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A CONSTRUCTION OF STRONGLY $\mathcal{P}$-REGULAR SEMIGROUPS

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Abstract: In this paper, the author studies the constructions of strongly $\mathcal{P}$-regular semigroups. A construction method of strongly $\mathcal{P}$-regular semigroups is obtained in terms of regular *-semigroups and some kinds of mappings satisfying certain conditions. The result generalizes and enriches some results on orthodox semigroups in the literatures.

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1 Introduction

Inverse semigroups play an important role in the algebraic theory of semigroups. Many remarkable results of inverse semigroups and their generalizations are obtained in the literatures. As generalizations of inverse semigroups, orthodox semigroups and regular *-semigroups were extensively studied (for example, see [1–5]). To give a common generalization of orthodox semigroups and regular *-semigroups, Yamada-Sen [7] introduced $\mathcal{P}$-regular semigroups. Many achievements on orthodox semigroups were generalized to $\mathcal{P}$-regular semigroups (for example, see [6–8]). In the procession of characterizing $\mathcal{P}$-regular semigroups, regular *-semigroups have played a similar role as that of inverse semigroups in characterizing orthodox semigroups.

Standard representations are important tools for the constructions and classifications of bands (for example, see [9]). Inspired by these facts, He-Guo-Shum [1] gave the standard representations of orthodox semigroups and investigated e-varieties of orthodox semigroups determined by their standard representations. Recently, He-Shum-Wang [10] generalized the results in [1] to a class of non-regular semigroups, namely good B-quasi-Ehresmann semigroups.

In this paper, we generalize some results obtained in [1] to $\mathcal{P}$-regular semigroups. In particular, we give a construction method of strongly $\mathcal{P}$-regular semigroups by considering their standard representations.

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Biography: Wang Shoufeng(1979–), male, born at Laiwu, Shandong, associated professor, major in semigroup theory and application.

2 Preliminaries

Let S be a semigroup. The set of all idempotents of S will be denoted by $E(S)$, and the set of all inverses of an element a from S will be denoted by $V(a)$. Recall that

$$V(a) = \{x \in S | xax = x, axa = a\}.$$

A semigroup S is called regular if $V(a) \neq \emptyset$ for each $a \in S$.

From Nordahl-Scheiblich [3], a regular semigroup T with a unary operation "$*$" denoted by $(T, ^*)$ is called a regular *-semigroup if it satisfies the following identities

$$(a^*)^* = a, \quad aa^*a = a, \quad (ab)^* = b^*a^*.$$

In this case, the set $F_T = \{aa^* | a \in T\}$ is called the set of projections of $(T, ^*)$.

Let S be a regular semigroup. From Yamada-Sen [7], a non-empty subset P of $E(S)$ is called a characteristic set (in brief, C-set) of S if it satisfies the following conditions:

(p.1) $P^2 \subseteq E(S)$;

(p.2) $(\forall q \in P)\ qPq \subseteq P$;

(p.3) $(\forall a \in S)(\exists a^+ \in V(a))\ aP^1a^+ \subseteq P,\ a^+P^1a \subseteq P$.

In this case, S is called a projectively regular semigroup with P as the set of projections or simply a $\mathcal{P}$-regular semigroup, and written as $S(P)$. The inverse a^+ of an element a from $S(P)$ in condition (p.3) is called a $\mathcal{P}$-inverse of a. The set of all $\mathcal{P}$-inverses of a will be denoted by $V_P(a)$.

A C-set P of a regular semigroup S is said to be strong, while the $\mathcal{P}$-regular semigroup $S(P)$ is said to be strongly $\mathcal{P}$-regular, if P satisfies the following condition:

$$(\forall p, q \in P) \qquad pq \in P \Longrightarrow qp \in P.$$

From Zhang-He [8], a $\mathcal{P}$-regular semigroup is always a strongly $\mathcal{P}$-regular semigroup with respect to some strong C-set. It is well known that regular *-semigroups and orthodox semigroups are strongly $\mathcal{P}$-regular semigroups.

Let $S(P)$ and $T(Q)$ be $\mathcal{P}$-regular semigroups. A homomorphism (resp., isomorphism) ξ from S to T is called a $\mathcal{P}$-homomorphism (resp., $\mathcal{P}$-isomorphism), if $P\xi = S\xi \cap Q$. As in the literatures, we use the notion "$\mathcal{P}$-congruences" for the congruences on $\mathcal{P}$-regular semigroups. If ρ is a $\mathcal{P}$-congruence on $S(P)$, then $S\rho^\natural(P\rho^\natural)$ is also a $\mathcal{P}$-regular semigroup, where $\rho^\natural$ is the natural homomorphism from S to S/ρ induced by ρ. Customarily, we write $S\rho^\natural(P\rho^\natural)$ as $S(P)/\rho$ and write $P\rho^\natural$ as $P\rho$. If ρ is a $\mathcal{P}$-congruence on $S(P)$ satisfying the following condition:

$$(\forall a, b \in S)(\forall a^+ \in V_P(a))(\forall b^+ \in V_P(b)) \ \ a\rho b \Longrightarrow a^+\rho b^+,$$

then we call ρ a strongly $\mathcal{P}$-congruence. In this case, $(S/\rho, ^*)$ is a regular *-semigroup with the operation "$*$" defined as follows: $(a\rho)^* = a^+\rho$ for all $a \in S$ and $a^+ \in V_P(a)$. Furthermore, $F_{(S(P)/\rho,\ ^*)} = P\rho$ in the case. If it makes sense, the least strongly $\mathcal{P}$-congruence on $S(P)$ will be denoted by γ_S or simply by γ.

To end this section, we recall some basic results on regular *-semigroups and strongly $\mathcal{P}$-regular semigroups, which will be used throughout this paper.

Lemma 2.1 [3, 5] Let $(T,^*)$ be a regular *-semigroup and $a \in T$.

(1) $F_T^2 \subseteq E(T)$ and $aF_T^1a^*, a^*F_T^1a \subseteq F_T$.

(2) $F_T = \{x^*x | x \in T\} = \{e \in E(T) | e^* = e\}$.

(3) If $x, y, xy \in F_T$, then $xy = yx$.

(4) If $x, y \in F_T$ and $x\mathcal{L}y$, then $x = y$.

(5) If $x, y \in F_T$ and $x\mathcal{R}y$, then $x = y$.

(6) $a \in E(T)$ if and only if $a^* \in E(T)$.

Lemma 2.2 [7, 8] Let $S(P)$ be a strongly $\mathcal{P}$-regular semigroup, $a, b \in S$ and $e \in E(S), p \in P$.

(1) $(a, b) \in \gamma$ if and only if $V_P(a) = V_P(b)$.

(2) $\gamma \cap \mathcal{H}$ is the equality relation on S.

(3) $V_P(e) \subseteq E(S)$ and $p \in V_P(p) \subseteq P$.

3 Main Results

The aim of this section is to give a structure theorem of strongly $\mathcal{P}$-regular semigroups. To this purpose, we need the following lemma which can be proved by direct calculations by using Lemma 2.1. For any regular *-semigroup $(T, ^*)$ and $x \in T$, we let $x^\ddagger = xx^*$ and $x^\dagger = x^*x$ in the sequel.

Lemma 3.3 Let $(T, ^*)$ be a regular *-semigroup, $x, y \in T$ and $\alpha, \beta \in F_T$.

(1) $x \in E(T)$ implies that $x^* \in E(T), x^\ddagger = x^\ddagger x^\dagger x^\ddagger$ and $x^\dagger = x^\dagger x^\ddagger x^\dagger$;

(2) $(y^\ddagger x^\dagger y^\ddagger)x^\dagger(y^\ddagger x^\dagger y^\ddagger) = y^\ddagger x^\dagger y^\ddagger, (xy)^\dagger = (xy)^\dagger y^\dagger (xy)^\dagger$ and $(y(xy)^\dagger)^\ddagger = y^\ddagger x^\dagger y^\ddagger$, $(xy^\ddagger x^\dagger y^\ddagger)^\ddagger = (xy)^\ddagger$;

(3) $(x^\dagger y^\ddagger x^\dagger)y^\ddagger(x^\dagger y^\ddagger x^\dagger) = x^\dagger y^\ddagger x^\dagger, (xy)^\ddagger = (xy)^\ddagger x^\ddagger (xy)^\ddagger$ and $((xy)^\ddagger x)^\dagger = x^\dagger y^\ddagger x^\dagger$, $(x^\dagger y^\ddagger x^\dagger y)^\dagger = (xy)^\dagger$;

(4) $\alpha = \alpha(xy)^\dagger\alpha$ implies that $\alpha = \alpha y^\dagger \alpha$ and $(y\alpha)^\ddagger = (y\alpha)^\ddagger x^\dagger (y\alpha)^\ddagger$;

(5) $\beta = \beta(xy)^\ddagger\beta$ implies that $\beta = \beta x^\ddagger \beta$ and $(\beta x)^\dagger = (\beta x)^\dagger y^\ddagger (\beta x)^\dagger$;

(6) $(x(y\alpha)^\ddagger)^\ddagger = (xy\alpha)^\ddagger$ and $((\beta x)^\dagger y)^\dagger = (\beta xy)^\dagger$.

Now, we use $\mathcal{F}(A, B)$ to denote the set of mappings which map from a set A to another set B. Let $\xi \in \mathcal{F}(A, B)$ and $b \in B$. By $[b]$ we mean the constant mapping which maps A into B with value b. If ξ is a constant mapping, we denote the value of ξ by $[\xi]$. We write $\mathcal{F}(A, B)$ specifically as $\mathcal{F}_r(A, B)$ if its members are acting on A from the right. Thus, for any $\xi \in \mathcal{F}_r(A, B)$ and $\eta \in \mathcal{F}_r(B, C)$, we can compose them by $x\xi\eta = (x\xi)\eta$, for all $x \in A$. Dually, we denote $\mathcal{F}(A, B)$ by $\mathcal{F}_l(A, B)$ if its member are acting on A from the left. Hence, for any $\xi \in \mathcal{F}_l(A, B)$ and $\eta \in \mathcal{F}_l(B, C)$, we can compose them by $\eta * \xi(x) = \eta(\xi(x))$, for all $x \in A$. In particular, we can write $\mathcal{F}_r(A, A)$ and $\mathcal{F}_l(A, A)$ by $\mathcal{T}_r(A)$ and $\mathcal{T}_l(A)$, respectively.

With the above notations, we have the following useful result on general regular semigroups.

Lemma 3.4 [2] If S is a regular semigroup, then the mapping $\theta:\ a \longmapsto (\rho_a, \delta_a)$ is a homomorphism from S to $\mathcal{T}_l(S/\mathcal{R}) \times \mathcal{T}_r(S/\mathcal{L})$, where ρ_a and δ_a are defined by

$$\rho_a(R_x) = R_{ax}, \qquad L_x\delta_a = L_{xa} \quad (x \in S).$$

At this stage, we can establish our main theorem.

Theorem 3.5 Let $(T, ^*)$ be a regular *-semigroup with the set of projections F_T. We associate each element α in F_T with two non-empty sets I_α and Λ_α such that $I_\alpha \cap I_\beta = \Lambda_\alpha \cap \Lambda_\beta = \emptyset$ whenever $\beta \neq \alpha$ in F_T. Form the following set

$$\bar{S} = \{(i, x, \lambda) | x \in T, i \in I_{x^\ddagger}, \lambda \in \Lambda_{x^\dagger}\}.$$

For any $(i, x, \lambda) \in \bar{S}$ with $\alpha = \alpha x^\dagger \alpha$ and $\beta = \beta x^\ddagger \beta$ in F_T, we define

$$\xi^{(i,x,\lambda)}_{\alpha,(x\alpha)^\ddagger} \in \mathcal{F}_l(I_\alpha, I_{(x\alpha)^\ddagger}) \quad \text{and} \quad \eta^{(i,x,\lambda)}_{\beta,(\beta x)^\dagger} \in \mathcal{F}_r(\Lambda_\beta, \Lambda_{(\beta x)^\dagger}).$$

Suppose that the following conditions hold for any $(i, x, \lambda), (j, y, \mu) \in \bar{S}$:

(C.1) (i) $\xi^{(i,x,\lambda)}_{x^\dagger,x^\ddagger} = [i]$ and $\eta^{(i,x,\lambda)}_{x^\ddagger,x^\dagger} = [\lambda]$;

(ii) if $(i, x, \lambda) \in \bar{S}$ and $x \in E(T)$, then $\xi^{(i,x,\lambda)}_{x^\ddagger,x^\ddagger}(i) = i$, $\ (\lambda)\eta^{(i,x,\lambda)}_{x^\dagger,x^\dagger} = \lambda$;

(iii) there exist $k \in I_{(xy)^\ddagger}$ and $\nu \in \Lambda_{(xy)^\dagger}$ such that

$$\xi^{(i,x,\lambda)}_{y^\ddagger x^\dagger y^\ddagger,(xy)^\ddagger} * \xi^{(j,y,\mu)}_{(xy)^\dagger,y^\ddagger x^\dagger y^\ddagger} = [k], \qquad \eta^{(i,x,\lambda)}_{(xy)^\ddagger,x^\dagger y^\ddagger x^\dagger} \eta^{(j,y,\mu)}_{x^\dagger y^\ddagger x^\dagger,(xy)^\dagger} = [\nu];$$

(iv) for any $\alpha = \alpha(xy)^\dagger\alpha$ and $\beta = \beta(xy)^\ddagger\beta$ in F_T, we have

$$\xi^{(k,xy,\nu)}_{\alpha,(xy\alpha)^\ddagger} = \xi^{(i,x,\lambda)}_{(y\alpha)^\ddagger,(xy\alpha)^\ddagger} * \xi^{(j,y,\mu)}_{\alpha,(y\alpha)^\ddagger}, \qquad \eta^{(k,xy,\nu)}_{\beta,(\beta xy)^\dagger} = \eta^{(i,x,\lambda)}_{\beta,(\beta x)^\dagger} \eta^{(j,y,\mu)}_{(\beta x)^\dagger,(\beta xy)^\dagger}.$$

Then, $\bar{S}$ forms a strongly $\mathcal{P}$-regular semigroup with strong C-set

$$\{(i, x, \lambda) \in \bar{S} | x \in F_T\}$$

with respect to the following operation "$\circ$":

$$(i, x, \lambda) \circ (j, y, \mu) = ([\xi^{(i,x,\lambda)}_{y^\ddagger x^\dagger y^\ddagger,(xy)^\ddagger} * \xi^{(j,y,\mu)}_{(xy)^\dagger,y^\ddagger x^\dagger y^\ddagger}], xy, [\eta^{(i,x,\lambda)}_{(xy)^\ddagger,x^\dagger y^\ddagger x^\dagger} \eta^{(j,y,\mu)}_{x^\dagger y^\ddagger x^\dagger,(xy)^\dagger}]).$$

Conversely, every strongly $\mathcal{P}$-regular semigroup can be constructed in this way.

Proof Let (i, x, λ), (j, y, μ) and (k, z, ν) be three arbitrary elements in $\bar{S}$. Then, by Lemma 3.3, we see that condition (C.1) is meaningful. By condition (C.1) (iii), the operation "$\circ$" on $\bar{S}$ is well-defined. Evidently, condition (C.1) (iv) is equivalent to the following condition:

(C.1) (iv$'$) for any $\alpha = \alpha(xy)^\dagger\alpha$ and $\beta = \beta(xy)^\ddagger\beta$ in F_T, we have

$$\xi^{(i,x,\lambda)\circ(j,y,\mu)}_{\alpha,(xy\alpha)^\ddagger} = \xi^{(i,x,\lambda)}_{(y\alpha)^\ddagger,(xy\alpha)^\ddagger} * \xi^{(j,y,\mu)}_{\alpha,(y\alpha)^\ddagger}, \qquad \eta^{(i,x,\lambda)\circ(j,y,\mu)}_{\beta,(\beta xy)^\dagger} = \eta^{(i,x,\lambda)}_{\beta,(\beta x)^\dagger} \eta^{(j,y,\mu)}_{(\beta x)^\dagger,(\beta xy)^\dagger}.$$

Since

$$z^\ddagger(xy)^\dagger z^\ddagger = (z^\ddagger(xy)^\dagger z^\ddagger)(xy)^\dagger(z^\ddagger(xy)^\dagger z^\ddagger), (xy(z^\ddagger(xy)^\dagger z^\ddagger))^\ddagger = (xyz)^\ddagger$$

and

$$(yz^{\ddagger}(xy)^{\dagger}z^{\ddagger})^{\ddagger}=(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},$$

we have

$$\xi^{(i,x,\lambda)\circ(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(xyz)^{\ddagger}}=\xi^{(i,x,\lambda)}_{(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},(xyz)^{\ddagger}}*\xi^{(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}.$$

Observe that

$$(xyz)^{\dagger}(yz)^{\dagger}(xyz)^{\dagger}=(xyz)^{\dagger},\quad (yz(xyz)^{\dagger})^{\ddagger}=(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},(z(xyz)^{\dagger})^{\ddagger}=z^{\ddagger}(xy)^{\dagger}z^{\ddagger},$$

it follows that

$$\xi^{(j,y,\mu)\circ(k,z,\nu)}_{(xyz)^{\dagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}=\xi^{(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}*\xi^{(k,z,\nu)}_{(xyz)^{\dagger},z^{\ddagger}(xy)^{\dagger}z^{\ddagger}}.$$

Thus, we deduce that

$$\begin{aligned}
&[\xi^{(i,x,\lambda)\circ(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(xyz)^{\ddagger}}*\xi^{(k,z,\nu)}_{(xyz)^{\dagger},z^{\ddagger}(xy)^{\dagger}z^{\ddagger}}]\\
=\ &[\xi^{(i,x,\lambda)}_{(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},(xyz)^{\ddagger}}*\xi^{(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}*\xi^{(k,z,\nu)}_{(xyz)^{\dagger},z^{\ddagger}(xy)^{\dagger}z^{\ddagger}}]\\
=\ &[\xi^{(i,x,\lambda)}_{(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},(xyz)^{\ddagger}}*\xi^{(j,y,\mu)\circ(k,z,\nu)}_{(xyz)^{\dagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}].
\end{aligned}\tag{3.1}$$

Similarly, we also have

$$[\eta^{(i,x,\lambda)\circ(j,y,\mu)}_{(xyz)^{\ddagger},(xy)^{\dagger}z^{\ddagger}(xy)^{\dagger}}\eta^{(k,z,\nu)}_{(xy)^{\dagger}z^{\ddagger}(xy)^{\dagger},(xyz)^{\dagger}}]=[\eta^{(i,x,\lambda)}_{(xyz)^{\ddagger},x^{\dagger}(yz)^{\ddagger}x^{\dagger}}\eta^{(j,y,\mu)\circ(k,z,\nu)}_{x^{\dagger}(yz)^{\ddagger}x^{\dagger},(xyz)^{\dagger}}].$$

Hence

$$\begin{aligned}
&((i,x,\lambda)\circ(j,y,\mu))\circ(k,z,\nu)\\
=\ &([\xi^{(i,x,\lambda)\circ(j,y,\mu)}_{z^{\ddagger}(xy)^{\dagger}z^{\ddagger},(xyz)^{\ddagger}}*\xi^{(k,z,\nu)}_{(xyz)^{\dagger},z^{\ddagger}(xy)^{\dagger}z^{\ddagger}}],(xy)z,[\eta^{(i,x,\lambda)\circ(j,y,\mu)}_{(xyz)^{\ddagger},(xy)^{\dagger}z^{\ddagger}(xy)^{\dagger}}\eta^{(k,z,\nu)}_{(xy)^{\dagger}z^{\ddagger}(xy)^{\dagger},(xyz)^{\dagger}}])\\
=\ &([\xi^{(i,x,\lambda)}_{(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger},(xyz)^{\ddagger}}*\xi^{(j,y,\mu)\circ(k,z,\nu)}_{(xyz)^{\dagger},(yz)^{\ddagger}x^{\dagger}(yz)^{\ddagger}}],x(yz),[\eta^{(i,x,\lambda)}_{(xyz)^{\ddagger},x^{\dagger}(yz)^{\ddagger}x^{\dagger}}\eta^{(j,y,\mu)\circ(k,z,\nu)}_{x^{\dagger}(yz)^{\ddagger}x^{\dagger},(xyz)^{\dagger}}])\\
=\ &(i,x,\lambda)\circ((j,y,\mu)\circ(k,z,\nu)).
\end{aligned}$$

This proves that $\bar{S}$ is indeed a semigroup with respect to the operation "$\circ$".

Let $(i,x,\lambda)\in\bar{S}$. If $(i,x,\lambda)\in E(\bar{S})$, then we immediately see that $x\in E(T)$. Conversely, if $x\in E(T)$, then by (1) of Lemma 3.3,

$$(i,x,\lambda)\circ(i,x,\lambda)=([\xi^{(i,x,\lambda)}_{x^{\ddagger},x^{\ddagger}}*\xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}],x,[\eta^{(i,x,\lambda)}_{x^{\ddagger},x^{\dagger}}\eta^{(i,x,\lambda)}_{x^{\dagger},x^{\dagger}}]).$$

By (i) and (ii) of condition (C.1), we can see that

$$([\xi^{(i,x,\lambda)}_{x^{\ddagger},x^{\ddagger}}*\xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}],x,[\eta^{(i,x,\lambda)}_{x^{\ddagger},x^{\dagger}}\eta^{(i,x,\lambda)}_{x^{\dagger},x^{\dagger}}])=(i,x,\lambda).$$

This shows that

$$E(\bar{S})=\{(i,x,\lambda)\in\bar{S}|x\in E(T)\}.$$

Let $\bar{P}=\{(i,x,\lambda)\in\bar{S}|x\in F_T\}$. Since $F_T^2\subseteq E(T)$ and $xF_Tx\subseteq F_T$ for each $x\in F_T$ by Lemma 2.1 (1), we can easily see that $\bar{P}^2\subseteq E(\bar{S})$ and $p\circ\bar{P}\circ p\subseteq\bar{P}$ for any $p\in\bar{P}$. Now, let

$(i,x,\lambda) \in \bar{S}$ and choose $(j,y,\mu) = (j,x^*,\mu) \in \bar{S}$. Then, we have $y^\ddagger = x^\dagger$ and $y^\dagger = x^\ddagger$. By condition (C.1) (i), and the fact (3.1) and its dual,

$$\begin{aligned}(i,x,\lambda)\circ(j,y,\mu)\circ(i,x,\lambda) &= ([\xi^{(i,x,\lambda)}_{x^\dagger,x^\ddagger} * \xi^{(j,y,\mu)}_{y^\dagger,y^\ddagger} * \xi^{(i,x,\lambda)}_{x^\dagger,x^\ddagger}], xyx, [\eta^{(i,x,\lambda)}_{x^\ddagger,x^\dagger}\eta^{(j,y,\mu)}_{y^\ddagger,y^\dagger}\eta^{(i,x,\lambda)}_{x^\ddagger,x^\dagger}])\\ &= (i,x,\lambda).\end{aligned}$$

A similar argument also shows that $(j,y,\mu)\circ(i,x,\lambda)\circ(j,y,\mu) = (j,y,\mu)$. Thus, $(j,x^*,\mu) \in V((i,x,\lambda))$. Since $xF^1_T x^* \subseteq F_T$ and $x^* F^1_T x \subseteq F_T$ by Lemma 2.1 (1), we have

$$(i,x,\lambda)\circ\bar{P}^1\circ(j,x^*,\mu), (j,x^*,\mu)\circ\bar{P}^1\circ(i,x,\lambda) \subseteq \bar{P}.$$

This implies that $(j,x^*,\mu) \in V_{\bar{P}}((i,x,\lambda))$.

At last, if $p = (i,x,\lambda), q = (j,y,\mu) \in \bar{P}$ and $p\circ q = (,xy,) \in \bar{P}$, then by the definition of $\bar{P}$, we have $x,y,xy \in F_T$, and so $yx = xy \in F_T$ by (3) of Lemma 2.1. This implies $q\circ p = (,yx,) \in \bar{P}$. By the definition of strongly $\mathcal{P}$-regular semigroups, $\bar{S}(\bar{P})$ is a strongly $\mathcal{P}$-regular semigroup.

Conversely, let $S(P)$ be a strongly $\mathcal{P}$-regular semigroup and $T = S(P)/\gamma$. Then T is a regular *-semigroup with the operation $(a\gamma)^* = a^+\gamma$ where $a^+ \in V_P(a)$, and $F_{(T,\ ^*)} = P\gamma$. Denote $F_{(T,\ ^*)}$ by F_T. For any $\alpha \in F_T$, we define

$$I_\alpha = \{R_a | \bar{a}^\ddagger = \alpha, a \in S\}, \qquad \Lambda_\alpha = \{L_a | \bar{a}^\dagger = \alpha, a \in S\},$$

where $\bar{a}$ denotes the γ-class containing a for each $a \in S$. By Lemma 2.1 (4), (5), we can see that $I_\alpha \cap I_\beta = \Lambda_\alpha \cap \Lambda_\beta = \emptyset$ whenever $\beta \neq \alpha$ in F_T. Denote

$$\bar{S} = \{(R_b, \bar{a}, L_c) \mid R_b \in I_{\bar{a}^\ddagger},\ L_c \in \Lambda_{\bar{a}^\dagger},\ a,b,c \in S\}.$$

Now, for any $(R_b,\bar{a},L_c) \in \bar{S}$, let $a_1 \in V_P(a), b_1 \in V_P(b)$ and $c_1 \in V_P(c)$. Then

$$\bar{b}\bar{b}_1 = \bar{a}^\ddagger = \bar{a}\bar{a}_1, \qquad \bar{c}_1\bar{c} = \bar{a}^\dagger = \bar{a}_1\bar{a}.$$

Hence, by (1) and (3) of Lemma 2.2 and the fact that $bb_1, aa_1, c_1c, a_1a \in P$, we have

$$bb_1, aa_1 \in V_P(bb_1) = V_P(aa_1),\ c_1c, a_1a \in V_P(c_1c) = V_P(a_1a).$$

Let $d = bb_1ac_1c$. Then, we have

$$ca_1d = ca_1bb_1ac_1c = ca_1(aa_1bb_1aa_1)ac_1c = ca_1ac_1c = c(c_1ca_1ac_1c) = c.$$

This yields that $c\mathcal{L}d$. Dually, $b\mathcal{R}d$. Further, we have $\bar{d} = \bar{b}\bar{b}_1\bar{a}\bar{c}_1\bar{c} = \bar{a}^\ddagger\bar{a}\bar{a}^\dagger = \bar{a}$. This implies that $(R_b,\bar{a},L_c) = (R_d,\bar{d},L_d)$ and whence $\bar{S} = \{(R_a,\bar{a},L_a) | a \in S\}$. Define

$$\phi:\quad S \to \bar{S},\quad a \mapsto (R_a,\bar{a},L_a).$$

Then, by (2) of Lemma 2.2, ϕ is a bijection from S onto $\bar{S}$. Let

$$(i,x,\lambda) = a\phi = (R_a,\bar{a},L_a), (j,y,\mu) = b\phi = (R_b,\bar{b},L_b)$$

be two arbitrary elements in $\bar{S}$, where $a, b \in S$. For any $\alpha = \alpha x^{\dagger}\alpha$ and $\beta = \beta x^{\ddagger}\beta$ in F_T, define

$$\xi^{(i,x,\lambda)}_{\alpha,(x\alpha)^{\ddagger}} = \rho_a|_{I_\alpha}, \qquad \eta^{(i,x,\lambda)}_{\beta,(\beta x)^{\dagger}} = \delta_a|_{\Lambda_\beta},$$

where ρ_a, δ_a are defined as in Lemma 3.4. Then, it is clear that

$$\xi^{(i,x,\lambda)}_{\alpha,(x\alpha)^{\ddagger}} \in \mathcal{F}_l(I_\alpha, I_{(x\alpha)^{\ddagger}}) \quad \text{and} \quad \eta^{(i,x,\lambda)}_{\beta,(\beta x)^{\dagger}} \in \mathcal{F}_r(\Lambda_\beta, \Lambda_{(\beta x)^{\dagger}}).$$

For any $R_c \in I_{x^{\dagger}}$ $(c \in S)$, we may choose $c_1 \in V_P(c)$ and $a_1 \in V_P(a)$. Since $\bar{c}\bar{c}_1 = \bar{c}^{\ddagger} = x^{\dagger} = \bar{a}^{\dagger} = \bar{a}_1\bar{a}$, $a_1 a \in V_P(a_1 a) = V_P(cc_1)$ by (1) and (3) of Lemma 2.2. Hence, $acc_1(a_1 a) = a(a_1 acc_1 a_1 a) = aa_1 a = a$, whence $acc_1\mathcal{R}a$ and so

$$\xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}(R_c) = \xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}(R_{cc_1}) = \rho_a(R_{cc_1}) = R_{acc_1} = R_a = i. \tag{3.2}$$

Therefore, $\xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}} = [i]$. Consequently, for any $R_d \in I_{(xy)^{\dagger}}$ $(d \in S)$, in view of the fact (3.2), we have

$$\xi^{(i,x,\lambda)}_{y^{\ddagger}x^{\dagger}y^{\ddagger},(xy)^{\ddagger}} * \xi^{(j,y,\mu)}_{(xy)^{\dagger},y^{\ddagger}x^{\dagger}y^{\ddagger}}(R_d) = \rho_a * \rho_b(R_d)$$
$$= \rho_{ab}(R_d) = \xi^{(ab)\phi}_{(xy)^{\dagger},(xy)^{\ddagger}}(R_d) = \xi^{(R_{ab},xy,L_{ab})}_{(xy)^{\dagger},(xy)^{\ddagger}}(R_d) = R_{ab}.$$

This leads to

$$\xi^{(i,x,\lambda)}_{y^{\ddagger}x^{\dagger}y^{\ddagger},(xy)^{\ddagger}} * \xi^{(j,y,\mu)}_{(xy)^{\dagger},y^{\ddagger}x^{\dagger}y^{\ddagger}} = [R_{ab}].$$

A similar argument shows that

$$\eta^{(i,x,\lambda)}_{x^{\ddagger},x^{\dagger}} = [\lambda] \quad \text{and} \quad \eta^{(i,x,\lambda)}_{(xy)^{\ddagger},x^{\dagger}y^{\ddagger}x^{\dagger}}\eta^{(j,y,\mu)}_{x^{\dagger}y^{\ddagger}x^{\dagger},(xy)^{\dagger}} = [L_{ab}].$$

This shows that the operation

$$(i,x,\lambda)\circ(j,y,\mu) = ([\xi^{(i,x,\lambda)}_{y^{\ddagger}x^{\dagger}y^{\ddagger},(xy)^{\ddagger}} * \xi^{(j,y,\mu)}_{(xy)^{\dagger},y^{\ddagger}x^{\dagger}y^{\ddagger}}], xy, [\eta^{(i,x,\lambda)}_{(xy)^{\ddagger},x^{\dagger}y^{\ddagger}x^{\dagger}}\eta^{(j,y,\mu)}_{x^{\dagger}y^{\ddagger}x^{\dagger},(xy)^{\dagger}}]) \tag{3.3}$$

on $\bar{S}$ is well-defined. Moreover, for any $\alpha = \alpha(xy)^{\dagger}\alpha$ in F_T, we have

$$\xi^{(i,x,\lambda)}_{(y\alpha)^{\ddagger},(xy\alpha)^{\ddagger}} * \xi^{(j,y,\mu)}_{\alpha,(y\alpha)^{\ddagger}} = (\rho_a * \rho_b)|_{I_\alpha} = \rho_{ab}|_{I_\alpha} = \xi^{(i,x,\lambda)\circ(j,y,\mu)}_{\alpha,(xy\alpha)^{\ddagger}}.$$

Similarly, we also have

$$\eta^{(i,x,\lambda)\circ(j,y,\mu)}_{\beta,(\beta xy)^{\dagger}} = \eta^{(i,x,\lambda)}_{\beta,(\beta x)^{\dagger}}\eta^{(j,y,\mu)}_{(\beta x)^{\dagger},(\beta xy)^{\dagger}}$$

for $\beta = \beta(xy)^{\ddagger}\beta$ in F_T.

At last, if $(i,x,\lambda) \in \bar{S}$, $x \in E(T)$ and $(i,x,\lambda) = a\phi$, then $(i,x,\lambda) = (R_a, \bar{a}, L_a)$ and $x = \bar{a} \in E(S/\gamma)$. So there exists $a_1 \in E(S)$ such that $(a_1, a) \in \gamma$. By (1) and (3) of Lemma 2.2, we have

$$V_P(a) = V_P(a_1) \subseteq E(S),$$

which implies that $a \in E(S)$ by (3) of Lemma 2.2 again. Thus

$$\xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}(i) = \xi^{(i,x,\lambda)}_{x^{\dagger},x^{\ddagger}}(R_a) = \rho_a(R_a) = R_{a^2} = R_a = i.$$

Dually, we have $(\lambda)\eta_{x^\dagger,x^\dagger}^{(i,x,\lambda)}=\lambda$.

Thus, by the proof of the direct part, we have shown that $\bar{S}$ forms a strongly $\mathcal{P}$-regular semigroup with strong C-set $\bar{P}=\{(i,x,\lambda)\in\bar{S}|x\in F_T\}$ with respect to the operation "$\circ$" as defined above.

Now, for any $p\in P$, since $\bar{p}\in F_T=P\gamma$, we have $p\phi\in\bar{P}$. On the other hand, if $c\in S$ and $c\phi=(k,z,\nu)\in\bar{P}$, then $\bar{c}=z\in F_T=P\gamma$ and so $\bar{c}=\bar{p}$ for some $p\in P$. This implies that $V_P(p)=V_P(c)$ by (1) of Lemma 2.2. Since $p\in V_P(p)$, it follows that $c\in V_P(p)\subseteq P$ from Lemma 2.2 (3). Thus, $P\phi=S\phi\cap\bar{P}$. Observe that

$$(ab)\phi=(R_{ab},\overline{ab},L_{ab})=(i,x,\lambda)\circ(j,y,\mu)=a\phi\circ b\phi$$

by the identity (3.3), the mapping ϕ is a $\mathcal{P}$-isomorphism from S onto $\bar{S}$. The proof is completed.

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强$\mathcal{P}$-正则半群的一个构造

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摘 要: 本文研究了强$\mathcal{P}$-正则半群的结构. 利用正则*-半群和一族满足某种条件的映射给出了强$\mathcal{P}$-正则半群的一个构造, 推广和丰富了相关文献中纯正半群的结果.

关键词: 强$\mathcal{P}$-正则半群; 正则*-半群; 结构

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