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DOUBLE SUBORDINATION PRESERVING PROPERTIES FOR THE LIU-OWA INTEGRAL OPERATOR

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Abstract: In the paper, we investigate subordination and superordination preserving problems for analytic and multivalent functions in the open unit disk, which are associated with the Liu-Owa integral operator. By using the method of differential subordination, we derive sandwich-type results of functions belonging to these classes, which generalize and improve some previous known results.

Keywords: analytic and multivalent function; differential subordination; superordination; Liu-Owa integral operator; sandwich-type result

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1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}.$$

Let f and g be two members of $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have (see [8] and [21])

$$f \prec g \ \ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.1 (see [8]) Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If $\mathfrak{p}$ is analytic in $\mathbb{U}$ and satisfies the following differential subordination

$$\phi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \prec h(z) \ \ (z \in \mathbb{U}), \tag{1.1}$$

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then $\mathfrak{p}$ is called a solution of the differential subordination (1.1). The univalent function $\mathfrak{q}$ is called a dominant of the solutions of the differential subordination (1.1), if $\mathfrak{p} \prec \mathfrak{q}$ for all $\mathfrak{p}$ satisfying (1.1). A dominant $\tilde{\mathfrak{q}}$ that satisfies $\tilde{\mathfrak{q}} \prec \mathfrak{q}$ for all dominants $\mathfrak{q}$ of (1.1) is said to be the best dominant.

Definition 1.2 (see [9]) Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If $\mathfrak{p}$ and $\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z))$ are univalent in $\mathbb{U}$ and satisfy the following differential superordination

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \ \ (z \in \mathbb{U}), \tag{1.2}$$

then $\mathfrak{p}$ is called a solution of the differential superordination (1.2). An analytic function $\mathfrak{q}$ is called a subordination of the solutions of the differential superordination (1.2), if $\mathfrak{q} \prec \mathfrak{p}$ for all $\mathfrak{p}$ satisfying (1.2). A univalent subordination $\tilde{\mathfrak{q}}$ that satisfies $\mathfrak{q} \prec \tilde{\mathfrak{q}}$ for all subordinations $\mathfrak{q}$ of (1.2) is said to be the best subordination.

Definition 1.3 (see [9]) We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \left\{ \xi : \xi \in \partial\mathbb{U} \ \text{and} \ \lim_{z \to \xi} f(z) = \infty \right\},$$

and are such that $f'(\xi) \neq 0$ ($\xi \in \partial\mathbb{U} \setminus E(f)$).

Let $\mathcal{A}(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ \ (p \in \mathbb{N}; \ z \in \mathbb{U}). \tag{1.3}$$

Motivated essentially by Jung et al. [4], Liu and Owa [5] introduced the integral operator $Q^{\alpha}_{\beta,p} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as follows:

$$Q^{\alpha}_{\beta,p} f(z) = \begin{pmatrix} p + \alpha + \beta - 1 \\ \\ p + \beta - 1 \end{pmatrix} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha - 1} t^{\beta - 1} f(t) dt \ \ (\alpha > 0; \beta > -1; p \in \mathbb{N}) \tag{1.4}$$

and

$$Q^{0}_{\beta,p} f(z) = f(z) \ \ (\alpha = 0; \ \beta > -1).$$

If $f \in \mathcal{A}(p)$ given by (1.3), then from (1.4), we deduce that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \frac{\Gamma(\alpha + \beta + p)}{\Gamma(\beta + p)} \sum_{n=1}^{\infty} \frac{\Gamma(\beta + p + n)}{\Gamma(\alpha + \beta + p + n)} a_{p+n} z^{p+n} \ \ (\alpha > 0; \beta > -1; p \in \mathbb{N}). \tag{1.5}$$

It is easily verified from definition (1.5) that (see [5])

$$z\left(Q^{\alpha}_{\beta,p} f(z)\right)' = (\alpha + \beta + p - 1) Q^{\alpha-1}_{\beta,p} f(z) - (\alpha + \beta - 1) Q^{\alpha}_{\beta,p} f(z). \tag{1.6}$$

We note that, for $p = 1$, we obtain the operator $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$ defined by Jung et al. [4], and studied by Aouf [16] and Gao et al. [6]. On the other hand, if we set $\alpha = 1, \beta = c$ in (1.5), we obtain the generalized Libera operator J_c $(c > -p)$ defined by (see [1, 13]; also [19, 20])

$$Q^{1}_{c,p}f(z) = J_c(f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \ \ (c > -p;\ z \in \mathbb{U}). \tag{1.7}$$

With the help of the principle of subordination, various subordination preserving properties involving certain integral operators for analytic functions in $\mathbb{U}$ were investigated by Bulbocă [2], Miller et al. [10], and Owa and Srivastava [14]. Moreover, Miller and Mocanu [9] considered differential superordinations, as the dual problem of differential subordinations (see also [3]). In the present paper, we investigate some subordination and superordination preserving properties of the integral operator $Q^{\alpha}_{\beta,p}$ defined by (1.4). Also, we obtain several sandwich-type results for these multivalent functions.

In order to establish our main results, we shall require the following lemmas.

Lemma 1.1 (see [11]) Suppose that the function $H : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies the following condition

$$\mathrm{Re}\,\{H(is,t)\} \leq 0$$

for all real s and $t \leq -\frac{n(1+s^2)}{2}$ $(n \in \mathbb{N})$. If the function $\mathfrak{p}(z) = 1 + \mathfrak{p}_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\,\{H(\mathfrak{p}(z), z\mathfrak{p}'(z))\} > 0 \ \ (z \in \mathbb{U}),$$

then $\mathrm{Re}\{\mathfrak{p}(z)\} > 0$ for $z \in \mathbb{U}$.

Lemma 1.2 (see [12]) Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = b$. If $Re\,\{\kappa h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the following differential equation

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \ \ (z \in \mathbb{U};\ q(0) = b)$$

is analytic in $\mathbb{U}$ and satisfies the inequality given by $\mathrm{Re}\,\{\kappa q(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.

Lemma 1.3 (see [8]) Let $\mathfrak{p} \in \mathcal{Q}$ with $\phi(0) = a$ and let the function $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \in \mathbb{N}$. If q is not subordinate to $\mathfrak{p}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \ \text{ and } \ \xi_0 \in \partial\mathbb{U} \setminus E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\mathbb{U}),\ q(z_0) = \mathfrak{p}(z_0) \ \text{ and } \ z_0 q'(z_0) = m\xi_0 \mathfrak{p}'(\xi_0) \ \ (m \geq n),$$

where $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(\cdot,t)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ $(z \in \mathbb{U};\ 0 \leq t_1 \leq t_2)$.

Lemma 1.4 (see [9]) Let $q \in \mathcal{H}[a,1]$ and $\varphi:\ \mathbb{C}^2 \longrightarrow \mathbb{C}$. Also let

$$\varphi(q(z), zq'(z)) = h(z) \ \ (z \in \mathbb{U}).$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $\mathfrak{p} \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \ \ (z \in \mathbb{U}),$$

implies that $q(z) \prec \mathfrak{p}(z)$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 1.5 (see [15]) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ with $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \ \ (z \in \mathbb{U};\ 0 \le t < \infty).$$

2 Main Results

First of all, we begin by proving the following subordination theorem involving the operator $Q^{\alpha}_{\beta,p}$ defined by (1.4). Unless otherwise mentioned, we assume throughout this paper that $\alpha \ge 1, \beta > -1, 0 < \lambda \le 1, \mu > 0, p \in \mathbb{N}$ and $z \in \mathbb{U}$.

Theorem 2.1 Let $f, g \in \mathcal{A}(p)$ and suppose that

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z) = (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}\right), \tag{2.1}$$

where

$$\delta = \frac{\lambda^2 + \mu^2(\alpha+\beta+p-1)^2 - |\lambda^2 - \mu^2(\alpha+\beta+p-1)^2|}{4\lambda\mu(\alpha+\beta+p-1)}. \tag{2.2}$$

Then the following subordination condition

$$(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \phi(z) \tag{2.3}$$

implies that

$$\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}.$$

Moreover, the function $\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}$ is the best dominant.

Proof Let us define the functions F and G, respectively, by

$$F(z) = \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \ \text{ and } \ G(z) = \left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}. \tag{2.4}$$

We first prove that, if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \ \ (z \in \mathbb{U}), \tag{2.5}$$

then $\mathrm{Re}\{q(z)\} > 0$ for $z \in \mathbb{U}$.

Taking the logarithmic differentiation on both sides of the second equation in (2.4) and using (1.6) for $g \in \mathcal{A}(p)$, we have

$$\phi(z) = G(z) + \frac{\lambda z G'(z)}{\mu(\alpha+\beta+p-1)}. \tag{2.6}$$

Differentiating both sides of (2.6) with respect to z yields

$$\phi'(z) = \left(1 + \frac{\lambda}{\mu(\alpha+\beta+p-1)}\right) G'(z) + \frac{\lambda z G''(z)}{\mu(\alpha+\beta+p-1)}. \tag{2.7}$$

Combining (2.5) and (2.7), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \mu(\alpha+\beta+p-1)/\lambda} = h(z) \ \ (z \in \mathbb{U}). \tag{2.8}$$

Thus, form (2.1) and (2.8), we see that

$$\mathrm{Re}\left\{h(z) + \frac{\mu(\alpha+\beta+p-1)}{\lambda}\right\} > 0 \ \ (z \in \mathbb{U}).$$

Also, in view of Lemma 1.2, we conclude that the differential equation (2.8) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $q(0) = h(0) = 1$.

Let us put

$$H(u, v) = u + \frac{v}{u + \mu(\alpha+\beta+p-1)/\lambda} + \delta, \tag{2.9}$$

where δ is given by (2.2). From (2.1), (2.8), together with (2.9), we obtain

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \ \ (z \in \mathbb{U}).$$

Now, we proceed to show that

$$\mathrm{Re}\{H(is, t)\} \leq 0 \ \ \left(s \in \mathbb{R};\ t \leq -\frac{1+s^2}{2}\right). \tag{2.10}$$

In fact, from (2.9), we have

$$\begin{aligned} \mathrm{Re}\{H(is,t)\} &= \mathrm{Re}\left\{is + \frac{t}{is + \mu(\alpha+\beta+p-1)/\lambda} + \delta\right\} \\ &= \frac{t\lambda\mu(\alpha+\beta+p-1)}{\lambda^2 s^2 + \mu^2(\alpha+\beta+p-1)^2} + \delta \\ &\leq -\frac{E_\delta(s)}{2[\lambda^2 s^2 + \mu^2(\alpha+\beta+p-1)^2]}, \end{aligned}$$

where

$$E_\delta(s) = [\lambda\mu(\alpha+\beta+p-1) - 2\delta\lambda^2]s^2 - 2\delta\mu^2(\alpha+\beta+p-1)^2 + \lambda\mu(\alpha+\beta+p-1). \tag{2.11}$$

For δ given by (2.2), we can prove easily that the expression $E_\delta(s)$ in (2.11) is greater than or equal to zero, which implies that (2.10) holds true. Therefore, by using Lemma 1.1, we conclude that $\mathrm{Re}\{q(z)\} > 0$ for $z \in \mathbb{U}$, that is, that the function G defined by (2.4) is convex (univalent) in $\mathbb{U}$.

Next, we prove that $F \prec G$ $(z \in \mathbb{U})$ holds for the functions F and G defined by (2.4). Without loss of generality, we assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and that $G'(\xi) \neq 0$ for $|\xi| = 1$.

Let us define the function $L(z,t)$ by

$$L(z,t) = G(z) + \frac{\lambda(1+t)}{\mu(\alpha+\beta+p-1)} zG'(z) \ \ (0 \leq t < \infty;\ z \in \mathbb{U}).$$

Then

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1+\frac{\lambda(1+t)}{\mu(\alpha+\beta+p-1)}\right) = 1+\frac{\lambda(1+t)}{\mu(\alpha+\beta+p-1)} \neq 0 \ (0 \leq t < \infty;\ z \in \mathbb{U}),$$

and this show that the function $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots$ satisfies the conditions $a_1(t) \neq 0$ for all $t \in [0,\infty)$ and $\lim\limits_{t\to\infty} |a_1(t)| = +\infty$.

Moreover, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\mu(\alpha+\beta+p-1)+\lambda(1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0 \ \ (0 \leq t < \infty),$$

because G is convex in $\mathbb{U}$. Hence, by virtue of Lemma 1.5, we deduce that $L(z,t)$ is a subordination chain. We notice from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{\lambda zG'(z)}{\mu(\alpha+\beta+p-1)} = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \ \ (0 \leq t < \infty),$$

which implies that

$$L(\xi,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \ \ (\xi \in \partial\mathbb{U};\ 0 \leq t < \infty). \tag{2.12}$$

Now, we suppose that F is not subordinate G, then by Lemma 1.3, there exist two points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U}$, such that

$$F(z_0) = G(\xi_0) \ \text{ and } \ z_0F'(z_0) = (1+t)\xi_0G'(\xi_0) \ \ (0 \leq t < \infty).$$

Thus, by means of subordination condition (2.3), we have

$$\begin{aligned} L(\xi_0,t) = G(\xi_0) + \frac{\lambda(1+t)\xi_0G'(\xi_0)}{\mu(\alpha+\beta+p-1)} &= F(z_0) + \frac{\lambda z_0F'(z_0)}{\mu(\alpha+\beta+p-1)} \\ = \quad (1-\lambda)\left(\frac{Q^\alpha_{\beta,p}f(z_0)}{z_0^p}\right)^\mu &+ \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z_0)}{Q^\alpha_{\beta,p}f(z_0)}\right)\left(\frac{Q^\alpha_{\beta,p}f(z_0)}{z_0^p}\right)^\mu \in \phi(\mathbb{U}), \end{aligned}$$

which contradicts to (2.12). Hence, we deduce that $F \prec G$. Considering $F = G$, we know that the function G is the best dominant. This completes the proof of Theorem 2.1.

We next derive a dual result of Theorem 2.1, in the sense that subordinations are replaced by superordinations.

Theorem 2.2 Let $f, g \in \mathcal{A}(p)$ and suppose that

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z) = (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}g(z)}{Q^{\alpha}_{\beta,p}g(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}\right),$$

where δ is given by (2.2). If the function

$$(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1] \cap \mathcal{Q}$. Then the following superordination condition

$$\phi(z) \prec (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}$$

implies that

$$\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}.$$

Moreover, the function $\left(\frac{Q^{\alpha}_{\beta,p}g(z)}{z^p}\right)^{\mu}$ is the best subordination.

Proof Let us define the functions F and G just as (2.4). We first observe that, if the function q is defined by (2.5), then we obtain from (2.6) that

$$\phi(z) = G(z) + \frac{\lambda z G'(z)}{\mu(\alpha+\beta+p-1)} = \varphi\left(G(z), zG'(z)\right). \tag{2.13}$$

By applying the same method as in the proof of Theorem 2.1, we can prove that $\operatorname{Re}\{q(z)\} > 0$ for $z \in \mathbb{U}$. That is, the function G defined by (2.4) is convex (univalent) in $\mathbb{U}$.

Next, we will show that $G \prec F$. For this purpose, we consider the function $L(z,t)$ defined by

$$L(z,t) = G(z) + \frac{\lambda t}{\mu(\alpha+\beta+p-1)} zG'(z) \ \ (0 \le t < \infty;\ z \in \mathbb{U}).$$

Since the function G is convex in $\mathbb{U}$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore, by Lemma 1.4, we conclude that $G \prec F$. Furthermore, since the differential equation (2.13) has the univalent solution G, it is the best subordination of the given differential superordination. We thus complete the proof of Theorem 2.2.

If we combine Theorems 2.1 and 2.2, then we get the following sandwich-type theorem.

Theorem 2.3 Let $f,\ g_j \in \mathcal{A}(p)$ $(j=1,2)$ and suppose that

$$\mathrm{Re}\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}g_j(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}g_j(z)}{Q^{\alpha}_{\beta,p}g_j(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}g_j(z)}{z^p}\right)^{\mu}\right), \tag{2.14}$$

where δ is given by (2.2). If the function

$$(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \in \mathcal{H}[1,1]\cap\mathcal{Q}$. Then the following subordination relationship

$$\phi_1(z) \prec (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \phi_2(z)$$

implies that

$$\left(\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}\right)^{\mu}.$$

Moreover, the functions $\left(\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p}\right)^{\mu}$ and $\left(\frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}\right)^{\mu}$ are, respectively, the best subordination and the best dominant.

Remark 2.1 By putting $\lambda = 1$ in Theorems 2.1–2.3, we obtain the results obtained by Aouf and Seoudy [18].

Remark 2.2 By taking $\lambda = \mu = 1$ in Theorems 2.1–2.3, we obtain the results obtained by Aouf and Seoudy [17].

3 Corollaries and Consequences

Since the assumption of Theorem 2.3 of the preceding section that the functions

$$(1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \quad \text{and} \quad \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}$$

need to be univalent in $\mathbb{U}$, is not so easy to check, we will replace these conditions by another simple conditions in the following result.

Corollary 3.1 Let $f,\ g_j \in \mathcal{A}(p)$ $(j=1,2)$. Suppose that the condition (2.14) is satisfied and

$$\mathrm{Re}\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta \quad \left(\psi(z) = (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}\right), \tag{3.1}$$

where δ is given by (2.2). Then the following subordination relationship

$$\phi_1(z) \prec (1-\lambda)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{Q^{\alpha-1}_{\beta,p}f(z)}{Q^{\alpha}_{\beta,p}f(z)}\right)\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \phi_2(z)$$

implies that

$$\left(\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu} \prec \left(\frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}\right)^{\mu}.$$

Moreover, the functions $\left(\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p}\right)^{\mu}$ and $\left(\frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}\right)^{\mu}$ are, respectively, the best subordination and the best dominant.

Proof To prove our result, we have to show that the condition (3.1) implies the univalence of ψ and $F(z) = \left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p}\right)^{\mu}$. Since δ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \delta \leq \frac{1}{2}$, condition (3.1) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [7]) and hence ψ is univalent in $\mathbb{U}$. Also, by using the same techniques as in the proof of Theorem 2.1, we can prove that F is convex (univalent) in $\mathbb{U}$, and so the details may be omitted. Therefore, by applying Theorem 2.3, we obtain the desired result.

Upon setting $\mu = 1$ in Theorem 2.3, we are easily led to the following result.

Corollary 3.2 Let $f,\ g_j \in \mathcal{A}(p)$ $(j = 1, 2)$ and suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = \frac{(1-\lambda)Q^{\alpha}_{\beta,p}g_j(z) + \lambda Q^{\alpha-1}_{\beta,p}g_j(z)}{z^p} \ \ (j = 1, 2);\ z \in \mathbb{U}\right),$$

where δ is given by (2.2) with $\mu = 1$. If the function

$$\frac{(1-\lambda)Q^{\alpha}_{\beta,p}f(z) + \lambda Q^{\alpha-1}_{\beta,p}f(z)}{z^p}$$

is univalent in $\mathbb{U}$ and $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^p} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$. Then the following subordination relationship

$$\phi_1(z) \prec \frac{(1-\lambda)Q^{\alpha}_{\beta,p}f(z) + \lambda Q^{\alpha-1}_{\beta,p}f(z)}{z^p} \prec \phi_2(z)$$

implies that

$$\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p} \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^p} \prec \frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}.$$

Moreover, the functions $\frac{Q^{\alpha}_{\beta,p}g_1(z)}{z^p}$ and $\frac{Q^{\alpha}_{\beta,p}g_2(z)}{z^p}$ are, respectively, the best subordination and the best dominant.

By putting $\alpha = 1$ and $\beta = c$ in Theorem 2.3, we can derive the following result involving the integral operator J_c defined by (1.7).

Corollary 3.3 Let $f,\ g_j \in \mathcal{A}(p)$ $(j = 1, 2)$ and suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = (1-\lambda)\left(\frac{J_c(g_j)(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{g_j(z)}{J_c(g_j)(z)}\right)\left(\frac{J_c(g_j)(z)}{z^p}\right)^{\mu}\right),$$

where

$$\delta = \frac{\lambda^2 + \mu^2(c+p)^2 - |\lambda^2 - \mu^2(c+p)^2|}{4\lambda\mu(c+p)} \quad (c > -p).$$

If the function

$$(1-\lambda)\left(\frac{J_c(f)(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{f(z)}{J_c(f)(z)}\right)\left(\frac{J_c(f)(z)}{z^p}\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(\frac{J_c(f)(z)}{z^p}\right)^\mu \in \mathcal{H}[1,1]\cap\mathcal{Q}$. Then the following subordination relationship

$$\phi_1(z) \prec (1-\lambda)\left(\frac{J_c(f)(z)}{z^p}\right)^\mu + \lambda\left(\frac{f(z)}{J_c(f)(z)}\right)\left(\frac{J_c(f)(z)}{z^p}\right)^\mu \prec \phi_2(z)$$

implies that

$$\left(\frac{J_c(g_1)(z)}{z^p}\right)^\mu \prec \left(\frac{J_c(f)(z)}{z^p}\right)^\mu \prec \left(\frac{J_c(g_2)(z)}{z^p}\right)^\mu.$$

Moreover, the functions $\left(\frac{J_c(g_1)(z)}{z^p}\right)^\mu$ and $\left(\frac{J_c(g_2)(z)}{z^p}\right)^\mu$ are, respectively, the best subordination and the best dominant.

Further, setting $\lambda = \mu = 1$ in Corollary 3.3, we have the following result.

Corollary 3.4 Let $f,\ g_j \in \mathcal{A}(p)$ $(j = 1,2)$ and suppose that

$$\mathrm{Re}\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z) = \frac{g_j(z)}{z^p}\ \ (j=1,2);\ z \in \mathbb{U}\right),$$

where

$$\delta = \frac{1+(c+p)^2 - |1-(c+p)^2|}{4(c+p)} \quad (c > -p).$$

If the function $\frac{f(z)}{z^p}$ is univalent in $\mathbb{U}$ and $\frac{J_c(f)(z)}{z^p} \in \mathcal{H}[1,1]\cap\mathcal{Q}$. Then the following subordination relationship

$$\frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p}$$

implies that

$$\frac{J_c(g_1)(z)}{z^p} \prec \frac{J_c(f)(z)}{z^p} \prec \frac{J_c(g_2)(z)}{z^p}.$$

Moreover, the functions $\frac{J_c(g_1)(z)}{z^p}$ and $\frac{J_c(g_2)(z)}{z^p}$ are, respectively, the best subordination and the best dominant.

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关于Liu-Owa积分算子的双重从属保持性质

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摘要: 本文研究了单位圆盘内关于Liu-Owa 积分算子的多叶解析函数类的从属和超从属保持问题. 利用微分从属的方法, 获得了该类函数的中间型结果, 推广和改进了一些已知结果.

关键词: 解析和多叶函数; 微分从属; 超从属; Liu-Owa 积分算子; 中间型结果

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