Vol. 35 (2015)
No. 4

数 学 杂 志
J. of Math. (PRC)

U-ABUNDANT SEMIGROUPS WITH LEFT CENTRAL IDEMPOTENTS

SUN Yan, REN Xue-ming, GONG Chun-mei

(*School of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China*)

Abstract: In this paper, we study the semilattice decomposition of U-abundant semigroups with left central idempotents. By using this semilattice decomposition, it is proved that a semigroup S is a U-abundant semigroup with left central idempotents if and only if it is a strong semilattice of a direct product $M_\alpha \times \Lambda_\alpha$, where M_α is a unipotent monoid and Λ_α is a right zero band. This result is the basis of the establishing of the structure theorem of U-abundant semigroups with left central idempotents.

Keywords: U-abundant semigroup; left central idempotent; unipotent monoid; $\sim$-Green's relation

2010 MR Subject Classification: 20M10

Document code: A **Article ID:** 0255-7797(2015)04-0833-08

1 Introduction

On a semigroup S the relation $\widetilde{\mathcal{L}}$ is defined by the rule that for any elements a, b of S, $a\widetilde{\mathcal{L}}b$ if and only if for all $e \in E$, $ae = a \Leftrightarrow be = b$. The relation $\widetilde{\mathcal{R}}$ is dually defined. The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ on a semigroup S are generalizations of the familiar $*$-Green's relations $\mathcal{L}^*$ and $\mathcal{R}^*$. As usual, the join of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ is denoted by $\widetilde{\mathcal{D}}$ and the intersection of them is denoted by $\widetilde{\mathcal{H}}$. Clearly, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ are equivalences, but $\widetilde{\mathcal{L}}$ is generally not right compatible and $\widetilde{\mathcal{R}}$ is generally not left compatible. The $\widetilde{\mathcal{L}}$-class containing the element a of the semigroup S is denoted by $\widetilde{L}_a$ or by $\widetilde{L}_a(S)$ in case of ambiguity. One can see that there is at most one idempotent contained in each $\widetilde{\mathcal{H}}$-class. Furthermore, if a and b are both regular elements of a semigroup S, then $(a, b) \in \widetilde{\mathcal{L}}$ if and only if $(a, b) \in \mathcal{L}$. In particular, if S itself is a regular semigroup, then $\widetilde{\mathcal{L}} = \mathcal{L}$ [1]. Dually, we also have $\widetilde{\mathcal{R}} = \mathcal{R}$ on a regular semigroup S.

A monoid is a semigroup with identity. A monoid is called unipotent if it does not contain any idempotents except identity. A semigroup in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent is called abundant [2]. A semigroup S is called U-semiabundant [3] if each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class of S contains at least one idempotent. All

* **Received date:** 2013-12-31 **Accepted date:** 2014-10-13

Foundation item: Supported by National Natural Science Foundation of China (11471255); Scientific Research Program Foundation of Shanxi Provincial Education Department (14JK1412); Fundamental Research Foundation of School (JC1219); Youth Foundation of School (QN1134).

Biography: Sun Yan(1981–), female, born at Pingdu, Shandong, lecturer, major in semigroup algebra.

abundant semigroups and U-semiabundant semigroups form two important classes of generalized regular semigroups. Moreover, we easily see that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$. Thus, abundant semigroups are obviously U-semiabundant semigroups, but U-semiabundant semigroups may not be abundant semigroups. This means that U-semiabundant semigroups are the generalizations of abundant semigroups in the range of generalized regular semigroups.

It is well known that a Clifford semigroup is a strong semilattice of groups [4]. And Fountain proved that an adequate semigroup with idempotents lying in the center is a strong semilattice of cancellative monoids [5]. Later on, the semilattice decomposition on abundant semigroups with left central idempotents has been investigated by Shum and Ren [6].

In this paper, we will extend the above results to the class of U-abundant semigroups in which all idempotents are left central. Thus, the results of Clifford , Fountain and Shum are all amplified. The main techniques that we use in the study are the $\sim$-Green's relations. For terminologies and notations not given in this paper, the reader is referred to Lawson [3] and Howie [7].

2 Preliminaries

In this section, we first give some basic definitions and results concerning U-abundant semigroups with left central idempotents.

Definition 2.1 An idempotent e of a semigroup S is called a left central idempotent if $xey = exy$ for all $x, y \in S^1$ and $y \neq 1$.

Definition 2.2 A U-semiabundant semigroup S is called a U-abundant semigroup if S satisfies the congruence condition, that is, $\widetilde{\mathcal{L}}$ is a right congruence and $\widetilde{\mathcal{R}}$ is a left congruence on S respectively.

Next, S is always a U-abundant semigroup with left central idempotents, that is, S is a U-abundant semigroup and all idempotents of S are left central.

Lemma 2.3 Each $\widetilde{\mathcal{L}}$-class of S contains a unique idempotent.

Proof Suppose $(e, f) \in \widetilde{\mathcal{L}}$ for $e, f \in E$. Then, we have $e\mathcal{L}f$. This leads to $ef = e$ and $fe = f$. Since f is a left central idempotent, we immediately have $ef = fef$. Thereby

$$e = ef = fef = f.$$

Remark We now denote a unique idempotent in the $\widetilde{\mathcal{L}}$-class containing the element a of S by a^*. Then we have the following lemmas.

Lemma 2.4 The relation $\widetilde{\mathcal{L}}$ is a congruence on S.

Proof Let $(a, b) \in \widetilde{\mathcal{L}}$ for $a, b \in S$. In order to show that $(ca, cb) \in \widetilde{\mathcal{L}}$ for any $c \in S$, we suppose that $cae = ca$ for any $e \in E$. By Lemma 2.3, then there exists a unique idempotent c^* in $\widetilde{L}_c$ such that $(c, c^*) \in \widetilde{\mathcal{L}}$. Since $\widetilde{\mathcal{L}}$ is a right congruence on U-abundant semigroups, we have $(ca, c^*a) \in \widetilde{\mathcal{L}}$ for $a \in S$. Thus, by the definition of $\widetilde{\mathcal{L}}$, we get $c^*ae = c^*a$. Thereby $ac^*a^*e = ac^*a^*$, because c^* is a left central idempotent and $a = aa^*$. Furthermore, since $(a, b) \in \widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}$ is a right congruence on a U-abundant semigroup, we also have $(ac^*a^*, bc^*a^*) \in \widetilde{\mathcal{L}}$. Thus, by the definition of $\widetilde{\mathcal{L}}$, we immediately have $bc^*a^*e = bc^*a^*$.

Hence, by the left centrality of c^*, we deduce that $c^*ba^*e = c^*ba^*$. Finally, and again, since $(c, c^*) \in \widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}$ is a right congruence on a U-abundant semigroup, we have $(cba^*, c^*ba^*) \in \widetilde{\mathcal{L}}$. Thus, by the definition of $\widetilde{\mathcal{L}}$, we can obtain that $cba^*e = cba^*$. As a result, we have $cbe = cb$, since $a^* = b^*$ and $b = bb^*$. Similarly, if $cbe = cb$ for any $e \in E$, then $cae = ca$. This leads to $(ca, cb) \in \widetilde{\mathcal{L}}$. Hence, $\widetilde{\mathcal{L}}$ is a left congruence on S. It is well known that $\widetilde{\mathcal{L}}$ is a right congruence on a U-abundant semigroup S. This shows that $\widetilde{\mathcal{L}}$ is a congruence on S.

Lemma 2.5 $(ab)^* = a^*b^*$ for any $a, b \in S$.

Proof It is trivial that $(b, b^*) \in \widetilde{\mathcal{L}}$ for any $b \in S$. Since $\widetilde{\mathcal{L}}$ is a congruence on a U-abundant semigroup with left central idempotents, we have $(ab, ab^*) \in \widetilde{\mathcal{L}}$ for any $a \in S$. Thus, it is obvious that $(ab)^* = (ab^*)^*$ by Lemma 2.3. Similarly, we know that $(ab^*, a^*b^*) \in \widetilde{\mathcal{L}}$. By using Lemma 2.3 again, we get $(ab^*)^* = (a^*b^*)^* = a^*b^*$. Consequently, $(ab)^* = (ab^*)^* = a^*b^*$.

Lemma 2.6 Define a relation σ on S by $a\sigma b$ if and only if $a^*b^* = b^*$ and $b^*a^* = a^*$ for any $a, b \in S$. Then the relation σ is a semilattice congruence on S.

Proof It is easy to see that σ is an equivalent relation on S. Suppose that $a\sigma b$ for $a, b \in S$. Then $a^*b^* = b^*$ and $b^*a^* = a^*$. By applying Lemma 2.5, we get $(ac)^*(bc)^* = a^*c^*b^*c^* = a^*b^*c^* = b^*c^* = (bc)^*$. By using similar arguments, we can also prove that $(bc)^*(ac)^* = (ac)^*$. This shows that $(ac, bc) \in \sigma$. Similarly, $(ca, cb) \in \sigma$. Hence, σ is a congruence on S. Now, by Lemma 2.5 and the left centrality of a^*, we have $(ab)^*(ba)^* = a^*b^*a^* = b^*a^*a^* = (ba)^*$ and $(ba)^*(ab)^* = (ab)^*$. Thus, we know immediately that $(ab, ba) \in \sigma$ by the definition of σ. In addition, it is obvious that $a\sigma a^*$ and $a\sigma a^2$ for any $a \in S$. We prove that σ is indeed a smilattice congruence on S.

Lemma 2.7 On a U-abundant semigroup S with left central idempotents, we have

(i) $\widetilde{\mathcal{L}} = \widetilde{\mathcal{H}}$ and $\sigma = \widetilde{\mathcal{R}} = \widetilde{\mathcal{D}}$;

(ii) $\widetilde{\mathcal{H}}, \widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{D}}$ are all congruences on S.

Proof (i) Since S is a U-abundant semigroup, there exists $e \in E$ such that $(a, e) \in \widetilde{\mathcal{R}}$ for any $a \in S$ and so $ea = a$. Thus, by Lemma 2.5, we have $ea^* = a^*$. On the other hand, since $(a, a^*) \in \widetilde{\mathcal{L}}$ and a^* is a left central idempotent, we deduce that $a = aa^* = aa^*a^* = a^*aa^* = a^*a$, that is, $a^*a = a$. Thus, by using $(a, e) \in \widetilde{\mathcal{R}}$ and the definition of $\widetilde{\mathcal{R}}$, we have $a^*e = e$. From this, together with $ea^* = a^*$, we can deduce that $(a^*, e) \in \mathcal{R}$. This leads to $(a^*, e) \in \widetilde{\mathcal{R}}$. Hence, $(a, a^*) \in \widetilde{\mathcal{R}}$. Again since $(a, a^*) \in \widetilde{\mathcal{L}}$, we have $(a, a^*) \in \widetilde{\mathcal{H}}$. Now let $(a, b) \in \widetilde{\mathcal{L}}$ for $a, b \in S$. Then, we have known that $(a, a^*) \in \widetilde{\mathcal{H}}$ and $(b, b^*) \in \widetilde{\mathcal{H}}$. By Lemma 2.3, we have $a^* = b^*$ and so $(a, b) \in \widetilde{\mathcal{H}}$. This shows that $\widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{H}}$. Clearly, $\widetilde{\mathcal{L}} = \widetilde{\mathcal{H}} \subseteq \widetilde{\mathcal{R}}$.

Let $(a, b) \in \widetilde{\mathcal{R}}$ for $a, b \in S$. Then $(a^*, b^*) \in \widetilde{\mathcal{R}}$, since $\widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{R}}$. Thus, we have $a^*b^* = b^*$and $b^*a^* = a^*$. Hence, we have $(a, b) \in \sigma$ by the definition of σ. This leads to $\widetilde{\mathcal{R}} \subseteq \sigma$.

On the other hand, let $(a, b) \in \sigma$ for $a, b \in S$. Then $a^*b^* = b^*$and $b^*a^* = a^*$. In order to show that $(a, b) \in \widetilde{\mathcal{R}}$, we suppose $ea = a$ for any $e \in E$. By Lemma 2.5, we have $ea^* = a^*$. Thus $ea^*b^* = a^*b^*$. This implies that $eb^* = b^*$. And again, since $(b, b^*) \in \widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{R}}$, we immediately have $(b, b^*) \in \widetilde{\mathcal{R}}$. Thus, by using the definition of $\widetilde{\mathcal{R}}$, we know that $eb = b$. Similarly, if $eb = b$ for any $e \in E$, then $ea = a$. As a result, we obtain $(a, b) \in \widetilde{\mathcal{R}}$. This leads to $\sigma \subseteq \widetilde{\mathcal{R}}$ and so $\sigma = \widetilde{\mathcal{R}}$. Now, it is easy to see that

$$\widetilde{\mathcal{D}} = \widetilde{\mathcal{L}} \vee \widetilde{\mathcal{R}} = \widetilde{\mathcal{H}} \vee \widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}.$$

This shows that $\sigma = \widetilde{\mathcal{R}} = \widetilde{\mathcal{D}}$.

(ii) By using Lemma 2.4 and Lemma 2.6, we immediately know that $\widetilde{\mathcal{H}}, \widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{D}}$ are all congruences on S.

3 Main Results

We are now going to give the characterization theorem for U-abundant semigroups with left central idempotents.

Theorem 3.1 Let S be a semigroup. Then the following statements are equivalent:

(i) S is a U-abundant semigroup with left central idempotents;

(ii) S is a semilattice of direct products $M_\alpha \times \Lambda_\alpha$, where M_α is a unipotent monoid and Λ_α is a right zero band for every $\alpha \in Y$. Moreover, E is a right normal band;

(iii) S is a strong semilattice of direct products $M_\alpha \times \Lambda_\alpha$, where M_α is a unipotent monoid and Λ_α is a right zero band for every $\alpha \in Y$.

Proof (i)$\Rightarrow$(ii) Clearly, $S = \bigcup\limits_{\alpha \in Y} S_\alpha$, where S_α is a σ-class of S on a semilattice Y. We know that $S_\alpha \bigcap E \neq \varnothing$, since $\sigma = \widetilde{\mathcal{R}}$ and S is a U-abundant semigroup. Let $M_\alpha = S_\alpha e_\alpha$ for some $e_\alpha \in S_\alpha \bigcap E$. To show that M_α is a unipotent monoid with an identity element e_α. Let $y = xe_\alpha \in M_\alpha$ for any $x \in S_\alpha$. Then $ye_\alpha = xe_\alpha = y$. On the other hand, by using $\sigma = \widetilde{\mathcal{R}}$, we get $(y, e_\alpha) \in \widetilde{\mathcal{R}}$ and so $e_\alpha y = y$. Hence, M_α is indeed a monoid with an identity element e_α. In particular, $M_\alpha = \widetilde{\mathcal{L}} = \widetilde{\mathcal{H}}$. This means that M_α contains a unique idempotent. Thus, we know that M_α also is a unipotent monoid. Now, let Λ_α be the set of all idempotents of S_α, that is, $\Lambda_\alpha = S_\alpha \bigcap E$. Then it is clear that $e\widetilde{\mathcal{R}}f$ for all $e, f \in \Lambda_\alpha$. This implies that Λ_α is a right zero band. We define a mapping $\varphi : M_\alpha \times \Lambda_\alpha \to S_\alpha$ by $\varphi(x, f) = xf$ for any $(x, f) \in M_\alpha \times \Lambda_\alpha$. Then, for any $(x, f), (y, k) \in M_\alpha \times \Lambda_\alpha$, we have

$$\varphi(x, f)\varphi(y, k) = xfyk = xyfk = xyk = \varphi[(x, f)(y, k)].$$

Thus, φ is a morphism. Furthermore, if $\varphi(x, f) = \varphi(y, k)$, that is, $xf = yk$, then $xfe_\alpha = yke_\alpha$. Since Λ_α is a right zero band and e_α is an identity element in M_α, we have $x = y$. In the meantime, we also have $xf = xk$. By using Lemma 2.5, we obtain that $x^*f = x^*k$. Again since Λ_α is a right zero band and $x^* \in \Lambda_\alpha$, we immediately have $f = k$. This shows that φ is a monomorphism as well. In order to show that the mapping φ is onto. Let any $a \in S_\alpha$. Then there exists a unique idempotent a^* such that $(a, a^*) \in \widetilde{\mathcal{L}}$. By Lemma 2.7, we know that $\widetilde{L} \subseteq \widetilde{R}$ and $\sigma = \widetilde{R}$. This means that $(a, a^*) \in \sigma$ such that $a^* \in S_\alpha$. Hence, $a^* \in \Lambda_\alpha = S_\alpha \bigcap E$. Moreover, by using $S_\alpha \bigcap E \neq \varnothing$, it is natural for us to know $ae_\alpha \in M_\alpha$ for some $e_\alpha \in S_\alpha \bigcap E$. In this case, we always have

$$(ae_\alpha, a^*)\varphi = ae_\alpha a^* = aa^* = a,$$

since Λ_α is a right zero band and $(a, a^*) \in \widetilde{\mathcal{L}}$. This shows that the mapping φ is onto. In conclusion, we prove that $S_\alpha \cong M_\alpha \times \Lambda_\alpha$.

Finally, since any $e \in E$ is a left central idempotent, we have $ehg = heg$ for all $e, h, g \in E$. This show that E is a right normal band.

(ii)⇒(iii) To show that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a strong semilattice of direct products $M_\alpha \times \Lambda_\alpha$, we pick any $\alpha, \beta \in Y$ with $\alpha \geq \beta$. First, let $a \in S_\alpha$ and $e_\beta \in S_\beta \bigcap E$. Then $e_\beta a \in S_\beta$. According to this fact, we define a mapping $\theta_{\alpha,\beta} : S_\alpha \to S_\beta$ by $a\theta_{\alpha,\beta} = e_\beta a$ for any $a \in S_\alpha$and some $e_\beta \in S_\beta \bigcap E$. Write $e_\beta a = (u, i) \in S_\beta$. Let $g = (1_\beta, i) \in S_\beta \bigcap E$ and $e_\beta = (1_\beta, j) \in S_\beta \bigcap E$, where 1_β is the identity element of M_β. Then

$$e_\beta a g = (u, i)(1_\beta, i) = (u, i) = e_\beta a \tag{2.1}$$

and

$$g = (1_\beta, i) = (1_\beta, j)(1_\beta, i) = e_\beta g. \tag{2.2}$$

Similarly, we let $b = (v, l) \in S_\alpha$ and $h = (1_\alpha, l) \in S_\alpha \bigcap E$. Then

$$hb = b. \tag{2.3}$$

Furthermore, since E is a right normal band, we also have

$$g e_\beta h = e_\beta g h. \tag{2.4}$$

By using (2.1), (2.2), (2.3) and (2.4), we can proof that

$$e_\beta a e_\beta b = e_\beta a g e_\beta h b = e_\beta a e_\beta g h b = e_\beta a g h b = e_\beta a b.$$

Thus, $a\theta_{\alpha,\beta} b\theta_{\alpha,\beta} = (ab)\theta_{\alpha,\beta}$. This shows that $\theta_{\alpha,\beta}$ is a morphism.

On the other hand, it is easy to prove that $\theta_{\alpha,\alpha} = 1_{S_\alpha}$ for any $\alpha \in Y$.

Now, we let $a = (w, k) \in S_\alpha$ and $p = (1_\alpha, k) \in S_\alpha \bigcap E$ for $\alpha \in Y$. Then we have $pa = a$. By the right normality of E, for some $e_\beta \in S_\beta \bigcap E$ and $e_\gamma \in S_\gamma \bigcap E$ with $\alpha \geq \beta \geq \gamma$, we can deduce that

$$e_\gamma e_\beta p = e_\beta e_\gamma p = e_\beta e_\gamma \cdot e_\gamma p = e_\gamma p$$

so that

$$a\theta_{\alpha,\beta}\theta_{\beta,\gamma} = e_\gamma(e_\beta a) = e_\gamma e_\beta p a = e_\gamma p a = e_\gamma a = a\theta_{\alpha,\gamma}.$$

Thus, $\theta_{\alpha,\gamma} = \theta_{\alpha,\beta}\theta_{\beta,\gamma}$.

Finally, let $a \in S_\alpha, b \in S_\beta$ for any $\alpha, \beta \in Y$. Then $ab = e_{\alpha\beta}(ab) \in S_{\alpha\beta}$ for some $e_{\alpha\beta} \in S_{\alpha\beta} \bigcap E$. Since $e_{\alpha\beta} a \in S_{\alpha\beta}$, by using (2.1), we known that there exists $f^2 = f \in S_{\alpha\beta}$ such that $e_{\alpha\beta} a f = e_{\alpha\beta} a$ for $e_{\alpha\beta} a \in S_{\alpha\beta}$. Moreover, by using (2.3), we also know that there exists $e^2 = e \in S_\beta$ such that $eb = b$ for $b \in S_\beta$. Thus, by the right normality of E again, we have

$$\begin{aligned} e_{\alpha\beta} a e_{\alpha\beta} b &= e_{\alpha\beta} a f e_{\alpha\beta} e b = e_{\alpha\beta} a e_{\alpha\beta} f e b \\ &= e_{\alpha\beta} a (e_{\alpha\beta} f) e b = e_{\alpha\beta} a f e b \\ &= e_{\alpha\beta} a b = ab. \end{aligned}$$

This shows that $ab = a\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta}$. Hence, $S = \bigcup\limits_{\alpha\in Y} S_\alpha$ is indeed a strong semilattice of direct products $S_\alpha = M_\alpha \times \Lambda_\alpha$, and denote it by $S = [Y; S_\alpha, \theta_{\alpha,\beta}]$.

(iii)⇒(i) Let $S = [Y; S_\alpha, \theta_{\alpha,\beta}]$ be a strong semilattice of direct products $M_\alpha \times \Lambda_\alpha$ for $\alpha \in Y$, where M_α is a unipotent monoid and Λ_α is a right zero band. And let $x, y \in S^1$ with $y \neq 1$ and $e \in E$. Then $x \in S^1_\alpha, y \in S^1_\beta$ and $e \in S_\gamma \bigcap E$ for some $\alpha, \beta, \gamma \in Y$. Write $\alpha\beta\gamma = \delta$, where $\delta \in Y$. Now, we suppose that

$$\begin{aligned} x\theta_{\alpha,\delta} &= (m, t) \in S_\delta, \\ y\theta_{\beta,\delta} &= (v, l) \in S_\delta, \\ e\theta_{\gamma,\delta} &= (1_\delta, q) \in S_\delta. \end{aligned}$$

Then

$$xey = x\theta_{\alpha,\delta}e\theta_{\gamma,\delta}y\theta_{\beta,\delta} = (m, t)(1_\delta, q)(v, l) = (mv, l).$$

Similarly, we have $exy = (mv, l)$. Thus, $xey = exy$. This shows that the element e is a left central idempotent for any $e \in E$. In other words, S is a semigroup with left central idempotents.

It remains to show that each $\widetilde{\mathcal{R}}$-class of S contains at least one idempotent. Let $a = (u, i) \in S_\alpha$ and $f = (1_\alpha, i) \in S_\alpha \bigcap E$ for any $\alpha \in Y$. In order to proof that $(a, f) \in \widetilde{\mathcal{R}}$, we assume that $ea = a$ for any $e \in E$. Then $e = (1_\gamma, k) \in S_\gamma \bigcap E$ for some $\gamma \in Y$. Because S is the semilattice of S_α, we can immediately see that $\alpha \leq \gamma$ for $\alpha, \gamma \in Y$. Thus,

$$ef = (1_\gamma, k)\theta_{\gamma,\alpha} \cdot (1_\alpha, i)\theta_{\alpha,\alpha} = (1_\alpha, i) = f.$$

Similarly, if $ef = f$ for any $e \in E$, then $ea = a$. Hence, we know that $(a, f) \in \widetilde{\mathcal{R}}$. This implies that each $\widetilde{\mathcal{R}}$-class of S contains at least one idempotent.

On the other hand, we need to proof that each $\widetilde{\mathcal{L}}$-class of S contains at least one idempotent. Let $b = (w, h) \in S_\alpha$ and $g = (1_\alpha, h) \in S_\alpha \bigcap E$ for any $\alpha \in Y$. In order to proof that $(b, g) \in \widetilde{\mathcal{L}}$. We assume that $be = b$ for any $e \in E$. Then $e = (1_\gamma, k) \in S_\gamma \bigcap E$ for some $\gamma \in Y$ and $\gamma \geq \alpha$. Again, since $be = b$, we have

$$(w, h)(1_\gamma, k) = (w, h)\theta_{\alpha,\alpha} \cdot (1_\gamma, k)\theta_{\gamma,\alpha} = (w, k\theta_{\gamma,\alpha}) = (w, h).$$

This implies that $k\theta_{\gamma,\alpha} = h$. Hence, we have

$$ge = (1_\alpha, h)(1_\gamma, k) = (1_\alpha, h)\theta_{\alpha,\alpha} \cdot (1_\gamma, k)\theta_{\gamma,\alpha} = (1_\alpha, k\theta_{\gamma,\alpha}) = (1_\alpha, h) = g.$$

Similarly, if $ge = g$ for any $e \in E$, then $be = b$. Thereby, we obtain that $(b, g) \in \widetilde{\mathcal{L}}$. This implies that each $\widetilde{\mathcal{L}}$-class of S contains at least one idempotent. Thus, S is a U-semiabundant semigroup.

Finally, we shall show that $\widetilde{\mathcal{L}}$ is a right congruence. We first let $(a, b) \in \widetilde{\mathcal{L}}$ for $a, b \in S$. Then a and b are the element of the same S_α for $\alpha \in Y$. Clearly, if $(a, b) \in \widetilde{\mathcal{L}}$ but

$a \in S_\alpha, b \in S_\beta$, since S is a U-semiabundant semigroup, there exist $f^2 = f \in S_\alpha \bigcap E$ and $g^2 = g \in S_\beta \bigcap E$ such that $(a, f) \in \widetilde{\mathcal{L}}$ and $(b, g) \in \widetilde{\mathcal{L}}$. This implies that $(f, g) \in \widetilde{\mathcal{L}}$. Thus, $fg = f$ and $gf = g$, where the elements $fg, gf \in S_{\alpha\beta}, f \in S_\alpha$ and $g \in S_\beta$. Hence, $\alpha = \beta$. According to this fact, we let $(a, b) \in \widetilde{\mathcal{L}}$ for $a = (u, i) \in S_\alpha, b = (w, h) \in S_\alpha$. To show that $(ac, bc) \in \widetilde{\mathcal{L}}$ for any $c \in S$, we assume that $ace = ac$ for any $e \in E$. Then $c = (v, j) \in S_\beta$ and $e = (1_\gamma, k) \in S_\gamma$ for some $\beta, \gamma \in Y$. In the meantime, we have $\alpha, \beta \le \gamma$ for $\alpha, \beta, \gamma \in Y$, since S is a strong semilattice of S_α. Furthermore, we obtain that

$$\begin{aligned}(u,i)(v,j)(1_\gamma,k) &= (u,i)\theta_{\alpha,\alpha\beta} \cdot (v,j)\theta_{\beta,\alpha\beta}(1_\gamma,k)\theta_{\gamma,\alpha\beta}\\ &= (u\theta_{\alpha,\alpha\beta} \cdot v\theta_{\beta,\alpha\beta}, k\theta_{\gamma,\alpha\beta})\\ &= (u\theta_{\alpha,\alpha\beta} \cdot v\theta_{\beta,\alpha\beta}, j\theta_{\beta,\alpha\beta}).\end{aligned}$$

This implies that $k\theta_{\gamma,\alpha\beta} = j\theta_{\beta,\alpha\beta}$. Hence, we immediately have that

$$\begin{aligned}bce &= (w,h)(v,j)(1_\gamma,k) = (w,h)\theta_{\alpha,\alpha\beta} \cdot (v,j)\theta_{\beta,\alpha\beta}(1_\gamma,k)\theta_{\gamma,\alpha\beta}\\ &= (w\theta_{\alpha,\alpha\beta} \cdot v\theta_{\beta,\alpha\beta}, k\theta_{\gamma,\alpha\beta}) = (w\theta_{\alpha,\alpha\beta} \cdot v\theta_{\beta,\alpha\beta}, j\theta_{\beta,\alpha\beta}) = bc.\end{aligned}$$

By using similar arguments, if $bce = bc$ for any $e \in E$, then $ace = ac$. Thereby, $(ac, bc) \in \widetilde{\mathcal{L}}$. This show that $\widetilde{\mathcal{L}}$ is a right congruence. Similarly, $\widetilde{\mathcal{R}}$ is also a left congruence. In other words, S is a U-abundant semigroup. Now summing up the above facts, we show that the semigroup S is indeed a U-abundant semigroup with left central idempotents. The proof is completed.

Remark We notice here that the above theorem extends a known result of Shum and Ren (see [6], Theorem 3.1).

The following theorem can be proved by using the similar method as Theorem 3.1.

Theorem 3.2 Let S be a semigroup. Then the following statements are equivalent:

(i) S is a regular semigroup with left central idempotents;

(ii) S is a strong semilattice of right groups;

(iii) S is a right Clifford semigroup and E is a right normal band.

Proof We first note that $\widetilde{\mathcal{L}} = \mathcal{L}$ and $\widetilde{\mathcal{R}} = \mathcal{R}$ on regular semigroups. If S is a regular semigroup with left central idempotents, then by the results of Lemma 2.7, we know immediately that $\mathcal{L} = \mathcal{H} \subseteq \mathcal{R} = \mathcal{D}$ and $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ are all congruences on S. Moreover, S is a completely regular semigroup. In particular, for any $a \in S$, a^* is a unique idempotent of the $\mathcal{H}$-class containing a. It following that each σ-class of S is a direct product of a group and a right zero band, which is called a right group. Hence, by using the same arguments as Theorem 3.1, we can prove that (i)$\Rightarrow$(ii). The details are omitted. For (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i), it is the dual of Theorem 4.1 in [8].

Remark We notice here that the above theorem extends a known result of Clifford (see [7], Theorem 2.1).

References

[1] Guo Yuqi, Shum K P, Gong Chunmei. On $(*, \sim)$-Green's relations and ortho-lc-monoids[J]. Comm. Algebra, 2011, 39: 5–31.

[2] Fountain J B. Abundant semigroups[J]. Proc. Lond. Math. Soc, 1982, 44(3): 103–129.

[3] Lawson M V. Rees matrix semigroups[J]. Proc. Edinburgh Math. Soc., 1990, 33: 23–37.

[4] Clifford A H. Semigroups admitting relatives inverse[J]. Ann. Math, 1941, 42: 1037–1049.

[5] Fountain J B. Adequate semigroups[J]. Proc. Edinburgh Math.Soc., 1979, 22: 113–125.

[6] Shum K P, Ren Xueming. Abundant semigroups with left central idempotents[J]. Pure Math. Appl., 1999, 10(1): 109–113.

[7] Howie J M. An introduction to semigroup theory[M]. London: Academic Press, 1976.

[8] Zhu Pinyu, Guo Yuqi, Shum K P. Structure and characterizations of left C-semigroups[J]. Sci. China, Series A, 1992, 6: 791–805.

具有左中心幂等元的U-富足半群

孙 燕, 任学明, 宫春梅

(西安建筑科技大学理学院, 陕西 西安 710055)

摘要: 本文研究了具有左中心幂等元的U-富足半群的半格分解. 利用半格分解, 证明了半群S为具有左中心幂等元的U-富足半群, 当且仅当S为直积$M_\alpha \times \Lambda_\alpha$的强半格, 其中$M_\alpha$是幂幺半群, Λ_α 是右零带. 这一结果为具有左中心幂等元的U-富足半群结构的建立奠定了基础.

关键词: U-富足半群; 左中心幂等元; 幂幺半群; $\sim$-格林关系

MR(2010)**主题分类号**: 20M10 **中图分类号**: O152.7