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(α,β)-γ-OPEN SETS AND (α,β)-γ-T_i SPACES

WU Yao-qiang

(*Department of Teachers Education, Suqian College, Suqian 223800, China*)

Abstract: In this paper, the concept of (α,β)-γ-open sets is introduced, some properties results are given. By using the concept of (α,β)-γ-T_i spaces and (α,β)-γ-T_i^* spaces ($i = 0, 1/2, 1, 2, 5/2$), their essential topological properties are generalized.

Keywords: α-open sets; (α,β)-open sets; (α,β)-γ-open sets; (α,β)-γ-T_i spaces; (α,β)-γ-T_i^* spaces

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1 Introduction and Preliminaries

Many concepts of generalized open sets were introduced by mathematicians during the last decades, which based on the open sets in topological space. Such as the concept of semi-open sets was introduced by Levine in 1963 (see [1]). The concepts of α-open (in the sense γ-open) sets and regular open sets were introduced by Njastad and Császár, respectively (see [2–3]). Umehara et al. (see [4]) defined the notion of (γ,γ')-open sets, furthermore, they introduced some new separation axioms and investigated their relationships in 1992. In fact, the theory of generalized open sets became an important topic in general topology and obtained a number of good results (see [1–16]). In this paper, we introduce and study the concept of (α,β)-γ-open sets and investigate (α,β)-γ-T_i separation axioms.

Throughout this paper, X always denotes a non-empty set, $(X,\mathcal{T})$ denotes topological space. Let us denote the collection of all open sets, closed sets in $(X,\mathcal{T})$ by $\mathcal{T}$ and $\mathcal{F}$, respectively. The closure of A and interior of A are denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. The terminology and symbol without definition in this paper may be found in [4–6].

Definition 1.1 [3] Let $(X,\mathcal{T})$ be a topological space, and $P(X)$ denote the collection of the power set of X. Then we say a mapping α is an operator associated to $\mathcal{T}$ on X (written in short as an operator on $\mathcal{T}$), if $\alpha : P(X) \to P(X)$ satisfies $V \subset \alpha(V)$ for each $V \in \mathcal{T}$. where $\alpha(V)$ denotes the value of α at V.

Definition 1.2 [3] Let $(X,\mathcal{T})$ be a topological space and α be an operator on $\mathcal{T}$, A non-empty subset A is said to be α-open if for each $x \in A$, there exists an open set U such that $x \in U$ and $\alpha(U) \subset A$. We suppose that the empty set is α-open and denote the

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Biography: Wu Yaoqiang(1973–), male, born at Suqian, Jiangsu, associate professor, major in general topology.

set of all α-open sets by $\mathcal{T}_\alpha$. The complement of α-open set is called α-closed. We denote $cl_\alpha(A) = \{x \in X|\alpha(U) \cap A \neq \emptyset$ for each open set U containing x $\}$.

Definition 1.3 [4] Let $(X, \mathcal{T})$ be a topological space and α, β be operators on $\mathcal{T}$, A non-empty subset is said to be (α, β)-open if for each $x \in A$, there exists a β-open set U such that $x \in U$ and $\alpha(U) \subset A$. We suppose that the empty set is (α, β)-open and denote the set of all (α, β)-open sets by $\mathcal{T}_{(\alpha,\beta)}$. The complement of (α, β)-open set is called (α, β)-closed.

Definition 1.4 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, A non-empty subset is said to be (α, β)-γ-open if for each $x \in A$, there exist γ-open sets U and V such that $x \in U$, $x \in V$ and $\alpha(U) \cup \beta(V) \subset A$. We suppose that the empty set is (α, β)-γ-open and denote the set of all (α, β)-γ-open sets by $\mathcal{T}_{(\alpha,\beta)-\gamma}$. The complement of (α, β)-γ-open set is called (α, β)-γ-closed. We denote $cl_{(\alpha,\beta)-\gamma}(A) = \{x \in X|(\alpha(U) \cup \beta(V)) \cap A \neq \emptyset$ for any γ-open set U and V containing x $\}$.

Furthermore, we denote the intersection of all (α, β)-γ-closed sets containing A by $CL_{(\alpha,\beta)-\gamma}(A)$. Namely, we defined as follows $CL_{(\alpha,\beta)-\gamma}(A) = \cap\{F|F$ is an (α, β)-γ-closed set in $(X, \mathcal{T})$ such that $A \subset F$ $\}$. It is obvious that $cl_{(\alpha,\beta)-\gamma}(A) \subset CL_{(\alpha,\beta)-\gamma}(A)$ for any subset A of $(X, \mathcal{T})$.

From the above Definitions 1.3 and 1.4, we know that a subset A is (α, β)-open in $(X, \mathcal{T})$, then it is (α, β)-γ-open, but the converse need not be true.

Example 1.1 Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. Consider the operators α, β, γ defined as follow: $\alpha : P(X) \to P(X)$, where $\alpha =$ id. that is id$(A) = A$ for every subset A of $(X, \mathcal{T})$; $\beta : P(X) \to P(X)$, where $\beta(A) = A$, if $b \in A$ and $\beta(A) = cl(A)$, if $b \notin A$; $\gamma : P(X) \to P(X)$, where $\gamma(A) = cl(A)$, if $b \notin A$ and $\gamma(A) = A$, if $b \in A$. We can see $\{b, c\}$ is (α, β)-open but not (α, β)-γ-open.

Furthermore, using Definitions 1.3 and 1.4, we have A is an (α, β)-γ-open set if and only if A is (α, γ)-open and (β, γ)-open, the proof follows.

Proof Sufficiency. Suppose A is an (α, β)-γ-open set. Let $x \in A$, then there exist γ-open sets U and V such that $x \in U$, $x \in V$ and $\alpha(U) \cup \beta(V) \subset A$. Thus implies $\alpha(U) \subset A$ and $\beta(V) \subset A$. Hence A is (α, γ)-open and (β, γ)-open.

Necessity. Let $x \in A$,since A is (α, γ)-open, then there exists a γ-open set U such that $x \in U$, $\alpha(U) \subset A$. Similarly, A is (β, γ)-open, then there exists a γ-open set V such that $x \in V$, $\beta(V) \subset A$, then $\alpha(U) \cup \beta(V) \subset A$. Hence A is an (α, β)-γ-open set.

Definition 1.5 [6] Let $(X, \mathcal{T})$ be a topological space and α be an operator. α is said to be regular if for each $x \in X$ and any open sets U and V containing x, there exists an open set W containing x such that $\alpha(W) \subset \alpha(U) \cap \alpha(V)$.

Definition 1.6 Let $(X, \mathcal{T})$ be a topological space and α, γ be operators on $\mathcal{T}$,(α, γ) is said to be (α, γ) -regular if for each $x \in X$ and any γ-open set U and V containing x,there exists a γ-open set W containing x such that $\alpha(W) \subset \alpha(U) \cap \alpha(V)$.

Definition 1.7 Let $(X, \mathcal{T})$ be a topological space and α, γ be operators on $\mathcal{T}$,$(X, \mathcal{T})$ is said to be (α, γ)-regular if for each $x \in X$ and every γ-open set U containing x, there exist γ-open sets W and S such that $\alpha(W) \cup \alpha(S) \subset U$.

Definition 1.8 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, $(X, \mathcal{T})$ is said to be (α, β)-γ-regular if for each $x \in X$ and every γ-open set U containing x, there exist γ-open sets W and S such that $\alpha(W) \cup \beta(S) \subset U$.

Definition 1.9 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, (α, β)-$\gamma : P(X) \to P(X)$ is said to be (α, γ)-pre open if for each $x \in X$ and every γ-open set U containing x,there exists an (α, β)-γ-open set W such that $x \in W$ and $W \subset \alpha(U)$.

Remark 1.1 By the above definition we can obtain the following statements:

Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$ and id: $P(X) \to P(X)$ be an identity operation, that is id$(A) = A$ for every subset A of $(X, \mathcal{T})$.

(1) If $\alpha = \beta = \gamma =$id, we have that A is an (α, β)-γ-open set if and only if (written in short as iff) A is an open set.

(2) If $\beta = \gamma =$id, we have that A is an (α, β)-γ-open set iff A is a γ-open set.

(3) If $\gamma =$id, we have that A is an (α, β)-γ-open set iff A is an (α, β)-open set.

Theorem 1.1 Let be $(X, \mathcal{T})$ a topological space and α, β, γ be operators on $\mathcal{T}$, then the following statements hold:

(1) Any union of α-open set is α-open.

(2) Any union of (α, β)-open set is (α, β) -open.

(3) Any union of (α, β)-γ-open set is (α, β)-γ-open.

Proof Statement (1) due to [3].

(2) Let $x \in \bigcup_{i\in I} A_i$. Then $x \in A_i$ for some i. Since each A_i is (α, β)-open, there exists a β-open set U_i such that $x \in U_i$ and $\alpha(U_i) \subset A_i \subset \bigcup_{i\in I} A_i$. Hence $\bigcup_{i\in I} A_i$ is an (α, β)-open set.

(3) Let $x \in \bigcup_{i\in I} A_i$. Then $x \in A_i$ for some i. Since each A_i is (α, β)-γ-open, there exist γ-open sets U_i and V_i such that $x \in U_i, x \in V_i$, $\alpha(U_i) \cup \beta(V_i) \subset A_i$, this implies that $\alpha(U_i) \cup \beta(V_i) \subset \bigcup_{i\in I} A_i$. Therefore $\bigcup_{i\in I} A_i$ is an (α, β)-γ-open set.

Theorem 1.2 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, then the following statements hold:

(1) $(X, \mathcal{T})$ is (α, β)-γ-regular iff $\mathcal{T}_{(\alpha,\beta)-\gamma} = \mathcal{T}_\gamma$.

(2) $(X, \mathcal{T})$ is (α, β)-γ-regular iff $(X, \mathcal{T})$ is (α, γ)-regular and (β, γ)-regular.

Proof (1) Sufficiency. We have $\mathcal{T}_{(\alpha,\beta)-\gamma} \subset \mathcal{T}_\gamma$ by Definition 1.5. Now we proof that $\mathcal{T}_{(\alpha,\beta)-\gamma} \supset \mathcal{T}_\gamma$. Let $A \in \mathcal{T}_\gamma$ and $x \in A$, since $(X, \mathcal{T})$ is (α, β)-γ-regular, then for γ-open set A containing x, there exist γ-open sets W and S such that $\alpha(W) \cup \beta(S) \subset A$. Hence A is (α, β)-γ-open, that is $A \in \mathcal{T}_{(\alpha,\beta)-\gamma}$, then $\mathcal{T}_{(\alpha,\beta)-\gamma} \supset \mathcal{T}_\gamma$. Thus $\mathcal{T}_{(\alpha,\beta)-\gamma} = \mathcal{T}_\gamma$.

Necessity. Let $x \in X$ and a subset A containing x, where A is a γ-open set. Suppose $\mathcal{T}_{(\alpha,\beta)-\gamma} = \mathcal{T}_\gamma$, then A is (α, β)-γ-open, thus there exist γ-open sets U and V such that $x \in U, x \in V$,$\alpha(U) \cup \beta(V) \subset A$. This implies $(X, \mathcal{T})$ is (α, β)-γ-regular.

(2) By using Theorem 1.2 (1) and Theorem 1.1 (3).

Proposition 1.1 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, If A and B are (α, β)-γ-open sets, (α, γ) and (β, γ) are regular operation. Then $A \cap B$ is (α, β)-γ-open.

Proof Suppose A and B are (α,β)-γ-open sets. Let $x\in A\cap B$. By Definition 1.4, there exist γ-open sets H and K such that $x\in H, x\in K$,$\alpha(H)\cup\beta(K)\subset A$. In the same way, there exist γ-open sets W and S such that $x\in W, x\in S$,$\alpha(W)\cup\beta(S)\subset B$. Furthermore, $(\alpha(H)\cup\beta(K))\cap(\alpha(W)\cup\beta(S))\subset A\cap B$, that is$(\alpha(H)\cap(\alpha(W))\cup(\beta(K)\cap\beta(S))\subset A\cap B$. Since (α,γ) and (β,γ) be regular operation. There exist γ-open sets U and V such that $\alpha(U)\subset\alpha(H)\cap(\alpha(W)$ and $\beta(V)\subset\beta(K)\cap\beta(S)$, respectively. Hence $\alpha(U)\cup\beta(V)\subset A\cap B$, this implies $A\cap B$ is (α,β)-γ-open.

Remark 1.2 If (α,γ) and (β,γ) are regular operation, it is obvious that $\mathcal{T}_{(\alpha,\beta)-\gamma}$ can form a topology on X by the above Theorem 1.1 (3) and Proposition 1.1.

Theorem 1.3 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$. Then the following statements hold:

(1) $x\in cl_{(\alpha,\beta)-\gamma}(A)$ iff $V\cap A\neq\emptyset$ for every (α,β)-γ-open V containing x.

(2) $A\subset cl_{(\alpha,\beta)-\gamma}(A)$ for every subset A of $(X,\mathcal{T})$.

(3) $cl_{(\alpha,\beta)-\gamma}(A)\subset cl_{(\alpha,\beta)-\gamma}(B)$ if $A\subset B$.

(4) A is $(\alpha,\beta)-\gamma$-closed iff $cl_{(\alpha,\beta)-\gamma}(A)=A$.

(5) $cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))=cl_{(\alpha,\beta)-\gamma}(A)$, that is $cl_{(\alpha,\beta)-\gamma}(A)$ is (α,β)-γ-closed.

(6) $cl_{(\alpha,\beta)-\gamma}(A)\cap G\subset cl_{(\alpha,\beta)-\gamma}(A\cap G)$ if (α,γ) and (β,γ) are regular operation for every subset A of $(X,\mathcal{T})$ and every (α,β)-γ-open set G.

(7) $cl_{(\alpha,\beta)-\gamma}(A\cup B)=cl_{(\alpha,\beta)-\gamma}(A)\cup cl_{(\alpha,\beta)-\gamma}(B)$ if (α,γ) and (β,γ) are regular operation for subsets $A,B\subset X$.

Proof Proof of (1)–(3) is obvious from Definition 1.4.

(4) Sufficiency. First we claim that $cl_{(\alpha,\beta)-\gamma}(A)\subset A$. Let $x\notin A$, by assumption A is (α,β)-γ-closed,then $x\in X-A$ and $X-A$ is (α,β)-γ-open. Hence there exist γ-open sets U and V such that $x\in U, x\in V$,$\alpha(U)\cup\beta(V)\subset X-A$. This implies $(\alpha(U)\cup\beta(V))\cap A=\emptyset$. Hence, by definition 1.4 we have that $x\notin cl_{(\alpha,\beta)-\gamma}(A)$. Thus $cl_{(\alpha,\beta)-\gamma}(A)\subset A$. From Theorem 1.3(2), therefore $cl_{(\alpha,\beta)-\gamma}(A)=A$.

Necessity. Let $x\in X-A$, that is $x\notin A$. By the assumption that $cl_{(\alpha,\beta)-\gamma}(A)=A$, then $x\notin cl_{(\alpha,\beta)-\gamma}(A)$. Hence there exist γ-open sets U and V containing x such that $(\alpha(U)\cup\beta(V))\cap A=\emptyset$, i.e., $\alpha(U)\cup\beta(V)\subset X-A$. Namely $X-A$ is (α,β)-γ-open. Therefore A is (α,β)-γ-closed.

(5) It is enough to proof $cl_{(\alpha,\beta)-\gamma}(A)\subset cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))$ by Theorem 1.3 (2) and Theorem 1.3 (3). Then we proof that $cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))\subset cl_{(\alpha,\beta)-\gamma}(A)$. Suppose $x\in cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))$, then for every (α,β)-γ-open set V containing x such that $V\cap cl_{(\alpha,\beta)-\gamma}(A)\neq\emptyset$. This implies $V\cap A\neq\emptyset$. Hence $x\in cl_{(\alpha,\beta)-\gamma}(A)$, this shows that $cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))\subset cl_{(\alpha,\beta)-\gamma}(A)$. Hence $cl_{(\alpha,\beta)-\gamma}(cl_{(\alpha,\beta)-\gamma}(A))=cl_{(\alpha,\beta)-\gamma}(A)$.

(6) Let $x\in cl_{(\alpha,\beta)-\gamma}(A)\cap G$, then $x\in cl_{(\alpha,\beta)-\gamma}(A)$ and $x\in G$. Let V be an (α,β)-γ-open set containing x. Then by Proposition 1.1, $V\cap G$ is also an (α,β)-γ-open set containing x. Since $x\in cl_{(\alpha,\beta)-\gamma}(A)$, then$(V\cap G)\cap A\neq\emptyset$. This implies that $V\cap(G\cap A)\neq\emptyset$ for every (α,β)-γ-open set V containing x. By Theorem 1.3 (1), we have $x\in cl_{(\alpha,\beta)-\gamma}(A\cap G)$. Hence $cl_{(\alpha,\beta)-\gamma}(A)\cap G\subset cl_{(\alpha,\beta)-\gamma}(A\cap G)$

(7) It is obvious that $cl_{(\alpha,\beta)-\gamma}(A)\cup cl_{(\alpha,\beta)-\gamma}B)\subset cl_{(\alpha,\beta)-\gamma}(A\cup(B)$ by Theorem 1.3 (3). Let $x\notin cl_{(\alpha,\beta)-\gamma}(A)\cup cl_{(\alpha,\beta)-\gamma}(B)$, this implies that $x\notin cl_{(\alpha,\beta)-\gamma}(A)$ and $x\notin cl_{(\alpha,\beta)-\gamma}(B)$. Then there exist γ-open sets U and V containing x such that $(\alpha(U)\cup\beta(V))\cap A=\emptyset$ and there exist γ-open sets W and S containing x such that $(\alpha(W)\cup\beta(S))\cap B=\emptyset$. Thus, we have $((\alpha(U)\cap\alpha(W))\cup(\beta(V)\cap\beta(S)))\cap(A\cup B)=\emptyset$. Since (α,γ) and (β,γ) are regular operation, then there exists a γ-open set E containing x such that $\alpha(E)\subset\alpha(U)\cap\alpha(W)$ and there exists γ-open set F containing x such that $\beta(F)\subset\beta(V)\cap\beta(S)$, respectively. Hence $((\alpha(E)\cup\beta(F))\cap(A\cup B)=\varnothing$. That is $x\notin cl_{(\alpha,\beta)-\gamma}(A\cup B)$, then $cl_{(\alpha,\beta)-\gamma}(A\cup(B)\subset cl_{(\alpha,\beta)-\gamma}(A)\cup cl_{(\alpha,\beta)-\gamma}B)$. Hence $cl_{(\alpha,\beta)-\gamma}(A)\cup cl_{(\alpha,\beta)-\gamma}(B)=cl_{(\alpha,\beta)-\gamma}(A\cup B)$.

Remark 1.3 From Definition 1.4, we have the following statements:

(1) $A\subset CL_{(\alpha,\beta)-\gamma}(A)$ for every subset A of $(X,\mathcal{T})$.

(2) $CL_{(\alpha,\beta)-\gamma}(A)\subset CL_{(\alpha,\beta)-\gamma}(B)$ if $A\subset B$.

(3) A is (α,β)-γ-closed iff $CL_{(\alpha,\beta)-\gamma}(A)=A$.

(4) $CL_{(\alpha,\beta)-\gamma}(CL_{(\alpha,\beta)-\gamma}(A))=CL_{(\alpha,\beta)-\gamma}(A)$, that is $CL_{(\alpha,\beta)-\gamma}(A)$ is (α,β)-γ-closed.

2 (α,β)-γ-T_i and (α,β)-γ-T_i^* $(i=0,1/2,1,2,5/2)$

Definition 2.1 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$.

(1) $(X,\mathcal{T})$ is said to be an (α,β)-γ-T_0 space if for every pair of distinct points $x,y\in X$, there exist two γ -open sets P_1 and P_2 such that $x\in P_1$, $y\notin\alpha(P_1)$ or $y\in P_2$, $x\notin\beta(P_2)$;

(2) $(X,\mathcal{T})$ is said to be an (α,β)-γ-$T_{1/2}$ space if every singleton set $\{x\}$ is (α,β)-γ-open or (α,β)-γ-closed where $x\in X$;

(3) $(X,\mathcal{T})$ is said to be an (α,β)-γ-T_1 space if for every pair of distinct points $x,y\in X$, there exist two γ-open sets P_1 and P_2 containing x,y, respectively, such that $x\notin\alpha(P_2)$ and $y\notin\beta(P_1)$;

(4) $(X,\mathcal{T})$ is said to be an (α,β)-γ-T_2 space if for every pair of distinct points $x,y\in X$, there exist two distinct γ-open sets P_1 and P_2 containing x,y, respectively, such that $\alpha(P_1)\cap\beta(P_2)=\emptyset$;

(5) $(X,\mathcal{T})$ is said to be an (α,β)-γ-$T_{5/2}$ space if for every pair of distinct points $x,y\in X$, there exist two distinct γ-open sets P_1 and P_2 containing x,y, respectively, such that $cl_{(\alpha,\beta)-\gamma}\alpha(P_1)\cap cl_{(\alpha,\beta)-\gamma}\beta(P_2)=\emptyset$.

Remark 2.1 From the above definition the following statements hold:

(1) If a space $(X,\mathcal{T})$ is an (α,β)-γ-T_1 space, then it is an (α,β)-γ-T_0 space;

(2) If a space $(X,\mathcal{T})$ is an (α,β)-γ-$T_{5/2}$ space, then it is an (α,β)-γ-T_2 space.

Theorem 2.1 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$. Suppose that (α,β)-γ $:P(X)\to P(X)$ is (α,γ)-preopen and (β,γ)-preopen. Then $(X,\mathcal{T})$ is an (α,β)-γ-T_0 space iff for every pair of distinct points x,y, we have $cl_{(\alpha,\beta)-\gamma}(\{x\})\neq cl_{(\alpha,\beta)-\gamma}(\{y\})$.

Proof Sufficiency. Suppose that $(X,\mathcal{T})$ is an (α,β)-γ-T_0 space,then for every pair of distinct points x,y, there exist two γ -open sets P_1 and P_2 such that $x\in P_1$, $y\notin\alpha(P_1)$ or $y\in P_2$, $x\notin\beta(P_2)$ by Definition 2.1. If $x\in P_1$, $y\notin\alpha(P_1)$, it follows from Definition 1.9 that there exists an (α,β)-γ-open set W such that $x\in W$ and $W\subset\alpha(P_1)$.

Hence $y \in X - \alpha(P_1) \subset X - W$. Because $X - W$ is an (α, β)-γ-closed set, we have that $cl_{(\alpha,\beta)-\gamma}(\{y\}) \subset cl_{(\alpha,\beta)-\gamma}(X - W)$ and so $cl_{(\alpha,\beta)-\gamma}(\{x\}) \neq cl_{(\alpha,\beta)-\gamma}(\{y\})$. Similarly, If $y \in P_2$, $x \notin \beta(P_2)$, it follows from Definition 1.9 that there exists an (α, β)-γ-open set S such that $y \in S$ and $S \subset \beta(P_2)$. Hence $x \in X - \beta(P_2) \subset X - S$. Because $X - S$ is an (α, β)-γ-closed set, we have that $cl_{(\alpha,\beta)-\gamma}(\{x\}) \subset cl_{(\alpha,\beta)-\gamma}(X - S)$ and so $cl_{(\alpha,\beta)-\gamma}(\{y\}) \neq cl_{(\alpha,\beta)-\gamma}(\{x\})$.

Necessity. Suppose that $x, y \in X$ and $x \neq y$, we have that $cl_{(\alpha,\beta)-\gamma}(\{x\}) \neq cl_{(\alpha,\beta)-\gamma}(\{y\})$. Then there exists a point $z \in cl_{(\alpha,\beta)-\gamma}(\{x\})$ but $z \notin cl_{(\alpha,\beta)-\gamma}(\{y\})$ or $z \in cl_{(\alpha,\beta)-\gamma}(\{y\})$ but $z \notin cl_{(\alpha,\beta)-\gamma}(\{x\})$. If $x \in cl_{(\alpha,\beta)-\gamma}(\{y\})$,then $cl_{(\alpha,\beta)-\gamma}(\{x\}) \subset cl_{(\alpha,\beta)-\gamma}(\{y\})$ by Theorem 1.3 (5). This implies that $z \in cl_{(\alpha,\beta)-\gamma}(\{y\})$. This contradiction shows that $x \notin cl_{(\alpha,\beta)-\gamma}(\{y\})$, i.e., there exist γ-open sets U, V such that $x \in U, V$ and $(\alpha(U) \cup \beta(V)) \cap \{y\} = \emptyset$ by Definition 1.4. Hence $x \in U, V$ and $y \notin \alpha(U), \beta(V)$. This implies that $(X, \mathcal{T})$ is an (α, β)-γ-T_0 space.

Theorem 2.2 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$. Then $(X, \mathcal{T})$ is an (α, β)-γ-T_1 space iff every singleton set $\{x\}$ is (α, β)-γ-closed.

Proof Sufficiency. Suppose that $(X, \mathcal{T})$ is an (α, β)-γ-T_1 space, then for every pair of distinct points $x, y \in X$, there exist two γ-open sets P_1 and P_2 containing x, y, respectively, such that $x \notin \alpha(P_2)$ and $y \notin \beta(P_1)$ by Definition 2.1. Similarly then for every pair of distinct points y, x, there exist two γ-open sets P_3 and P_4 containing x, y respectively such that $y \notin \alpha(P_3)$ and $x \notin \beta(P_4)$. Therefore we have $y \in P\alpha(P_2) \cup \beta(P_4) \subset X - \{x\}$. This implies $X - \{x\}$ is an (α, β)-γ-open set, i.e., $\{x\}$ is (α, β)-γ-closed.

Necessity. Suppose that $x, y \in X$ and $x \neq y$. By the assumption $X - \{x\}$ is an (α, β)-γ-open set containing y and $X - \{y\}$ is an (α, β)-γ-open set containing x. Then there exist two-open sets P_1 and P_2 containing y such that $y \in \alpha(P_1) \cup \beta(P_2) \subset X - \{x\}$ by Definition 1.4. Similarly there exist two -open sets P_3 and P_4 containing x such that $x \in \alpha(P_3) \cup \beta(P_4) \subset X - \{y\}$. This implies $x \notin \alpha(P_1)$, $x \notin \beta(P_2)$ and $y \notin \alpha(P_3)$, $y \notin \beta(P_4)$. Therefore there exist two γ-open sets P_1 and P_4 or P_2 and P_3 containing x, y respectively such that $x \notin \alpha(P_1)$ and $y \notin \beta(P_4)$ or $x \notin \beta(P_2)$ and $y \notin \alpha(P_3)$. Hence $(X, \mathcal{T})$ is an (α, β)-γ-T_1 space.

Theorem 2.3 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$, the following relations hold: (α, β)-γ-$T_{5/2} \Rightarrow (\alpha, \beta)$-$\gamma$-$T_2 \Rightarrow (\alpha, \beta)$-$\gamma$-$T_1 \Rightarrow (\alpha, \beta)$-$\gamma$-$T_{1/2} \Rightarrow (\alpha, \beta)$-$\gamma$-$T_0$, where $A \Rightarrow B$ represents A implies B.

Proof It is clear by above Theorems 2.1, 2.2 and Remark 2.1.

Definition 2.2 Let $(X, \mathcal{T})$ be a topological space and α, β, γ be operators on $\mathcal{T}$.

(1) $(X, \mathcal{T})$ is said to be an (α, β)-γ-T_0^* space if for every pair of distinct points $x, y \in X$, there exists an (α, β)-γ-open set P such that $x \in P$, $y \notin P$ or $y \in P$,$x \notin P$;

(2) $(X, \mathcal{T})$ is said to be an (α, β)-γ-T_1^* space if for every pair of distinct points $x, y \in X$, there exist two (α, β)-γ-open sets P_1 and P_2 containing x, y, respectively, such that $x \in P_1, y \notin P_1$ or $y \in P_2, x \notin P_2$;

(3) $(X, \mathcal{T})$ is said to be an (α, β)-γ-T_2^* space if for every pair of distinct points $x, y \in X$, there exist two (α, β)-γ-open sets P_1 and P_2 containing x, y, respectively, such that $P_1 \cap P_2 =$

$\emptyset$;

(4) $(X,\mathcal{T})$ is said to be an (α,β)-γ-$T^*_{5/2}$ space if for every pair of distinct points $x,y\in X$, there exist two (α,β)-γ-open sets P_1 and P_2 containing x,y, respectively, such that $CL_{(\alpha,\beta)-\gamma}(P_1)\cap CL_{(\alpha,\beta)-\gamma}(P_2)=\emptyset$.

Theorem 2.4 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$, Suppose that $(\alpha,\beta)-\gamma : P(X)\to P(X)$ is (α,γ)-preopen and (β,γ)-preopen. Then $(X,\mathcal{T})$ is an (α,β)-γ-T^*_0 space iff for every pair of distinct points x,y, we have $CL_{(\alpha,\beta)-\gamma}(\{x\})\neq CL_{(\alpha,\beta)-\gamma}(\{y\})$.

Proof Sufficiency. Suppose that $(X,\mathcal{T})$ is an (α,β)-γ-T^*_0 space, then for every pair of distinct points x,y, there exists an (α,β)-γ-open set P such that $x\in P$, $y\notin P$ or $y\in P$,$x\notin P$ by Definition 2.2. If $x\in P$, $y\notin P$, then $y\in X-P$, that is $\{y\}\subset X-P$. We have that $X-P$ is (α,β)-γ-closed. Thus $CL_{(\alpha,\beta)-\gamma}(\{y\})\subset X-P$. This implies $CL_{(\alpha,\beta)-\gamma}(\{x\})\neq CL_{(\alpha,\beta)-\gamma}(\{y\})$. Similarly, if $y\in P$,$x\notin P$, it follows that $\{x\}\subset X-P$. We have that $X-P$ is (α,β)-γ-closed. Thus $CL_{(\alpha,\beta)-\gamma}(\{x\})\subset X-P$. This implies $CL_{(\alpha,\beta)-\gamma}(\{x\})\neq CL_{(\alpha,\beta)-\gamma}(\{y\})$.

Necessity. Suppose that $x,y\in X$ and $x\neq y$, we have that

$$CL_{(\alpha,\beta)-\gamma}(\{x\})\neq CL_{(\alpha,\beta)-\gamma}(\{y\}).$$

Then there exists a point $z\in CL_{(\alpha,\beta)-\gamma}(\{x\})$ but $z\notin CL_{(\alpha,\beta)-\gamma}(\{y\})$ or $z\in CL_{(\alpha,\beta)-\gamma}(\{y\})$ but $z\notin CL_{(\alpha,\beta)-\gamma}(\{x\})$. If $x\in CL_{(\alpha,\beta)-\gamma}(\{y\})$, then $CL_{(\alpha,\beta)-\gamma}(\{x\})\subset CL_{(\alpha,\beta)-\gamma}(\{y\})$ by Remark 1.3 (2), (3). This implies that $z\in cl_{(\alpha,\beta)-\gamma}(\{y\})$. This contradiction shows that $x\notin CL_{(\alpha,\beta)-\gamma}(\{y\})$, then $X-CL_{(\alpha,\beta)-\gamma}(\{y\})$ is an (α,β)-γ-open set containing x but not y. This implies that $(X,\mathcal{T})$ is an (α,β)-γ-T^*_0 space.

Theorem 2.5 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$. Then $(X,\mathcal{T})$ is an (α,β)-γ-T^*_1 space iff every singleton set $\{x\}$ is (α,β)-γ-closed.

Proof Sufficiency. Suppose that $(X,\mathcal{T})$ is an (α,β)-γ-T^*_1 space, then for every pair of distinct points $x,y\in X$, there exist two (α,β)-γ-open sets P_1 and P_2 containing x,y, respectively, such that $x\in P_1$ and $y\notin P_1$ or $y\in P_2$ and $x\notin P_2$. Hence $y\in P_2\subset X-\{x\}$. This implies $X-\{x\}$ is an (α,β)-γ-open set, i.e., $\{x\}$ is (α,β)-γ-closed.

Necessity. Suppose that $x,y\in X$ and $x\neq y$. By the assumption $X-\{x\}$ is an (α,β)-γ-open set containing y and $X-\{y\}$ is an (α,β)-γ-open set containing x. Then $X-\{x\}$ and $X-\{y\}$ are the satisfied (α,β)-γ-open sets such that $y\in X-\{x\}$, $x\notin X-\{x\}$ and $x\in X-\{y\}$, $y\notin X-\{y\}$. Hence $(X,\mathcal{T})$ is an (α,β)-γ-T^*_1 space.

Remark 2.2 From Theorem 2.2 and Theorem 2.5 we obtain that (α,β)-γ-T_1 space is an (α,β)-γ-$^*T_1$ space, the converse is true.

Theorem 2.6 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$. The following relations hold: (α,β)-γ-$T^*_{5/2}\Rightarrow(\alpha,\beta)$-$\gamma$-$T^*_2\Rightarrow(\alpha,\beta)$-$\gamma$-$T^*_1\Rightarrow(\alpha,\beta)$-$\gamma$-$T^*_0$.

Proof It is clear by above Theorem 2.4, 2.5 and Definition 2.2.

Theorem 2.7 Let $(X,\mathcal{T})$ be a topological space and α,β,γ be operators on $\mathcal{T}$.The following statements hold:

(1) Every (α,β)-γ-T_0^* space is an (α,β)-γ-T_0 space;

(2) Every (α,β)-γ-T_1^* space is an (α,β)-γ-T_1 space;

(3) Every (α,β)-γ-T_2^* space is an (α,β)-γ-T_2 space.

Proof (1), (3) proofs are obvious by Definition 2.2. The proof for (2) is obtained by Remark 2.2.

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(α,β)-γ开集与(α,β)-γ-T_i空间

吴耀强

(宿迁学院教师教育系, 江苏 宿迁 223800)

摘要: 本文首先给出(α,β)-γ开集定义, 获得了(α,β)-γ开集性质; 然后引入了(α,β)-γ-T_i空间和(α,β)-γ-T_i^*空间概念$(i=0,1/2,1,2,5/2)$,并得到它们更广泛的拓扑性质.

关键词: α开集; (α,β)开集; (α,β)-γ开集; (α,β)-γ-T_i空间; (α,β)-γ-T_i^*空间

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