2}\rfloor+1}+1}{2}.$$

Next consider the upper bound.

We define the signed clique edge dominating function $f$ of graph $G$ as follows:

For $n=3$, let

$$f(e)=\begin{cases}+1, & \text{when } e\in K_3\bigcup\{v_iu_j|i=1,2,3,j\equiv 0\pmod 2\};\\ -1, & \text{otherwise.}\end{cases}$$

For $n=4$, let

$$f(e)=\begin{cases}+1, & \text{when } e\in K_4\bigcup\{v_iu_j|i=3,4,j\equiv 0\pmod 2\};\\ -1, & \text{otherwise.}\end{cases}$$

For $n=5$, let

$$f(e)=\begin{cases}+1, & \text{when } e\in K_5\bigcup\{v_iu_j|i=5,j\equiv 0\pmod 2\};\\ -1, & \text{otherwise.}\end{cases}$$

For $n=3,4,5$, every non-trivial clique $K_{n+2}$ in $G$, we have

$$\begin{aligned}\sum_{e\in E(K_{n+2})} f(e) &= \sum_{e\in E_1\bigcap E(K_{n+2})} f(e) - \sum_{e\in E_2\bigcap E(K_{n+2})} f(e)\\ &= \frac{n(n-1)}{2}+(6-n)-(3n-5)\\ &= \frac{n(n-1)}{2}-4n+11\geq 1.\end{aligned}$$

Hence $\gamma_{scl}^{'}(G)\leq \sum\limits_{e\in E(G)} f(e)=2(6-n)\lfloor\frac{m}{2}\rfloor-(n+1)m+\frac{n(n-1)}{2}+1$.

For $n\geq 6$, let the number of +1 edges in $K_n$ is $\lfloor\frac{(n+2)(n+1)}{4}\rfloor+1$. All other edges are assigned -1. For every non-trivial clique $K_{n+2}$ in $G$, we have

$$\begin{aligned}\sum_{e\in E(K_{n+2})} f(e) &= \sum_{e\in E_1\bigcap E(K_{n+2})} f(e) - \sum_{e\in E_2\bigcap E(K_{n+2})} f(e)\\ &= 2\lfloor\frac{(n+2)(n+1)}{4}\rfloor+2-\frac{(n+2)(n+1)}{2}\\ &= \begin{cases} 1, & n\equiv 0,1 \pmod 4;\\ 2, & n\equiv 2,3 \pmod 4).\end{cases}\end{aligned}$$

Hence

$$\gamma_{scl}^{'}(G)\leq \sum_{e\in E(G)} f(e)=-(n+1)m+2n+3+\frac{(-1)^{\lfloor\frac{n}{2}\rfloor+1}+1}{2}.$$

**Theorem 2.2** For any positive integer $n\geq 3$ and $m\geq 3$,

$$\gamma_{scl}^{'}(K_n\vee C_m)=\begin{cases} 2(6-n)\lceil\frac{m}{2}\rceil-(n+1)m+\frac{n(n-1)}{2}, & \text{when } n=3,4,5\\ -(n+1)m+2n+2+\frac{(-1)^{\lfloor\frac{n}{2}\rfloor+1}+1}{2}, & \text{when } n\geq 6.\end{cases}$$

**Proof** Let $f$ be a signed clique edge dominating function of graph $G=K_n\vee C_m$ such that $\gamma_{scl}^{'}(G)=f(E)=\sum\limits_{e\in E} f(e)$. The vertices of $K_n$ are $v_1,v_2,\cdots v_n$ in this order, and the vertices of $C_m$ are $u_1,u_2,\cdots u_m$ in this order. Then $|E(G)|=\frac{n(n-1)}{2}+(n+1)m$. Write

$$A=\{v_iu_j|i=1,2,\cdots n,j=1,2,\cdots m\}\bigcup\{u_iu_{i+1}|i=1,2,\cdots,m-1\}\bigcup\{u_1u_n\}.$$

$n=3,4,5$, we first prove lower bound.

$$\gamma_{scl}^{'}(G)\geq 2(6-n)\lceil\frac{m}{2}\rceil-(n+1)m+\frac{n(n-1)}{2}. \tag{2.2}$$

Let $s$ (respectively $t$) be the number of +1 (respectively -1 ) edges of $G$. Thus

$$\frac{n(n-1)}{2}+(n+1)m=s+t,\quad \gamma_{scl}^{'}(G)=s-t.$$

Suppose that (2.2) does not hold. Then $\gamma_{scl}^{'}(G)<2(6-n)\lceil\frac{m}{2}\rceil-(n+1)m+\frac{n(n-1)}{2}$. Hence $t>(n+1)m-(6-n)\lceil\frac{m}{2}\rceil$. Let the number of -1 edges in $A$ be $r$.

**Case 1** $m \equiv 0 \pmod 2$.

Suppose $3(n-2)\frac{m}{2} + m < r \leq (n+1)m$ . By the pigeonhole principle, there exists a clique $K_{n+2} \in G$, That the number of -1 edges is at least $3n-4$, such that $\sum\limits_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

If

$$(n+1)m - (6-n)\frac{m}{2} - \frac{n(n-1)}{2} + 1 \leq r \leq 3(n-2)\frac{m}{2} + m.$$

Then the number of -1 edges in $E(G) \setminus A$ is at least 1. By the pigeonhole principle, there exists a $K_{n+2} \in G$, such that $\sum\limits_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

**Case 2** $m \equiv 1 \pmod 2$.

Suppose $n\lfloor \frac{m}{2} \rfloor + 2(n-3)\lceil \frac{m}{2} \rceil + m < r \leq (n+1)m$. By the pigeonhole principle, there exists a clique $K_{n+2} \in G$, that the number of -1 edges is at least $3n-4$, such that $\sum\limits_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

If

$$(n+1)m - (6-n)\lceil\frac{m}{2}\rceil - \frac{n(n-1)}{2} + 1 \leq r \leq n\lfloor\frac{m}{2}\rfloor + 2(n-3)\lceil\frac{m}{2}\rceil + m,$$

then the number of -1 edges in $E(G) \setminus A$ is at least 1. By the pigeonhole principle, there exists a $K_{n+2} \in G$, such that $\sum\limits_{e\in E(K_{n+2})} f(e) \leq 0$. This is a contradiction.

In summary,

$$\gamma'_{scl}(G) \geq 2(6-n)\lceil\frac{m}{2}\rceil - (n+1)m + \frac{n(n-1)}{2}.$$

Next we consider the upper bound. The upper bound is obtained by specifying a signed clique edge dominating function. We define the signed clique edge dominating function $f$ of $G$ as follows:

For $n = 3$, let

$$f(e) = \begin{cases} +1, & \text{when } e \in K_3 \bigcup \{v_iu_j | i = 1,2,3, j \equiv 1 \pmod 2\}; \\ -1, & \text{otherwise.} \end{cases}$$

For $n = 4$, let

$$f(e) = \begin{cases} +1, & \text{when } e \in K_4 \bigcup \{v_iu_j | i = 3,4, j \equiv 1 \pmod 2\}; \\ -1, & \text{otherwise.} \end{cases}$$

For $n = 5$, let

$$f(e) = \begin{cases} +1, & \text{when } e \in K_5 \bigcup \{v_iu_j | i = 5, j \equiv 1 \pmod 2\}; \\ -1, & \text{otherwise.} \end{cases}$$

For $n = 3, 4, 5$, $m \equiv 1 \pmod 2$, consider clique $K_{n+2}$ of include edge $u_1u_m$,

$$\begin{aligned}\sum_{e\in E(K_{n+2})} f(e) &= \sum_{e\in E_1} f(e) - \sum_{e\in E_2} f(e)\\ &= \frac{n(n-1)}{2} + 2(6-n) - (4n-11)\\ &= \frac{n(n-1)}{2} - 6n + 23 \geq 1.\end{aligned}$$

We have

$$\gamma'_{scl}(G) \leq \sum_{e\in E(G)} f(e) = 2(6-n)\lceil\frac{m}{2}\rceil - (n+1)m + \frac{n(n-1)}{2}.$$

For other cases the proof is similar to Theorem 2.1.

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# 图的符号团边控制数

敖国艳[1,2,3] , 吉日木图[2,3], 赵凌琪[2]

(1.呼伦贝尔学院数学科学学院, 内蒙古 海拉尔 021008)

(2.内蒙古民族大学离散数学研究所, 内蒙古 通辽 028043)

(3.内蒙古民族大学数学学院, 内蒙古 通辽 028043)

**摘要:** 本文研究了图的符号团边控制数的问题. 利用鸽巢原理, 获得了图$K_n \vee P_m$和$K_n \vee C_m$的符号团边控制数, 推广了已有的结果.

**关键词:** 图; 符号团边控制数; 符号团边控制函数

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