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ON NONSINGULARITY AND GROUP INVERSE OF LINEAR COMBINATIONS OF FOUR TRIPOTENT MATRICES

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Abstract: In this paper, some conditions for the nonsingularity and group inverses of linear combinations of four tripotent matrices is mainly established. By using the method of matrix decomposition, we obtain some formulae for the inverses and group inverses of them, which perfects the theory of nonsingularity of linear combinations of k-potent matrices.

Keywords: inverse; group inverse; linear combination; tripotent matrices

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1 Introduction

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C} - 0$. For a positive integer n ,let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices over $\mathbb{C}$. The symbols rank(A), A^*, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ stand for the rank, conjugate transpose, the range space, and the null space of $A \in \mathcal{M}_n$, respectively. Recall that a matrix$A \in \mathcal{M}_n$ is tripotent if $A^3 = A$.

The nonsingularity of linear combinations of idempotent matrices and k-potent matrices was studied in, for example [1–4]. The nonsingularities of the combinations $c_1P+c_2Q-c_3PQ$ and $c_1P + c_2Q - c_3PQ - c_4QP - c_5PQP$ of two idempotent matrices P, Q were investigated in [5] and [6], respectively. The considerations of this paper are inspired by Benítez et al. [7]. They established necessary and sufficient conditions for the nonsingularity of combinations $T = c_1T_1 + c_2T_2 + c_3T_3 - c_4(T_1T_2 + T_3T_1 + T_2T_3)$ of three trioptent matrices and gave some formulae for the inverse of $T = c_1T_1 + c_2T_2 + c_3T_3 - c_4(T_1T_2 + T_3T_1 + T_2T_3)$ under some conditions.

In this paper we consider a combination of the form

$$T = c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4, \tag{1.1}$$

where $c_1, c_2, c_3, c_4 \in \mathbb{C}^*$ and $T_1, T_2, T_3, T_4 \in \mathcal{M}_n$ are three tripotent matrices. The purpose of this paper is mainly twofold: first, to establish necessary and sufficient conditions for the

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nonsingularity of combinations of form (1.1); second, to give some formulae for the inverse of them.

Now, let us give the following additional concepts and properties. A given matrix $A \in \mathcal{M}_n$ is said to be group invertible if there exists a matrix$X \in \mathcal{M}_n$ such that

$$AXA = A; XAX = X; AX = XA \tag{1.2}$$

hold. If such an X$\in \mathcal{M}_n$ exists, then it is unique, customarily denoted by$A^\sharp$ (see e.g. [8]). A matrix $A \in \mathcal{M}_n$ is group invertible if and only if there exist nonsingular $S \in \mathcal{M}_n$, $C \in \mathcal{M}_r$ such that $A = S(C \oplus 0)S^{-1}$,r being the rank of A (see [9], Exercise 5.10.12). In this situation, one has $A^\sharp = S(C^{-1} \oplus 0)S^{-1}$. This latter representation implies that any diagonalizable matrix is group invertible. Moreover, it is well known that A $\in \mathcal{M}_n$ is nonsingular if and only if $\mathcal{N}(A) = 0$. Furthermore, if $A \in \mathcal{M}_n$ and k is a natural number greater than 1, then A satisfies$A^k = A$ if and only if A is diagonalizable and the spectrum of A is contained in $\sqrt[k-1]{1} \cup 0$ (see e.g. [10]).

Special types of matrices, such as idempotents, tripotents, etc., are very useful in many contexts and they have been extensively studied in the literature. For example, quadratic forms with idempotent matrices are used extensively in statistical theory. So it is worth to stress and spread these kinds of results. Evidently, if T is a tripotent matrix, then T is group invertible and $T^\sharp = T$. Many of the results given in this work will be given in terms of group invertible matrices.

2 Main Results and Proofs

If $A \in \mathcal{M}_n$ satisfying $A^2 = I_n$, We call A an involutory matrix. On the inverse of linear combinations of involutory matrices, we have the following results.

Lemma 2.1 Let $A, B, C, D \in \mathcal{M}_n$ be involutory matrices and they are mutually commuting, $a, b, c, d \in \mathbb{C}^*$ and

$$\begin{gathered}(a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)\\(a-b-c+d)(a-b+c-d)(a+b-c-d)(a-b-c-d) \neq 0,\end{gathered}$$

then

$$\begin{aligned}&(aA+bB+cC+dD)^{-1}\\=\ &\frac{1}{m}(x_1A+x_2B+x_3C+x_4D+z_1ABC+z_2ABD+z_3ACD+z_4BCD), \end{aligned}\tag{2.1}$$

where

$$\begin{aligned}m &= ax_1+bx_2+cx_3+dx_4 = (a+b+c+d)(a-b+c+d)(a+b-c+d)\\&\quad (a+b+c-d)(a-b-c+d)(a-b+c-d)(a+b-c-d)(a-b-c-d),\\z_1 &= 2bc(a^5-2a^3b^2+ab^4-2a^3c^2-2ab^2c^2+ac^4+2a^3d^2+2ab^2d^2+2ac^2d^2-3ad^4),\\z_2 &= 2bd(a^5-2a^3b^2+ab^4+2a^3c^2+2ab^2c^2-3ac^4-2a^3d^2-2ab^2d^2+2ac^2d^2+ad^4),\\z_3 &= 2cd(a^5+2a^3b^2-3ab^4-2a^3c^2+2ab^2c^2+ac^4-2a^3d^2+2ab^2d^2-2ac^2d^2+ad^4),\end{aligned}$$

$$\begin{aligned}
z_4 &= 2bcd(-3a^4+2a^2b^2+b^4+2a^2c^2-2b^2c^2+c^4+2a^2d^2-2b^2d^2-2c^2d^2+d^4),\\
x_1 &= -(-a^7+3a^5b^2-3a^3b^4+ab^6+3a^5c^2-2a^3b^2c^2-ab^4c^2-3a^3c^4-ab^2c^4+ac^6+3a^5d^2\\
&\quad -2a^3b^2d^2-ab^4d^2-2a^3c^2d^2+10ab^2c^2d^2-ac^4d^2-3a^3d^4-ab^2d^4-ac^2d^4+ad^6),\\
x_2 &= -b(a^6-3a^4b^2+3a^2b^4-b^6-a^4c^2-2a^2b^2c^2+3ab^4c^2-a^2c^4-3b^2c^4+c^6-a^4d^2\\
&\quad -2a^2b^2d^2+3b^4d^2+10a^2c^2d^2-2b^2c^4d^2-c^4d^2-a^2d^4-3b^2d^4-c^2d^4+d^6),\\
x_3 &= -c(a^6-a^4b^2-a^2b^4+b^6-3a^4c^2-2a^2b^2c^2-3b^4c^2+3a^2c^4+3b^2c^4-c^6-a^4d^2\\
&\quad +10a^2b^2d^2-b^4d^2-2a^2c^2d^2-2b^2c^2d^2+3c^4d^2-a^2d^4-b^2d^4-3c^2d^4+d^6),\\
x_4 &= d(-a^6+a^4b^2+a^2b^4-b^6+a^4c^2-10a^2b^2c^2+b^4c^2+a^2c^4+b^2c^4-c^6+3a^4d^2+2a^2b^2d^2\\
&\quad +3b^4d^2+2a^2c^2d^2+2b^2c^2d^2+3c^4d^2-3a^2d^4-3b^2d^4-3c^2d^4+d^6).
\end{aligned}$$

Corollary 2.1 Let $A, B, C \in \mathcal{M}_n$ be involutiory matrices and they are mutually commuting, $a, b, c \in \mathbb{C}^*$ and $(a+b+c)(a-b+c)(a+b-c)(a-b-c) \neq 0$, then

$$(aA+bB+cC)^{-1} = \frac{1}{m}(xA+yB+zC+wABC), \tag{2.2}$$

where

$$\begin{aligned}
&m = ax_1+bx_2+cx_3+dx_4 = (a+b+c)(a-b+c)(a+b-c)(a-b-c),\\
&x = a^3-ab^2-ac^2, y = b^3-bc^2-ba^2, z = c^3-ca^2-cb^2, w = 2abc.
\end{aligned}$$

Proof In Lemma 2.1, put $D=0$ and $d=0$, we will obtain Corollary 2.1.

About group inverses of linear combinations of three tripotent matrices, we give the following Lemmas.

Lemma 2.2 (see [7], Theorem 2.2) Let $T_1, T_2, T_3 \in \mathcal{M}_n \setminus \{0\}$ be three mutually commuting tripotent matrices and $c_1, c_2, c_3 \in \mathbb{C}^*$ such that $(c_i+c_j)(c_i-c_j) \neq 0$ $(i, j = 1, 2, 3$ and $i \neq j)$ and

$$(c_1+c_2+c_3)(c_1-c_2+c_3)(c_1+c_2-c_3)(c_1-c_2-c_3) \neq 0,$$

then

$$\mathcal{R}(T_1^2+T_2^2+T_3^2) = \mathcal{R}(c_1T_1+c_2T_2+c_3T_3), \mathcal{N}(T_1^2+T_2^2+T_3^2) = \mathcal{N}(c_1T_1+c_2T_2+c_3T_3),$$

$c_1T_1+c_2T_2+c_3T_3$ is group invertible, and

$$\begin{aligned}
(c_1T_1+c_2T_2+c_3T_3)^\sharp &= q(T_1,T_2,T_3)T_1^2T_2^2T_3^2 + p_{c_1,c_2}(T_1,T_2)T_1^2T_2^2(I_n-T_3^2)\\
&\quad + p_{c_1,c_3}(T_1,T_3)T_1^2(I_n-T_2^2) + p_{c_2,c_3}(T_2,T_3)(I_n-T_1^2), \quad (2.3)
\end{aligned}$$

where $p_{a,b} : \mathbb{C}^2 \to \mathbb{C}$ and $q : \mathbb{C}^3 \to \mathbb{C}$ are the following complex polynomials,

$$\begin{aligned}
p_{a,b}(z,w) &= \frac{b^2}{a(a^2-b^2)}zw^2 + \frac{a^2}{b(b^2-a^2)}z^2w + \frac{1}{a}z + \frac{1}{b}w \ \ (a, b \in \mathbb{C}^*, a^2 \neq b^2),\\
q(z,w,u) &= \frac{(c_1^3-c_1c_2^2-c_1c_3^2)z + (c_2^3-c_2c_3^2-c_2c_1^2)w + (c_3^3-c_3c_1^2-c_3c_2^2)u + 2c_1c_2c_3zwu}{(c_1+c_2+c_3)(c_1-c_2+c_3)(c_1+c_2-c_3)(c_1-c_2-c_3)}.
\end{aligned} \tag{2.4}$$

In particular, if $T_1^2+T_2^2+T_3^2$ is nonsingular, then $c_1T_1+c_2T_2+c_3T_3$ is nonsingular and $(c_1T_1+c_2T_2+c_3T_3)^{-1}$ is given by (2.3).

Lemma 2.3 (see [7], Theorem 2.3) Let $T_1, T_2, T_3 \in \mathcal{M}_n$ be three mutually commuting tripotent matrices, then $T_1+T_2+T_3$ is nonsingular if and only if

$$I_n+T_1T_2+T_2T_3+T_3T_1+T_1T_2T_3$$

and $T_1^2+T_2^2+T_3^2$ are nonsingular.

Now we give the nonsingularity and group inverses of linear combinations of four tripotent matrices. And we denote

$$\begin{aligned}&(aA+bB+cC+dD)^{-1}\\ =\ &\frac{1}{m}(x_1A+x_2B+x_3C+x_4D+z_1ABC+z_2ABD+z_3ACD+z_4BCD)\\ =\ &h_{a,b,c,d}(A,B,C,D)\end{aligned}$$

in (2.1), $(c_1T_1+c_2T_2+c_3T_3)^\sharp=k_{c_1,c_2,c_3}(T_1,T_2,T_3)$ in (2.3), $p_{a,b}(z,w)$ and $q(z,w,u)$ are the same in (2.4).

Theorem 2.1 Let $T_1, T_2, T_3, T_4 \in \mathcal{M}_n \setminus \{0\}$ be four mutually commuting tripotent matrices and $c_1, c_2, c_3, c_4 \in \mathbb{C}^*$ such that

$$\begin{aligned}&(c_i+c_j)(c_i-c_j)\neq 0 \ \ (i,j=1,2,3,4 \ \text{and} \ i\neq j),\\ &(c_i+c_j+c_k)(c_i-c_j+c_k)(c_i+c_j-c_k)(c_i-c_j-c_k)\neq 0 \ \ (i,j=1,2,3,4 \ \text{and} \ i<j<k),\\ &(c_1+c_2+c_3+c_4)(c_1-c_2+c_3+c_4)(c_1+c_2-c_3+c_4)(c_1+c_2+c_3-c_4)\\ &(c_1-c_2-c_3+c_4)(c_1-c_2+c_3-c_4)(c_1+c_2-c_3-c_4)(c_1-c_2-c_3-c_4)\neq 0,\end{aligned}$$

then

$$\begin{aligned}\mathcal{R}(T_1^2+T_2^2+T_3^2+T_4^2)&=\mathcal{R}(c_1T_1+c_2T_2+c_3T_3+c_4T_4),\\ \mathcal{N}(T_1^2+T_2^2+T_3^2+T_4^2)&=\mathcal{N}(c_1T_1+c_2T_2+c_3T_3+c_4T_4),\end{aligned}$$

$c_1T_1+c_2T_2+c_3T_3+c_4T_4$ is group invertible, and

$$\begin{aligned}&(c_1T_1+c_2T_2+c_3T_3+c_4T_4)^\sharp=h_{c_1,c_2,c_3,c_4}(T_1,T_2,T_3,T_4)T_1^2T_2^2T_3^2T_4^2\\ &+q(T_1,T_2,T_3)T_1^2T_2^2T_3^2(I_n-T_4^2)+k_{c_1,c_2,c_4}(T_1,T_2,T_4)T_1^2T_2^2(I_n-T_3^2)\\ &+k_{c_1,c_3,c_4}(T_1,T_3,T_4)T_1^2(I_n-T_2^2)+k_{c_2,c_3,c_4}(T_2,T_3,T_4)(I_n-T_1^2).\end{aligned}\tag{2.5}$$

Proof By (see [9], Exercise 5.10.12), there exist nonsingular matrices $S_1 \in \mathcal{M}_n$ and $X_1 \in \mathcal{M}_{n-t}$ such that $T_1=S_1(X_1\oplus 0)S_1^{-1}$. The tripotency of T_1 and the nonsingularity of X_1 lead to $X_1^2=I_{n-t}$ as $T_1T_j=T_jT_1, j=2,3,4$. We can write matrices T_2, T_3 and T_4 as follows

$$T_2=S_1\begin{pmatrix}X_2&0\\0&E_2\end{pmatrix}S_1^{-1},\quad T_3=S_1\begin{pmatrix}X_3&0\\0&E_3\end{pmatrix}S_1^{-1},\quad T_4=S_1\begin{pmatrix}X_4&0\\0&E_4\end{pmatrix}S_1^{-1}\tag{2.6}$$

with $E_2, E_3, E_4 \in \mathcal{M}_t$, and

$$X_iX_j = X_jX_i \qquad (i, j = 1, 2, 3, 4). \tag{2.7}$$

Let us notice that matrices $X_2, X_3, X_4, E_2, E_3, E_4$ are tripotent because T_2, T_3 and T_4 are tripotent. By applying again (see [9], Exercise 5.10.12), there exist nonsingular matrices $S_2 \in \mathcal{M}_{n-t}$ and $Y_2 \in \mathcal{M}_{n-t-s}$ such that $X_2 = S_2(Y_2 \oplus 0)S_2^{-1}$,where $Y_2^2 = I_{n-t-s}$. From (2.7) we can write

$$X_1 = S_2 \begin{pmatrix} Y_1 & 0 \\ 0 & D_1 \end{pmatrix} S_2^{-1}, \quad X_3 = S_2 \begin{pmatrix} Y_3 & 0 \\ 0 & D_3 \end{pmatrix} S_2^{-1}, \quad X_4 = S_2 \begin{pmatrix} Y_4 & 0 \\ 0 & D_4 \end{pmatrix} S_2^{-1}. \tag{2.8}$$

Observe that $Y_1^2 = I_{n-t-s}, D_1^2 = I_s, Y_i^3 = Y_i, D_i^3 = D_i, (i, j = 3, 4)$ and

$$Y_iY_j = Y_jY_i \;\; D_iD_j = D_jD_i \;\; (i, j = 1, 2, 3, 4). \tag{2.9}$$

By applying again (see [9], Exercise 5.10.12), there exist nonsingular matrices $S_3 \in \mathcal{M}_{n-t-s}$ and $Z_3 \in \mathcal{M}_{n-t-s-r}$ such that $Y_3 = S_3(Z_3 \oplus 0)S_3^{-1}$, where $Z_3^2 = I_{n-t-s-r}$. From (2.8) we can write

$$Y_1 = S_3 \begin{pmatrix} Z_1 & 0 \\ 0 & C_1 \end{pmatrix} S_3^{-1}, \quad Y_2 = S_3 \begin{pmatrix} Z_2 & 0 \\ 0 & C_2 \end{pmatrix} S_3^{-1}, \quad Y_4 = S_3 \begin{pmatrix} Z_4 & 0 \\ 0 & C_4 \end{pmatrix} S_3^{-1}. \tag{2.10}$$

Observe that $Z_1^2 = Z_2^2 = I_{n-t-s-r}, C_1^2 = C_2^2 = I_r, Z_4^3 = Z_4, C_4^3 = C_4$ and $Z_iZ_j = Z_jZ_i, C_iC_j = C_jC_i \;\; (i, j = 1, 2, 3, 4)$.

Finally, utilize again (see [9], Exercise 5.10.12) to matrix Z_4 to obtain nonsingular matrices $S_4 \in \mathcal{M}_{n-t-s-r}$ and $A_4 \in \mathcal{M}_u$ such that $Z_4 = S_4(A_4 \oplus 0)S_4^{-1}$ with $A_4^2 = I_u$. By carrying out the same routine as before, we can write

$$Z_1 = S_4 \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} S_4^{-1}, Z_2 = S_4 \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix} S_4^{-1}, Z_3 = S_4 \begin{pmatrix} A_3 & 0 \\ 0 & B_3 \end{pmatrix} S_4^{-1}. \tag{2.11}$$

where $A_i^2 = I_{n-t-s-r-u}, B_i^2 = I_u \;\; (i = 1, 2, 3)$ and $A_iA_j = A_jA_i, B_iB_j = B_jB_i \;\; (i, j = 1, 2, 3, 4)$.

Let us define $m = n-t-r-u$. By setting $S = S_1(S_2 \oplus I_t)(S_3 \oplus I_s \oplus I_t)(S_4 \oplus I_r \oplus I_s \oplus I_t)$, one easily has

$$T_1 = S(A_1 \oplus B_1 \oplus C_1 \oplus D_1 \oplus 0)S^{-1}, T_2 = S(A_2 \oplus B_2 \oplus C_2 \oplus 0 \oplus E_2)S^{-1},$$
$$T_3 = S(A_3 \oplus B_3 \oplus 0 \oplus D_3 \oplus E_3)S^{-1},\; T_4 = S(A_4 \oplus 0 \oplus C_4 \oplus D_4 \oplus E_4)S^{-1},$$

and the matrices $A_i^2 = I_m \;\; (i = 1, 2, 3, 4)$; $B_i^2 = I_u \;\; (i = 1, 2, 3)$; $C_i^2 = I_r \;\; (i = 1, 2), C_4^3 = C_4$; $D_1^2 = I_s, D_i^3 = D_i (i = 3, 4)$; $E_i^3 = E_i \;\; (i = 2, 3, 4)$. In addition, the families $\{A_i\}_{i=1,2,3,4}$, $\{B_i\}_{i=1,2,3}$, $\{C_i\}_{i=1,2,4}$, $\{D_i\}_{i=1,3,4}$,$\{E_i\}_{i=2,3,4}$ are commutative.

Observe that

$$T_1^2+T_2^2+T_3^2+T_4^2 = S[4I_m\oplus 3I_u\oplus(C_1^2+C_2^2+C_4^2)\oplus(D_1^2+D_3^2+D_4^2)\oplus(E_2^2+E_3^2+E_4^2)]S^{-1} \quad (2.12)$$

and

$$\begin{aligned} &c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4 = S[(c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4) \oplus (c_1B_1 + c_2B_2 + c_3B_3) \\ &\oplus(c_1C_1 + c_2C_2 + c_4C_4) \oplus (c_1D_1 + c_3D_3 + c_4D_4) \oplus (c_2E_2 + c_3E_3 + c_4E_4)]S^{-1}. \end{aligned} \quad (2.13)$$

By Lemma 2.1, Corollary 2.1 and Lemma 2.2, we have that $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$ and $c_1B_1 + c_2B_2 + c_3B_3$ are nonsingular and

$$\begin{aligned} &(c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4)^{-1} = h_{c_1,c_2,c_3,c_4}(A_1, A_2, A_3, A_4), \\ &(c_1B_1 + c_2B_2 + c_3B_3)^{-1} = q(B_1, B_2, B_3). \end{aligned}$$

Since $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$ and $c_1B_1 + c_2B_2 + c_3B_3$ are nonsingular, then

$$\mathcal{N}(c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4) = \mathcal{N}(4I_m), \mathcal{N}(c_1B_1 + c_2B_2 + c_3B_3) = \mathcal{N}(3I_u).$$

Lemma 2.2 leads to

$$\begin{aligned} &\mathcal{N}(c_1C_1 + c_2C_2 + c_4C_4) = \mathcal{N}(C_1^2 + C_2^2 + C_4^2), \\ &\mathcal{N}(c_1D_1 + c_3D_3 + c_4D_4) = \mathcal{N}(D_1^2 + D_3^2 + D_4^2), \\ &\mathcal{N}(c_2E_2 + c_3E_3 + c_4E_4) = \mathcal{N}(E_2^2 + E_3^2 + E_4^2), \end{aligned}$$

and analogous identities for the range space. By considering (2.12) and (2.13), and Lemma 2.2 we get that the null space (range space) of $c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4$ equals to the null space (range space)$T_1^2 + T_2^2 + T_3^2 + T_4^2$.

By Lemma 2.2 we have the group invertibility of $c_1C_1+c_2C_2+c_4C_4$, $c_1D_1+c_3D_3+c_4D_4$, and $c_2E_2 + c_3E_3 + c_4E_4$, we get

$$\begin{aligned} &(c_1C_1 + c_2C_2 + c_4C_4)^\sharp = k_{c_1,c_2,c_4,}(C_1, C_2, C_4), \\ &(c_1D_1 + c_3D_3 + c_4D_4)^\sharp = k_{c_1,c_3,c_4}(D_1, D_3, D_4), \\ &(c_2E_2 + c_3E_3 + c_4E_4)^\sharp = k_{c_2,c_3,c_4}(E_2, E_3, E_4). \end{aligned}$$

The second part of Lemma 2.3 leads to the group invertibility of $c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4$ and

$$\begin{aligned} &(c_1T_1 + c_2T_2 + c_3T_3 + c_4T_4)^\sharp = S(h_{c_1,c_2,c_3,c_4}(A_1, A_2, A_3, A_4) \oplus q(B_1, B_2, B_3) \\ &\oplus k_{c_1,c_2,c_4,}(C_1, C_2, C_4) \oplus k_{c_1,c_3,c_4}(D_1, D_3, D_4) \oplus k_{c_2,c_3,c_4}(E_2, E_3, E_4))S^{-1}. \end{aligned} \quad (2.14)$$

Now, observe that

$$\begin{aligned} &S(I_m \oplus 0 \oplus 0 \oplus 0 \oplus 0)S^{-1} = T_1^2T_2^2T_3^2T_4^2, \\ &S(0 \oplus I_u \oplus 0 \oplus 0 \oplus 0)S^{-1} = T_1^2T_2^2T_3^2(I_n - T_4^2), \end{aligned}$$

$$
\begin{aligned}
S(0\oplus 0\oplus I_r\oplus 0\oplus 0)S^{-1} &= T_1^2T_2^2(I_n-T_3^2),\\
S(0\oplus 0\oplus 0\oplus I_s\oplus 0)S^{-1} &= T_1^2(I_n-T_2^2),\\
S(0\oplus 0\oplus 0\oplus 0\oplus I_t)S^{-1} &= I_n-T_1^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& S(h_{c_1,c_2,c_3,c_4}(A_1,A_2,A_3,A_4)\oplus q(B_1,B_2,B_3)\oplus 0\oplus 0\oplus 0)S^{-1}\\
= \ & h_{c_1,c_2,c_3,c_4}(T_1,T_2,T_3,T_4)T_1^2T_2^2T_3^2T_4^2+q(T_1,T_2,T_3)T_1^2T_2^2T_3^2(I_n-T_4^2), \qquad (2.15)\\
& S(0\oplus 0\oplus k_{c_1,c_2,c_4,}(C_1,C_2,C_4)\oplus 0\oplus 0)S^{-1}=k_{c_1,c_2,c_4,}(T_1,T_2,T_4)T_1^2T_2^2(I_n-T_3^2), \qquad (2.16)\\
& S(0\oplus 0\oplus 0\oplus k_{c_1,c_3,c_4}(D_1,D_3,D_4)\oplus 0)S^{-1}=k_{c_1,c_3,c_4}(T_1,T_3,T_4)T_1^2(I_n-T_2^2), \qquad (2.17)\\
& S(0\oplus 0\oplus 0\oplus 0\oplus k_{c_2,c_3,c_4}(E_2,E_3,E_4))S^{-1}=k_{c_2,c_3,c_4}(T_2,T_3,T_4)(I_n-T_1^2). \qquad (2.18)
\end{aligned}
$$

Considering (2.14)–(2.18) finishes the proof.

Corollary 2.2 Let $T_1,T_2,T_3,T_4\in\mathcal{M}_n\setminus\{0\}$ be four mutually commuting tripotent matrices and $c_1,c_2,c_3,c_4\in\mathbb{C}^*$,if $T_1^2+T_2^2+T_3^2+T_4^2$ is nonsingular,then $c_1T_1+c_2T_2+c_3T_3+c_4T_4$ is nonsingular and $(c_1T_1+c_2T_2+c_3T_3+c_4T_4)^{-1}$ is given by (2.5).

Theorem 2.2 Let $T_1,T_2,T_3,T_4\in\mathcal{M}_n\setminus\{0\}$ be four mutually commuting tripotent matrices, then $T_1+T_2+T_3+T_4$ is nonsingular if and if only

$$I_n+T_1T_2+T_1T_3+T_1T_4+T_2T_3+T_2T_4+T_3T_4+T_1T_2T_3+T_1T_2T_4+T_1T_3T_4+T_2T_3T_4+T_1T_2T_3T_4$$

and $T_1^2+T_2^2+T_3^2+T_4^2$ are nonsingular.

Proof Since T_1,T_2,T_3 and T_4 are tripotent and mutually commutating, they are simultaneously diagonalizable (see, e.g. [11], p.52). Hence there is a single similarity matrix $S\in\mathcal{M}_n$ such that $T_1=S\mathrm{diag}(\lambda_1,\cdots,\lambda_n)S^{-1},T_2=S\mathrm{diag}(\mu_1,\cdots,\mu_n)S^{-1}$, $T_3=S\mathrm{diag}(\gamma_1,\cdots,\gamma_n)S^{-1}$, $T_4=S\mathrm{diag}(\tau_1,\cdots,\tau_n)S^{-1}$ being $\{\lambda_i\}_{i=1}^n$, $\{\mu_i\}_{i=1}^n$, $\{\gamma_i\}_{i=1}^n$ and $\{\tau_i\}_{i=1}^n$ are the ordered sets of eigenvalues of T_1,T_2,T_3 and T_4 with proper multiplicities, respectively. On the other hand,

$$
\begin{aligned}
& T_1+T_2+T_3+T_4=S\mathrm{diag}(\lambda_1+\mu_1+\gamma_1+\tau_1,\cdots,\lambda_n+\mu_n+\gamma_n+\tau_n)S^{-1} \qquad (2.19)\\
& I_n+T_1T_2+T_1T_3+T_1T_4+T_2T_3+T_2T_4+T_3T_4+T_1T_2T_3\\
& +T_1T_2T_4+T_1T_3T_4+T_2T_3T_4+T_1T_2T_3T_4\\
= \ & S\mathrm{diag}(g(\lambda_1+\mu_1+\gamma_1+\tau_1),\cdots,g(\lambda_n+\mu_n+\gamma_n+\tau_n))S^{-1} \qquad (2.20)
\end{aligned}
$$

and

$$T_1^2+T_2^2+T_3^2+T_4^2=S\mathrm{diag}(\lambda_1^2+\mu_1^2+\gamma_1^2+\tau_1^2,\cdots,\lambda_n^2+\mu_n^2+\gamma_n^2+\tau_n^2)S^{-1}, \qquad (2.21)$$

where $g:\mathbb{C}^4\to\mathbb{C}$ is given by

$$g(x,y,z,w)=1+xy+xz+xw+yz+yw+zw+xyz+xyw+xzw+yzw+xyzw.$$

Assume that $T_1+T_2+T_3+T_4$ is nonsingular. From (2.19) we get $\lambda_i+\mu_i+\gamma_i+\tau_i\neq 0$ for any $i=1,2,\cdots,n$ and hence $(\lambda_i,\mu_i,\gamma_i,\tau_i)\in\Phi^4\setminus\Omega$ for all $i=1,2,\cdots,n$, where

$$\begin{aligned}
\Phi &= \{-1,0,1\},\\
\Omega &= \{(0,0,1,-1),(0,0,-1,1),(0,1,0,-1),(0,-1,0,1),(0,1,-1,0),(0,-1,1,0),\\
&\quad (1,0,0,-1),(-1,0,0,1),(1,0,-1,0),(-1,0,1,0),(1,-1,0,0),(-1,1,0,0),(1,1,-1,-1),\\
&\quad (1,-1,1,-1),(1,-1,-1,1),(-1,1,1,-1),(-1,1,-1,1),(-1,-1,1,1),(0,0,0,0)\}.
\end{aligned}$$

Therefore, it is obtained that $g(\lambda_i+\mu_i+\gamma_i+\tau_i)\neq 0$ and $\lambda_i^2+\mu_i^2+\gamma_i^2+\tau_i^2\neq 0$ for all $i=1,2,\cdots,n$. In view of (2.20) and (2.21) it is seen that

$$\begin{aligned}
&I_n+T_1T_2+T_1T_3+T_1T_4+T_2T_3+T_2T_4+T_3T_4\\
&+T_1T_2T_3+T_1T_2T_4+T_1T_3T_4+T_2T_3T_4+T_1T_2T_3T_4
\end{aligned}$$

and $T_1^2+T_2^2+T_3^2+T_4^2$ are nonsingular.

Now, assume that

$$\begin{aligned}
&I_n+T_1T_2+T_1T_3+T_1T_4+T_2T_3+T_2T_4+T_3T_4\\
&+T_1T_2T_3+T_1T_2T_4+T_1T_3T_4+T_2T_3T_4+T_1T_2T_3T_4
\end{aligned}$$

and $T_1^2+T_2^2+T_3^2+T_4^2$ are nonsingular. From the nonsingularity of the first matrix, we get

$$1+\lambda_i\mu_i+\lambda_i\gamma_i+\lambda_i\tau_i+\mu_i\gamma_i+\mu_i\tau_i+\gamma_i\tau_i+\lambda_i\mu_i\gamma_i+\lambda_i\mu_i\tau_i+\mu_i\gamma_i\tau_i+\lambda_i\mu_i\gamma_i\tau_i\neq 0$$

for all $i=1,2,\cdots,n$. If $T_1+T_2+T_3+T_4$ were singular, then there would exist some $j\in\{1,2,\cdots,n\}$ such that $\lambda_i+\mu_i+\gamma_i+\tau_i=0$. So, the unique solution satisfying simultaneously these two equations would be $(\lambda_i,\mu_i,\gamma_i,\tau_i)=(0,0,0,0)$. Hence

$$\lambda_i^2+\mu_i^2+\gamma_i^2+\tau_i^2=0,$$

which would contradict to the assumption of the nonsingularity of

$$T_1^2+T_2^2+T_3^2+T_4^2.$$

So the proof is completed.

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关于四个三幂等阵线性组合的可逆性和群逆

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摘要: 本文研究了四个三幂等阵线性组合的可逆性及群逆. 利用矩阵分解的方法, 获得了它们可逆及群逆的一些条件, 并得到其逆和群逆的计算公式, 这些结论完善了k幂等阵可逆性理论.

关键词: 逆; 群逆; 线性组合; 三幂等阵

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