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# MEAN ERGODICITY OF COMPOSITION OPERATORS BETWEEN BANACH SPACES OF ANALYTIC FUNCTIONS

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**Abstract:** In this paper, we study mean ergodicity of some composition operators. By the classical theory of operators, we get the characterization of mean ergodicity for composition operators between given weighted Banach spaces of analytic functions defined on the unit disk. The same characterization on Hardy space of unit disk is also discussed. Our characterizations can be regarded as generalization of existing results for multiplication operators.

**Keywords:** Banach space; analytic functions; composition operators; mean ergodicity

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## 1 Introduction

This paper focuses on mean ergodicity of composition operators between given weighted Banach spaces of analytic functions defined on the unit disk and Hardy spaces of unit disk.

For a locally convex Hausdorff space, the space of all continuous linear operators from X into itself is denoted by $L(X)$. Equipping $L(X)$ with its strong operator topology, we write $L_s(X)$. Given $T \in L(X)$, its Cesàro means (see [1]) are defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N} \tag{1.1}$$

from which one could routinely verify

$$\frac{1}{n}T^n = T_{[n]} - \frac{(n-1)}{n}T_{[n-1]}, \quad n \in \mathbb{N}, \tag{1.2}$$

where $T_{[0]}$ is the identity operator on $X$.

An operator $T$ is mean ergodic if $\{T_{[n]}\}_{n=0}^{\infty}$ is a convergent sequence in $L_s(X)$ (see [1]).

Motivated by [1] in which the mean ergodicity of multiplication operators in weighted Banach spaces of holomorphic functions is examined, in this paper, the mean ergodicity of

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composition operators between given weighted Banach spaces of analytic functions defined on the unit disk and Hardy spaces of unit disk will be examined.

## 2 Weighted Spaces of Analytic Functions on the Unit Disk

The classic space $H^\infty$ is the space of all bounded analytic functions $f$ on on the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the norm

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|. \tag{2.3}$$

Let $S$ denote the subset of $H^\infty$ consisting of the analytic selfmaps of $\mathbb{U}$.

For every $w \in \mathbb{U}$, $C_w$ denotes the composition operators of constant symbol $w$. For every $n = 1, 2, \cdots$, denote $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$, then $C_{\varphi^{[n]}} = \varphi^{[n]}$.

Let $v$ be strictly positive bounded continuous functions (weights) on $\mathbb{U}$. We are interested in radial weights, i.e., weights which satisfy $v(z) = v(|z|)$. Especially interesting ones are weights $v$ which satisfy the following condition

$$\exists q > 0 : \frac{1}{v(1-\frac{1}{t})t^q} \text{ is almost increasing, } t \geq 1. \tag{2.4}$$

Let $H_v^\infty$ denote the Banach spaces of analytic functions defined on $\mathbb{U}$ endowed with the norm

$$\|f\|_v = \sup_{|z|<1} v(z)|f(z)|. \tag{2.5}$$

The boundedness and weak compactness of composition operator $C_\varphi f = f \circ \varphi$ on $H_v^\infty$ were investigated by several authors. We refer to [2] and [10].

In this section the mean ergodicity of composition operators between the spaces $H_v^\infty$ is considered. To present the result, first we need some auxiliary results. Recall that for any $z \in \mathbb{U}$, $\varphi_z(w)$ is the Möbius transformation of $\mathbb{U}$ which interchanges the origin and $z$, i.e.,

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}, \ w \in \mathbb{U}.$$

The psudohyperbolic distance $\rho(z, w)$ for all $z, w \in \mathbb{U}$ is defined by $\rho(z, w) = \left|\frac{z-w}{1-\bar{z}w}\right|$.

For $z, w \in \mathbb{U}$, the hyperbolic distance from $z$ to $w$ to be the "length of the shortest curve from $z$ to $w$", that is

$$\rho_{\mathbb{U}}(z, w) := \inf_\gamma \ell_{\mathbb{U}}(\gamma),$$

where on the right, $\gamma$ runs through all piecewise $C^1$ curve from $z$ to $w$ (see p.151 in [3] for details).

The relation between the psudohyperbolic distance and hyperbolic distance is reflected by the following:

**Lemma 2.1** For $z, w \in \mathbb{U}$ we have

$$\rho_{\mathbb{U}}(z, w) = \ell_{\mathbb{U}}(\gamma) = \log \frac{1+\rho(z, w)}{1-\rho(z, w)},$$

where $\gamma$ is the unique arc that joins $z$ and $w$, and lies on a circle perpendicular to the unit circle.

**Proof** See p.153 in [3].

**Lemma 2.2** Let $\varphi$ be a holomorphic self-map of $\mathbb{U}$ with a fixed point $p$, then for any $z \in \mathbb{U}$,

$$\rho_{\mathbb{U}}(\varphi^{[n]}(z), p) \leq \|\varphi\|_\infty^n \rho_{\mathbb{U}}(z, p) \tag{2.6}$$

for all $z, p \in \mathbb{U}$.

**Proof** From Exercises 5–7 on p.171 in [3], we have

$$\rho_{\mathbb{U}}(\varphi(z), \varphi(p)) \leq \|\varphi\|_\infty \rho_{\mathbb{U}}(z, p)$$

for all $z \in \mathbb{U}$. Since $p$ is a fixed point of $\varphi$, we have $\varphi^{[n]}(p) = p$ for all $n = 1, 2, 3, \cdots$. Iteration yields

$$\rho(\varphi_{\mathbb{U}}^{[n]}(z), p) \leq \|\varphi\|_\infty^n \rho_{\mathbb{U}}(z, p).$$

**Lemma 2.3** Let $v$ be a weight such that $v$ is radial and satisfying (2.4). For every $f \in H_v^\infty$ there exists a constant $C$ (depending on the weight $v$) such that

$$|f(z) - f(p)| \leq C_v \|f\|_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\} \rho(z, p) \tag{2.7}$$

for all $z, p \in \mathbb{U}$.

**Proof** Adaptation of the case $N = 1$ in Lemma 3.2 from [10] gives the proof.

The main result of this section is as follows:

**Theorem 2.1** Let $v$ be a weight such that $v$ is radial and satisfies (2.4). If $\varphi \in \mathrm{S}$ is a holomorphic self-map of $\mathbb{U}$ with a fixed point, satisfying

$$\lim_{n\to\infty} \|\varphi\|_\infty^n = 0, \tag{2.8}$$

then $C_\varphi$ is mean ergodic in $H_\infty^v$.

**Proof** For any $f(z)$ holomorphic in $\mathbb{U}$, denote $C_p f = f(p)$ where $p$ is the fixed point of $\varphi$. It is obviously that $C_p$ is a bounded composition operator. We are going to verify

$$\|C_{[n]} - C_p\|_v \to 0. \tag{2.9}$$

For any $f \in H_v^\infty$, $\|f\|_v = 1$, by (2.7) in Lemma 2.3, we have

$$\begin{aligned} \|C_{\varphi^{[n]}} f(z) - C_p f(z)\|_v &= \sup_{|z|<1} v(z) |f(\varphi^{[n]}(z)) - f(p)| \\ &\leq \sup_{|z|<1} v(z) C_v \max\left\{\frac{1}{v(\varphi^{[n]}(z))}, \frac{1}{v(p)}\right\} \rho(\varphi^{[n]}(z), p). \end{aligned}$$

Combining Lemma 2.1 with (2.6) in Lemma 2.2 yields

$$\|C_{\varphi^{[n]}} f(z) - C_p f(z)\|_v \leq C_v' \|\varphi\|_\infty^n, \tag{2.10}$$

where $C'_v = \sup_{|z|<1} v(z)C_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\} \log \frac{1+\rho(z,p)}{1-\rho(z,p)}$ is a positive constant depends only on $v$. Hence, by the definition of $C_{[n]}$ in (1.1),

$$\begin{aligned}\|C_{[n]} - C_p\|_v &= \sup_{\|f\|_v=1} \|C_{[n]}f(z) - C_pf(z)\|_v \\ &\le \frac{C'_v \sum_{k=1}^n \sup_{\|f\|_v=1} \|C_{\varphi^{[k]}}f(z) - C_pf(z)\|_v}{n}.\end{aligned}$$

By (2.10), we have

$$\|C_{[n]} - C_p\|_v \le \frac{1 - \|\varphi\|_\infty^n}{n(1 - \|\varphi\|_\infty)},$$

thus (2.8) follows, proving Theorem 2.1.

## 3 Hardy Spaces of Analytic Functions on the Unit Disk

The classical Hardy spaces $H^p$ $(0 < p < \infty)$ over the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the collection of functions analytic on the unit disc $\mathbb{U}$, satisfying

$$\|f\|_p = (\sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^p \mathrm{d}m(\xi))^{1/p} < \infty,$$

where $\mathbb{T}$ is the unit circle $|z| = 1$, $m$ is the normalized arc-length measure on $\mathbb{T}$.

If $\varphi$ is a function holomorphic on $\mathbb{U}$ with $\varphi(\mathbb{U}) \subset \mathbb{U}$, then $\varphi$ induces a linear composition operator $C_\varphi$ on the space $\mathrm{Hol}(\mathbb{U})$ of all functions holomorphic on $\mathbb{U}$ (see [3]) as follows

$$C_\varphi f = f \circ \varphi \quad (f \in \mathrm{Hol}(\mathbb{U})).$$

Various prospects of such operators were extensively studied (see [3]). In this section, we will examine mean ergodicity of composition operator in Hardy space of $\mathrm{Hol}(\mathbb{U})$. The main result of this section is as follows:

**Theorem 3.1** Let $\varphi \in S$ be a non-inner function. If for some $w \in \mathbb{U}$, $\varphi(w) = w$, then $C_\varphi$ is mean ergodic in $H^p$ $(0 < p < \infty)$.

**Proof** From the proof of Proposition 2.4 in [5] and Theorem 3.1 in [6], we know that only the case $p = 2$ needs to be verified. We will follow the proof of Theorem 1 in [8].

If $w = 0$, denote $H_0^2 = \{f \in H^2\}$, recall that $\|C_\varphi|H_0^2\| = \delta < 1$ (see [4]). For any $f \in H^2$, $\|f\|_2 = 1$,

$$\|C_\varphi f - C_0 f\|_2 = \|C_\varphi(f - f(0))\|_2 \le \delta\|f - f(0)\|_2,$$

thus,

$$\|C_\varphi^n f - C_0 f\|_2 = \|C_\varphi(f \circ \varphi^{[n-1]} - f(0))\|_2 \le \delta\|f \circ \varphi^{[n-1]} - f(0)\|_2.$$

Iterating yields

$$\|C_\varphi^n - C_0\| \le \delta^n. \tag{3.11}$$

Recall Cesàro means in (1.1), combining with (3.11), we have

$$\|C_{[n]} - C_0\| = \Big\|\frac{1}{n}\sum_{m=1}^{n} C_\varphi^m - C_0\Big\| \le \frac{\sum_{m=1}^{n}\|C_\varphi^m - C_0\|}{n} \le \frac{\delta - \delta^n}{n(1-\delta)},$$

thus the Cesàro means of $C_\varphi$ converges and Theorem 2.1 is proved in this case.

Consider the selfinverse conformal automorphism $\alpha_w(z) = (w - z)/(1 - \overline{w}z)$ and set $\psi = \alpha_w \circ \varphi \circ \alpha_w$. It is obvious that $\psi(0) = 0$. Thus, $\|C_\psi^n - C_\psi^m\| \to 0$ if $m, n \to \infty$. For each $k$, we have $C_\varphi^k = C_{\alpha_w} C_\psi^k C_{\alpha_w}$ such that $\|C_\varphi^n - C_\varphi^m\| \le \|C_{\alpha_w}\|^2 \|C_\psi^n - C_\psi^m\| \to 0$ if $m, n \to \infty$. This shows that $\{C_\varphi^n\}$ is norm-convergent. From Theorem 3.1 in [6], $C_{\varphi^{[n]}} = \varphi^{[n]} \to w$ converges weakly in $H^2$. Since the set of all composition operators is weakly sequentially compact (see [9] or Remark 1 in [8]), we conclude that $\|C_\varphi^n - C_w\| \to 0$. From (3.11) and since $\alpha_w(z)$ is a selfinverse conformal automorphism, we have

$$\begin{aligned}
\|C_{[n]} - C_w\| &= \Big\|\frac{1}{n}\sum_{m=1}^{n} C_\varphi^m - C_w\Big\| \\
&\le \frac{\sum_{m=1}^{n}\|C_{\alpha_w} C_\psi^m C_{\alpha_w} - C_w\|}{n} \\
&\le \frac{\|C_{\alpha_w}\|^2 \sum_{m=1}^{n}\|C_\psi^m - C_{\alpha_w} C_w C_{\alpha_w}\|}{n} \\
&= \frac{\|C_{\alpha_w}\|^2 \sum_{m=1}^{n}\|C_\psi^m - C_0\|}{n} \le \frac{\delta - \delta^n}{n(1-\delta)},
\end{aligned}$$

thus the Cesàro means of $C_\varphi$ converges and Theorem 2.1 is proved.

The space $H^2_{d,\beta}$ (see [7]) over the unit ball $\mathbb{B}_{\mathbb{N}} = \{z = (z_1, z_2, \cdots, z_N) \in \mathbb{C}^N : |z| = (\sum_{k=1}^{N} |z_k|^2)^{1/2} < 1\}$ is the collection of functions analytic on the unit ball $\mathbb{B}^N$ with reproducing kernel

$$k_\beta(z, w) = \frac{1}{(1 - \langle z, w\rangle)^\beta},$$

where $\mathbb{T}^N$ is the unit circle $|z| = 1$, $m$ is the normalized arc-length measure on $\mathbb{T}^N$.

The Schur-Agler class is the set of all holomorphic mapping $\varphi : \mathbb{B}_{\mathbb{N}} \to \mathbb{B}_{\mathbb{N}}$ for which the Hermitian kernel,

$$k^\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w)\rangle}{(1 - \langle z, w\rangle)^\beta}$$

is positive semidefinite (see [7]).

A holomorphic self-map $\varphi$ of $\mathbb{B}^N$ will be called hyperbolic if $\varphi$ has no fixed point and dilation coefficient $\alpha < 1$ (see [7]). The main result of this section is as follows:

**Theorem 3.2** If $\varphi$ is a hyperbolic self-map of $\mathbb{B}^N$ in the Schur-Agler class, then $C_\varphi$ is not mean ergodic in $H^2_{d,\beta}$.

**Proof** Our proof is an argument by contradiction. If $C_\varphi$ were mean ergodic in $H^2_{d,\beta}$, by Corollary 1.5 in [7], we have the following

$$\|C_\varphi^n\| \ge A(1 - |\varphi_n(0)|)^{-\beta/2}, \tag{3.12}$$

where $A$ is some positive constant independent of $n$. If $\varphi$ is a hyperbolic self-map of $\mathbb{B}^N$, we have

$$\lim_{n\to\infty}(1-|\varphi_n(0)|^2)^{1/n}=\alpha \tag{3.13}$$

from Theorem 3.5 in [7]. Recall Cesàro means in (1.1) and (1.2), combining with (3.12) and (3.13), for some fixed $\varepsilon_0>0$ satisfying

$$\alpha+\varepsilon_0<1, \tag{3.14}$$

there exists some $M>0$, if $n>M$, we have

$$\left\|C_{[n]}-\frac{(n-1)}{n}C_{[n-1]}\right\|=\frac{1}{n}\left\|C_\varphi^n\right\|\geq A\frac{(1/(\alpha+\varepsilon_0))^{n\beta/2}}{n}.$$

Thus, by (3.14), let $n\to\infty$, we have $\left\|C_{[n]}-\frac{(n-1)}{n}C_{[n-1]}\right\|\to\infty$ which contradicts to the norm convergence of $C_{[n]}$, proving Theorem 3.1.

## References

[1] Bonet J, Ricker W J. Mean ergodicity of multiplication operators in weighted spaces of holomorphic functions[J]. Arch. Math., 2009, 429–437.

[2] Bonet J, Domański P, Lindström M. Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions[J]. Canad. Math. Bull., 1999, 139–148.

[3] Shapiro J H. Composition operators and classical function theory[M]. Berlin: Springer-Verlag, 1993.

[4] Shapiro J H. What do composition operators know about inner functions[J]. Monatsh. Math., 2000, 57–70.

[5] Shapiro J H, Smith W. Hardy spaces that support no compact compositon operators[J]. J. Functional Analysis, 2003, 62–89.

[6] Bourdon P S, Matache V, Shapiro J H. On convergence to the Denjoy-Wolff point[J]. Illinois J. Math., 2005, 405–430.

[7] Jury M T. Norms and spectral radii of linear fractional composition operators on the ball[J]. J. Functional. Anal., 2008, 2387–2400.

[8] Matache V. Convergent sequences of composition operators[J]. J. Math. Anal. Appl., 2005, 659–668.

[9] Schwartz H G. Composition operators on $H^p$[D]. Toledo: Univ. of Toledo, 1963.

[10] Wolf E. Difference of compostion operators between weighted Banach spaces of holomorphic functions on the unit polydisk[J]. Result. Math., 2008, 361–372.

# 解析函数空间中复合算子的平均遍历性

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**摘 要**: 本文研究了复合算子的遍历性问题. 利用经典算子理论的方法, 获得了单位圆上加权Banach空间中复合算子遍历性的结果, 我们还得到单位圆上Hardy空间中类似的结果，我们的结论是对乘子算子相关结果的推广.

**关键词**: Banach空间; 解析函数; 复合算子; 平均遍历性

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