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BOUNDED VARIATION SOLUTIONS FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper, bounded variation solution for a class of retarded functional differential equations is studied. By using the Henstock-Kurzweil integral and Schauder fixed-point theorem, the existence theorem of bounded variation solutions for this class of retarded functional differential equations is obtained in Henstock-Kurzweil integral setting, which generalizes some related results.

Keywords: Henstock-Kurzweil integral; retarded functional differential equations; bounded variation solutions

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1 Introduction

The Henstock-Kurzweil integral encompasses the Riemann, and Lebesgue integrals [1]. A particular feature of this integral is that integrals of highly oscillating function such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0, 1], F(0) = 0$ can be defined. The integral was introduced by Henstock and Kurzweil independently in 1957–1958 and was proved useful in the study of ordinary differential equations (see [2]). In this paper, an existence theorem for bounded variation solutions to retarded functional differential equations (RFDEs) is extended using the Henstock-Kurzweil integral.

Let r and σ be nonnegative real numbers and t_0 some real number. Let $x \in \mathbb{R}^n$ be some function defined on $[t_0 - r, t_0 + \sigma]$. For any $t \in [t_0, t_0 + \sigma]$, the function $x_t \in \mathbb{R}^n$ is defined as $x_t(\theta) = x(t + \theta)$, where $\theta \in [-r, 0]$. A detailed account of the existence of solutions to

$$\dot{x}(t) = f(t, x_t) \tag{1.1}$$

with some initial function

$$x_{t_0} = \phi$$

can be found in Hale [3]. This result was established under certain assumption concerning the continuity of f and ϕ. Moreover in [3–5], the authors assumed that the indefinite integral

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of f satisfies Carathéodory- and Lipschitz-type properties. Also the mapping $t \to f(t, x_t)$ is Lebesgue integrable.

Finding a solution to (1.1) is equivalent to solving the integral equation

$$\begin{cases} x(t) = \phi(0) + \int_{t_0}^{t} f(s, x_s) ds, \qquad t \geq t_0, \\ x_{t_0} = \phi, \end{cases} \tag{1.2}$$

which is meaningful for a more general class of integrands. Hale notes that the existence results can be extended for integrands f satisfying a Carathéodory condition. The integral equation is formulated using the lebesgue integral.

In this paper, the conditions we assume on the righthand sides of the RFDEs are more general than those considered in [3–5]. We consider that the integrands f is Henstock-Kurzweil integrable and ϕ is a regulated function.

This paper is organized as follows: In Section 2 we recall Henstock-Kurzweil integral and some basic known results. In Section 3 we review retarded functional differential equations and present some results. The existence theorem of bounded variation solutions for retarded functional differential equations is established in Section 4.

2 Henstock-Kurzweil Integrals

In this section, we briefly recall Henstock-Kurzweil integral and some basic known results, which will be used in the sequel.

Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $\|\cdot\|$ be a norm in $\mathbb{R}^n$.

Let D be a finite collection of interval-point pairs $\{([t_{i-1}, t_i], \xi_i)\}_{i=1}^n$, where $\{[t_{i-1}, t_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a, b]$. Let $\delta(\xi)$ be a positive function on $[a, b]$, i.e. $\delta(\xi) : [a, b] \to \mathbb{R}^+$. We say $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^n$ is δ-fine of $[a, b]$ if $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, 2, \cdots, n$.

Definition 2.1 A function $u : [a, b] \to \mathbb{R}^n$ is said to be Henstock-Kurzweil integrable on $[a, b]$ if there exists an $I \in \mathbb{R}^n$ such that for every $\varepsilon > 0$, there exists $\delta(\xi) : [a, b] \to \mathbb{R}^+$ such that for every δ-fine partition $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^n$, we have

$$\|\sum_{i=1}^{n} u(\xi_i)(t_i - t_{i-1}) - I\| < \varepsilon.$$

We denote the Henstock-Kurzweil integral (also write as H-K integral) I by (H-K)$\int_a^b u(s)ds$.

This integral was discovered independently by Henstock and Kurzweil in 1957–1958. It extends the Riemann, improper Riemann, Lebesgue and Newton integrals. For a detailed discussion of Henstock-Kurzweil integral see [1], [7] and [12]. Unless otherwise stated, all notations can be found in [1].

The relationship between the Henstock-Kurzweil and the Lebesgue integrals is detailed in the following lemma:

Lemma 2.2 (see [7]) If u is Lebesgue integrable on the interval $[a, b]$, then it is H-K integrable on this interval.

Lemma 2.3 (see [7]) If u is H-K integrable on $[a, b]$ and nonnegative, then it is Lebesgue integrable there.

Using Theorem 1.29 in [8] and Theorem 7 in [6] the following equivalent form of the convergence theorem for Henstock-Kurzweil integrals of $\mathbb{R}^n$-valued functions can be given.

Theorem 2.4 Let $u, u_m : [a, b] \to \mathbb{R}^n, m = 1, 2, \cdots$ and $\{u_m\}$ is a sequence of H-K integrable function on $[a, b]$ satisfying the following conditions:

(i) There exists a positive function $\delta : [a, b] \to \mathbb{R}^+$ such that for every $\varepsilon > 0$ there exist a $p : [a, b] \to \mathbb{N}$ and a positive superadditive interval function Φ defined for closed intervals $J \in [a, b]$ with $\Phi([a, b]) < \varepsilon$ such that for every $\tau \in [a, b]$ we have

$$||[u_m(\tau) - u(\tau)]|J||| \leq \Phi(J) \tag{2.1}$$

provided $m > p(\tau)$ and (τ, J) is an δ-fine tagged interval with $\tau \in J$.

(ii) There exists a compact set $S \subset \mathbb{R}^n$ and $\theta : [a, b] \to \mathbb{R}^+$ such that for all θ-fine partitions $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^k$ and natural number $m = m(\xi_i), i = 1, 2, \cdots, k$, we have

$$\sum_{i=1}^{k} u_m(\xi_i)(t_i - t_{i-1}) \in S. \tag{2.2}$$

Then u is H-K integrable on $[a, b]$ and

$$\lim_{k\to\infty} \int_a^b u_m(s)ds = \int_a^b u(s)ds. \tag{2.3}$$

Proof Clearly a convergence result for integrals of $\mathbb{R}^n$-valued functions holds if and only if it holds for every component of the functions. Therefore without loss of generality we can consider sequence of real-valued functions only. Assume that u is a real valued function. By (2.1), we have

$$|[u_m(\tau) - u(\tau)]|J|| \leq \Phi(J).$$

Since S from (2.2) is a compact set in $\mathbb{R}^n$ there is $A > 0$ such that $S \subset [-A, A]^n \subset \mathbb{R}^n$, where $[-A, A]^n$ is the n-dimensional cube centered at the origin in $\mathbb{R}^n$ with the edge length $2A$. Let $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^k$ be an arbitrary θ-fine partition of $[a, b]$ and natural number $m = m(\xi_i), i = 1, 2, \cdots, k$. By (2.2), we have

$$-A \leq \sum_{i=1}^{k} u_m(\xi_i)(t_i - t_{i-1}) \leq A.$$

By Theorem 1.29 in [8] the conclusion of this theorem holds.

3 Retarded Functional Differential Equations

In this section, we review retarded functional differential equations and present some results. We start this section by recalling the concept of a regulated function. Let $G([a,b],\mathbb{R}^n)$ be the space of regulated functions $x:[a,b]\to\mathbb{R}^n$, that is, the lateral limits $x(t+)=\lim\limits_{\rho\to 0+}x(t+\rho), t\in[a,b)$, and $x(t-)=\lim\limits_{\rho\to 0-}x(t+\rho), t\in(a,b]$, exist and are finite. $G([a,b],\mathbb{R}^n)$ which is a Banach space when endowed with the norm $\|\phi\|=\sup\limits_{a\le t\le b}\|\phi(t)\|$ for all $\phi\in G([a,b],\mathbb{R}^n)$. Also, any function in $G([a,b],\mathbb{R}^n)$ is the uniform limit of step functions (see [9]). Define

$$G^-([a,b],\mathbb{R}^n)=\{u\in G([a,b],\mathbb{R}^n): u \text{ is left continuous at every } t\in(a,b]\}.$$

In $G^-([a,b],\mathbb{R}^n)$, we consider the norm induced by $G^-([a,b],\mathbb{R}^n)$. We denote by $BV([a,b],\mathbb{R}^n)$ the space of functions $x:[a,b]\to\mathbb{R}^n$ which are of bounded variation. In $BV([a,b],\mathbb{R}^n)$, we consider the variation norm given by $\|x\|_{BV}=\|x(a)\|+\mathrm{Var}_a^b x$, where $\mathrm{Var}_a^b x$ stands for the variation of x in the interval $[a,b]$. Then $(BV([a,b],\mathbb{R}^n),\|\cdot\|_{BV})$ is a Banach space and $BV([a,b],\mathbb{R}^n)\subset G([a,b],\mathbb{R}^n)$. When $x\in BV([a,b],\mathbb{R}^n)$ is also left continuous, we write $x\in BV^-([a,b],\mathbb{R}^n)$.

It is clear that for a function $x\in G^-([t_0-r,t_0+\sigma],\mathbb{R}^n)$, we have $x_t\in G^-([-r,0],\mathbb{R}^n)$ for all $t\in[t_0,t_0+\sigma]$.

Let us consider the initial value problem for RFDEs (1.1)

$$\begin{cases}\dot{x}(t)=f(t,x_t),\\ x_{t_0}=\phi,\end{cases}\tag{3.1}$$

where $\phi\in G^-([-r,0],\mathbb{R}^n), r\ge 0$, and $f(t,\phi)$ maps some open subset of $[t_0,t_0+\sigma]\times G^-([-r,0],\mathbb{R}^n)$ to $\mathbb{R}^n$. It is known that system (3.1) is equivalent to the integral equation (1.2) when the integral exists in the Henstock-Kurzweil sense.

Let us recall the concept of a solution of problem (3.1).

Definition 3.1 (see [7]) A function $x\in G^-([t_0-r,t_0+\sigma],\mathbb{R}^n)$ such that $(t,x_t)\in[t_0,t_0+\sigma]\times G^-([-r,0],\mathbb{R}^n)$ for all $t\in[t_0,t_0+\sigma]$ and moreover,

(i) $\dot{x}(t)=f(t,x_t)$, almost everywhere,

(ii) $x_{t_0}=\phi$

are satisfied is called a (local) solution of (3.1) in $[t_0,t_0+\sigma]$ (or sometimes also in $[t_0-r,t_0+\sigma]$) with initial condition (t_0,ϕ).

Let $G_1\subset G^-([t_0-r,t_0+\sigma],\mathbb{R}^n]$ with the following property: if $x=x(t), t\in[t_0-r,t_0+\sigma]$, is an element of G_1 and $\bar{t}\in[t_0-r,t_0+\sigma]$, then $\bar{x}$ given by

$$\bar{x}(t)=\begin{cases}x(t), & t_0-r\le t\le\bar{t},\\ x(\bar{t}+), & \bar{t}<t<t_0+\sigma\end{cases}$$

also belongs to G_1.

Let $H_1\subset G^-([-r,0],\mathbb{R}^n)$ be such that $\{x_t|t\in[t_0,t_0+\sigma], x\in G_1\}\subset H_1$ and assume $f(t,x_t):[t_0,t_0+\sigma]\times H_1\to\mathbb{R}^n$ satisfy the following conditions:

(A) There exists a positive function $\delta(\tau):[t_0,t_0+\sigma]\to\mathbb{R}^+$ such that for every $[u,v]$ satisfy $\tau\in[u,v]\subset(\tau-\delta(\tau),\tau+\delta(\tau))\subset[t_0,t_0+\sigma]$ and $x\in G_1$, we have

$$\|f(\tau,x_\tau)(v-u)\|\le|h(v)-h(u)|.$$

(B) For every $[u,v]$ satisfy $\tau\in[u,v]\subset(\tau-\delta(\tau),\tau+\delta(\tau))\subset[t_0,t_0+\sigma]$ and $x,y\in G_1$, we have

$$\|f(\tau,x_\tau)-f(\tau,y_\tau)\|(v-u)\le\omega(\|x_\tau-y_\tau\|)|h(v)-h(u)|,$$

where $h:[t_0,t_0+\sigma]\to\mathbb{R}$ is a nondecreasing function and continuous the left. $\omega:[0,\infty]\to\mathbb{R}$ is a continuous and increasing function with $\omega(0)=0,\omega(r)>0$ for $r>0$.

Let $\Omega\subset[t_0,t_0+\sigma]\times H_1$ be open, we have the following the results.

Definition 3.2 Assume that function $f:\Omega\to\mathbb{R}^n$ is a Carathéodory function and belongs to the class $F(\Omega,h,\omega)$ if f satisfys the condition (A), (B).

Theorem 3.3 Assume that $f\in F(\Omega,h,\omega)$ is given and that $x:[\alpha,\beta]\to\mathbb{R}^n,[\alpha,\beta]\subseteq[t_0,t_0+\sigma]$ is the pointwise limit of a sequence $\{x^k\}_{k\in N}$ of functions $x^k:[\alpha,\beta]\to\mathbb{R}^n$ such that $(x_s,s)\in\Omega,((x^k)_s,s)\in\Omega$ for every $k\in\mathbb{N}$ and $s\in[\alpha,\beta]$ and that (H-K)$\displaystyle\int_\alpha^\beta f(s,(x^k)_s)\mathrm{d}s$ exists for every $k\in\mathbb{N}$. Then the integral (H-K)$\displaystyle\int_\alpha^\beta f(s,x_s)\mathrm{d}s$ exists and

$$\int_\alpha^\beta f(s,x_s)\mathrm{d}s=\lim_{k\to\infty}\int_\alpha^\beta f(s,(x^k)_s)\mathrm{d}s.$$

Proof Assume that $\varepsilon>0$ is given. By condition (B), we have

$$\|f(\tau,(x^k)_\tau)-f(\tau,x_\tau)\|(t_2-t_1)\le\omega(\|(x^k)_\tau-x_\tau\|)(h(t_2)-h(t_1))\tag{3.2}$$

for every $\tau\in[\alpha,\beta]\subset[t_0,t_0+\sigma],t_1\le\tau\le t_2,[t_1,t_2]\subset[\alpha,\beta]$. Let us set

$$\mu(t)=\frac{\varepsilon}{h(\beta)-h(\alpha)+1}h(t)\ \text{ for }\ t\in[\alpha,\beta].$$

The function $\mu:[\alpha,\beta]\to\mathbb{R}$ is nondecreasing and $\mu(\beta)-\mu(\alpha)<\varepsilon$. Since

$$\lim_{k\to\infty}\|(x^k)_\tau-x_\tau\|=0$$

for every $\tau\in[\alpha,\beta]$ and the function ω is continuous at 0, there is a $p(\tau)\in\mathbb{N}$ such that for $k\ge p(\tau)$ we have

$$\omega(\|(x^k)_\tau-x_\tau\|)\le\frac{\varepsilon}{h(\beta)-h(\alpha)+1}.$$

Let $\Phi(J)=\mu(t_2)-\mu(t_1),J=[t_1,t_2]$, for $k\ge p(\tau)$ the inequality (3.2) can be rewritten to the form

$$\|f(\tau,(x^k)_\tau-f(\tau,x_\tau)\|(t_2-t_1)\le\frac{\varepsilon}{h(\beta)-h(\alpha)+1}(h(t_2)-h(t_1))=\Phi(J),$$

where $\tau \in J \subset (\tau - \delta(\tau), \tau + \delta(\tau)) \subset [\alpha, \beta]$.

By condition (A), then there exists $\theta : [\alpha, \beta] \to R^+$ such that for every θ-fine partition $D = \{([t_{i-1}, t_i], \xi_i)\}_{i=1}^k$ of $[\alpha, \beta]$,

$$\|\sum_{i=1}^{k} f(\xi_i, (x^k)_{\xi_i})(t_i - t_{i-1})\| \le \sum_{i=1}^{k} |h(t_i) - h(t_{i-1})| < h(\beta) - h(\alpha),$$

and this means that the sum $\sum\limits_{i=1}^{k} f(\xi_i, (x^k)_{\xi_i})(t_i - t_{i-1})$ belongs to the compact ball

$$S = \{x \in \mathbb{R}^n; \|x\| < h(\beta) - h(\alpha)\}$$

in $\mathbb{R}^n$. By Theorem 2.4 the integral $\displaystyle\int_\alpha^\beta f(s, x_s)\mathrm{d}s$ exists and the conclusion of the theorem holds.

Corollary 3.4 Assume that $f \in F(\Omega, h, \omega)$ is given and that $x : [\alpha, \beta] \to \mathbb{R}^n, [\alpha, \beta] \subseteq [t_0, t_0 + \sigma]$ is the pointwise limit of a sequence $\{\psi^k\}_{k\in\mathbb{N}}$ of finite step functions $\psi^k : [\alpha, \beta] \to \mathbb{R}^n$ such that $(s, x_s) \in \Omega, (s, (\psi^k)_s) \in \Omega$ for every $s \in [\alpha, \beta]$ and $k \in \mathbb{N}$ and that (H-K)$\displaystyle\int_\alpha^\beta f(s, (\psi^k)_s)\mathrm{d}s$ exists for every $k \in \mathbb{N}$. Then the integral (H-K)$\displaystyle\int_\alpha^\beta f(s, x_s)\mathrm{d}s$ exists and

$$\int_\alpha^\beta f(s, x_s)\mathrm{d}s = \lim_{k\to\infty} \int_\alpha^\beta f(s, (\psi^k)_s)\mathrm{d}s. \tag{3.3}$$

Proof For every $k \in \mathbb{N}$, the integral $\displaystyle\int_\alpha^\beta f(s, (\psi^k)_s)\mathrm{d}s$ exists, and by Theorem 3.3, we get the integral (H-K)$\displaystyle\int_\alpha^\beta f(s, x_s)\mathrm{d}s$ and (3.3) holds.

Corollary 3.5 If $f \in F(\Omega, h, \omega)$ is given and $x : [\alpha, \beta] \to \mathbb{R}^n, [\alpha, \beta] \subseteq [t_0, t_0 + \sigma]$ is a function of bounded variation on $[\alpha, \beta]$ such that $(s, x_s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral (H-K)$\displaystyle\int_\alpha^\beta f(s, x_s)\mathrm{d}s$ exists.

Proof The result follows from Corollary 3.1 because every function of bounded variation is the uniform limit of finite step functions [9].

Theorem 3.6 Assume that $f \in F(\Omega, h, \omega)$. If $[\alpha, \beta] \subset [t_0 - r, t_0 + \sigma]$ and $x : [\alpha, \beta] \to \mathbb{R}^n$ is such that $(t, x_t) \in \Omega$ for every $t \in [\alpha, \beta]$ and if the integral (H-K)$\displaystyle\int_\alpha^\beta f(t, x_t)\mathrm{d}t$ exists, then for every $s_1, s_2 \in [\alpha, \beta]$ the inequality

$$\left\| \int_{s_1}^{s_2} f(t, x_t)\mathrm{d}t \right\| \le |h(s_2) - h(s_1)| \tag{3.4}$$

is satisfied.

Proof Let an arbitrary $\varepsilon > 0$ be given, since the integral (H-K)$\displaystyle\int_\alpha^\beta f(t, x_t)\mathrm{d}t$ exists. Then the integral (H-K)$\displaystyle\int_{s_1}^{s_2} f(t, x_t)\mathrm{d}t$ exists for every $s_1, s_2 \in [\alpha, \beta]$. By Definition 2.1 and condition (A), there exists $\delta(\tau)$ such that for every δ-fine partition $D = \{([u_i, v_i], \tau_i)\}_{i=1}^n$ of the interval $[s_s, s_2]$, we have

$$\begin{aligned}
\Big\| \int_{s_1}^{s_2} f(t, x_t)\mathrm{d}t \Big\| \;\; &\leq \;\; \Big\| \int_{s_1}^{s_2} f(t, x_t)\mathrm{d}t - \sum_{i=1}^{n} [f(\tau_i, x_{\tau_i})(v_i - u_i)] \Big\| \\
&\quad + \Big\| \sum_{i=1}^{n} [f(\tau_i, x_{\tau_i})(v_i - u_i)] \Big\| \\
&< \;\; \varepsilon + \sum_{i=1}^{n} \Big\| [f(\tau_i, x_{\tau_i})(v_i - u_i)] \Big\| \\
&< \;\; \varepsilon + \sum_{i=1}^{n} |h(v_i) - h(u_i)| \\
&\leq \;\; \varepsilon + |h(s_2) - h(s_1)|.
\end{aligned}$$

Since $\varepsilon > 0$ can be arbitrary, we have

$$\Big\| \int_{s_1}^{s_2} f(t, x_t)\mathrm{d}t \Big\| \leq |h(s_2) - h(s_1)|.$$

4 Retarded Functional Differential Equations

In this section, we discuss the existence of bounded variation solutions for retarded functional differential equations and establish the existence theorem.

Let us recall the concept of a bounded variation solution of RFDEs (1.1).

Definition 4.1 A function $x \in BV^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \subset G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is called a solution of RFDEs (1.1) with initial function ϕ at t_0 if there exists an $\sigma > 0$ such that

$$\dot{x}(t) = f(t, x_t)$$

for almost all $t \in [t_0, t_0 + \sigma]$ and $x_{t_0} = \phi$.

Theorem 4.2 Assume that $f \in F(\Omega, h, \omega)$. If $[\alpha, \beta] \subset [t_0 - r, t_0 + \sigma]$ and $x : [\alpha, \beta] \to \mathbb{R}^n$ is a solution of equation (1.1), then x is bounded variation on $[\alpha, \beta]$ and

$$\mathrm{Var}_\alpha^\beta x \leq h(\beta) - h(\alpha) < +\infty.$$

Proof Let $\alpha = t_0 < t_1 < \cdots < t_k = \beta$ be an arbitrary division of the interval $[\alpha, \beta]$. By (3.4), we have

$$\sum_{i=1}^{k} \|x(t_i) - x(t_{i-1})\| = \sum_{i=1}^{k} \Big\| \int_{t_{i-1}}^{t_i} f(t, x_t)\mathrm{d}t \Big\| \leq \sum_{i=1}^{k} |h(t_i) - h(t_{j-1})| < h(\beta) - h(\alpha).$$

To prove our basic existence theorem, we need the Schauder fixed-point theorem. The following we state the Schauder theorem from Lemma 2.4 in [3].

Lemma 4.3 (Schauder fixed-point theorem) If U is a closed bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.

It is convenient here to introduce an auxiliary function $\widehat{x}$: if $x \in G^-([t_0, t_0+\sigma], \mathbb{R}^n)$ with $x(t_0) = \phi(0)$, the function $\widehat{x} \in G^-([t_0 - r, t_0 + \sigma])$ is defined as

$$\widehat{x}(t) = \begin{cases} x(t), & t \in [t_0, t_0 + \sigma], \\ \phi(t - t_0), & t \in [t_0 - r, t_0]. \end{cases}$$

Note that the above definition ensures that $\widehat{x}_{t_0} = \phi$ on $[-r, 0]$.

Theorem 4.4 Let ϕ be some fixed function in H_1, $f \in F(\Omega, h, \omega)$ and for $\theta_1, \theta_2 \in [-r, 0]$ such that

$$\|\phi(\theta_1) - \phi(\theta_2)\| \leq |h(\theta_1) - h(\theta_2)| \tag{4.1}$$

are fulfilled. Then for every $(t_0, \phi) \in \Omega$, there exists a $\Delta > 0$ such that on the interval $[t_0 - r, t_0 + \Delta] \subset [t_0 - r, t_0 + \sigma]$ there exists a solution $\widehat{x} \in BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n) \subset G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ to the REDEs (1.1) with initial function ϕ at t_0.

Proof We will consider two cases: when t_0 is a point of continuity of $h : [t_0, t_0+\sigma] \to \mathbb{R}$ and otherwise.

At first, let t_0 be a point of continuity of the function h; i.e., $h(t_0+) = h(t_0)$. Since G_1 is open, there is a $\Delta > 0$ such that if $t \in [t_0, t_0 + \Delta] \subset [t_0, t_0 + \sigma)$ and $x \in \mathbb{R}^n$ is such that $\|x(t) - \phi(0)\| = \|x(t) - x(t_0)\| < |h(t) - h(t_0)|$ then $(t, x_t) \in \Omega$.

Let the set $Q \subset BV^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n) \subset G_1$ is defined as

$$Q = \{\widehat{x} \in BV^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n) : \|x - \phi(0)\| \leq b, \text{ and for all } t \in [t_0, t_0 + \Delta], \|\widehat{x}_t - \phi\| \leq b\}$$

(Here b is positive number and ϕ is fixed in H_1).

It is also easy to check that the set Q is convex, i.e., if $\widehat{x}, \widehat{y} \in Q, \alpha \in [0, 1]$, then $\alpha\widehat{x} + (1 - \alpha)\widehat{y} \in Q$.

Next, let us show that Q is a closed subset of $BV^-([t_0-r, t_0+\Delta], \mathbb{R}^n)$. Let $\widehat{z}^k \in Q, k \in \mathbb{N}$ be sequence which converges in $BV([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ to a function $\widehat{z}$. Since

$$\|\widehat{z}^k - \widehat{z}\| \leq \|\widehat{z}^k - \widehat{z}\|_{BV},$$

we have

$$\lim_{k\to\infty} \|z^k - z\| = \lim_{k\to\infty} \|\widehat{z}^k - \widehat{z}\| = 0$$

uniformly for $t \in [t_0, t_0 + \Delta]$. Therefore we have

$$\|z - \phi(0)\| \leq \|z^k - z\| + \|z^k - \phi(0)\| < \varepsilon + b$$

for any $\varepsilon > 0$ whenever $k \in \mathbb{N}$ is sufficiently large. This yields

$$\|z - \phi(0)\| \leq b.$$

Similarly, we can show that

$$\|(\widehat{z}^k)_t - \widehat{z}_t\| \leq \|(\widehat{z}^k)_t - \widehat{z}_t\|_{BV}$$

for every $t \in [t_0, t_0 + \Delta]$, we have

$$\lim_{k\to\infty} \|(\widehat{z}^k)_t - \widehat{z}_t\| = 0$$

uniformly for $t \in [t_0, t_0 + \Delta]$. Therefore we have

$$\|\widehat{z}_t - \phi\| \leq \|(\widehat{z}^k)_t - \widehat{z}_t\| + \|(\widehat{z}^k)_t - \phi\| \leq \varepsilon + b$$

for any $\varepsilon > 0$ whenever $k \in \mathbb{N}$ is sufficiently large and $t \in [t_0, t_0 + \Delta]$. This yields

$$\|\widehat{z}_t - \phi\| \leq b$$

for $t \in [t_0, t_0 + \Delta]$ and therefore for the limit $\widehat{z}$ we have $\widehat{z} \in Q$ and Q is closed.

For $\widehat{x} \in Q$ define the map

$$T\widehat{x}(t) = \begin{cases} \phi(0) + \displaystyle\int_{t_0}^{t} f(s, x_s)\mathrm{d}s, & t \in [t_0, t_0 + \sigma], \\ \phi(t - t_0), & t \in [t_0 - r, t_0]. \end{cases}$$

The map T is well-defined because by Corollary 3.2 the integral $\displaystyle\int_{t_0}^{t} f(s, x_s)\mathrm{d}s$ exists for every $t \in [t_0, t_0 + \Delta]$.

For $t \in [t_0, t_0 + \Delta]$, by (3.4) we have

$$\|Tx(t) - \phi(0)\| = \|T\widehat{x}(t) - \phi(0)\| = \|\int_{t_0}^{t} f(s, x_s)\mathrm{d}s\| \leq |h(t_0 + \Delta) - h(t_0)|.$$

For every $t \in [t_0, t_0 + \sigma], \theta \in [-r, 0]$, let $s = t + \theta \in [t_0 - r, t_0], y(s) = T\widehat{x}(s)$, by (4.1), we get

$$\|y_t - \phi\| = \|T\widehat{x}(s) - \phi\| = \|\phi(s - t_0) - \phi(\theta)\| \leq |h(\theta_1) - h(\theta)|,$$

where $\theta_1 = s - t_0 \in [-r, 0]$. Let $b = \max\{|h(t_0 + \Delta) - h(t_0)|, |h(\theta_1) - h(\theta)|\}$. Hence $T\widehat{x} \subset Q$ for $\widehat{x} \in Q$, i.e., T maps Q into itself.

Let us show that $T : Q \to Q$ is continuous. If $\widehat{z}, \widehat{v} \in Q$ then

$$\begin{aligned} \|T\widehat{z} - T\widehat{v}\|_{BV} &= \|Tz - Tv\|_{BV} = \|Tz(t_0) - Tv(t_0)\| + \mathrm{Var}_{t_0}^{t_0+\Delta}(Tz - Tv) \\ &\leq 2\mathrm{Var}_{t_0}^{t_0+\Delta}(Tz - Tv) \end{aligned} \tag{4.2}$$

for every $t \in [t_0, t_0 + \Delta]$. Take $t_0 \leq s_1 < s_2 \leq t_0 + \Delta$ and $u, v \in Q$, we obtain

$$\|Tu(s_2) - Tv(s_2) - Tu(s_1) + Tv(s_1)\| = \Big\| \int_{s_1}^{s_2} [f(t, u_t) - f(t, v_t)]\mathrm{d}t \Big\|. \tag{4.3}$$

Since (H-K)$\displaystyle\int_{s_1}^{s_2}[f(t,u_t)-f(t,v_t)]\mathrm{d}t$ exists, then given $\varepsilon>0$, there exists $\delta:[s_1,s_2]\to\mathbb{R}^+$ such that for every δ-fine partition $D=\{([t_{i-1},t_i],\tau_i)\}_{i=1}^m$ of $[s_1,s_2]$, we have

$$\begin{aligned}\Big\|\int_{s_1}^{s_2}[f(t,u_t)-f(t,v_t)]\mathrm{d}t\Big\| &\le \Big\|\int_{s_1}^{s_2}[f(t,u_t)-f(t,v_t)]\mathrm{d}t\\ &\quad -\sum_{i=1}^m[f(\tau_i,u_{\tau_i})-f(\tau_i,v_{\tau_i})](t_i-t_{i-1})\Big\|\\ &\quad +\Big\|\sum_{i=1}^m[f(\tau_i,u_{\tau_i})-f(\tau_i,v_{\tau_i})](t_i-t_{i-1})\Big\|\\ &< \frac{\varepsilon}{2}+\sum_{i=1}^m\omega(\|u_{\tau_i}-v_{\tau_i}\|)|h(t_i)-h(t_{i-1})|\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^m\omega(\|u_{\tau_i}-v_{\tau_i}\|)(h(t_i)-h(t_{i-1})) &\le \Big\|\sum_{i=1}^m\omega(\|u_{\tau_i}-v_{\tau_i}\|)(h(t_i)-h(t_{i-1}))\\ &\quad -\int_{s_1}^{s_2}\omega(\|u_t-v_t\|)\mathrm{d}h(t)\Big\|+\int_{s_1}^{s_2}\omega(\|u_t-v_t\|)\mathrm{d}h(t)\\ &< \frac{\varepsilon}{2}+\int_{s_1}^{s_2}\omega(\|u_t-v_t\|)\mathrm{d}h(t).\end{aligned}$$

Hence

$$\|Tu(s_2)-Tv(s_2)-Tu(s_1)+Tv(s_1)\|<\varepsilon+\int_{s_1}^{s_2}\omega(\|u_t-v_t\|)\mathrm{d}h(t).$$

Since $\varepsilon>0$ be arbitrary, we have

$$\|Tu(s_2)-Tv(s_2)-Tu(s_1)+Tv(s_1)\|<\int_{s_1}^{s_2}\omega(\|u_t-v_t\|)\mathrm{d}h(t)$$

and

$$\mathrm{Var}_{t_0}^{t_0+\Delta}(Tu-Tv)<\int_{t_0}^{t_0+\Delta}\omega(\|u_t-v_t\|)\mathrm{d}h(t). \tag{4.4}$$

Assume that $z,z^k\in Q,k\in\mathbb{N}$ and $\lim\limits_{k\to\infty}\|z^k-z\|_{BV}=0$, then

$$\lim_{k\to\infty}\|(z^k)_t-z_t\|=0$$

uniformly for $t\in[t_0,t_0+\Delta]$ and by the function ω is continuous at 0 and $\omega(0)=0$, we obtain

$$\lim_{k\to\infty}\int_{t_0}^{t_0+\Delta}\omega(\|(z^k)_t-z_t\|)\mathrm{d}h(t)=0.$$

Therefore by (4.4),

$$\lim_{k\to\infty}\mathrm{Var}_{t_0}^{t_0+\Delta}(Tz^k-Tz)=0$$

and (4.2) yields

$$\lim_{k\to\infty} \|Tz^k - Tz\|_{BV} = 0.$$

On the other hand, for $t \in [t_0 - r, t_0]$ we have

$$(Tz^k)(t) = \phi(t - t_0) = (Tz)(t).$$

Then $Tz^k \to Tz \quad (k \to \infty)$, i.e., T is a continuous map.

Finally we show that $T(Q) \subset Q$ is sequentially compact in the Banach space $BV([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$. Let $\widehat{x}^k \in Q, k \in \mathbb{N}$ be an arbitrary sequence in Q. The sequence $\{\widehat{x}^k\}_{k=1}^{\infty}$ consists of equally bounded functions of equibounded variation and therefore Helly's Choice theorem (see [10]) yields that this sequence contains a pointwise convergent subsequence which we again denote by $\{\widehat{x}^k\}_{k=1}^{\infty}$. Hence we have $\lim\limits_{k\to\infty} \widehat{x}^k(t) = \widehat{x}(t)$ for every $t \in [t_0 - r, t_0 + \Delta]$, the values of $\widehat{x} \in BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$. Moreover, let $t = s + \theta, \theta \in [-r, 0], s \in [t_0, t_0 + \Delta]$, we have $\lim\limits_{k\to\infty} (\widehat{x}^k)_s = \widehat{x}_s$.

Put

$$y(t) = T\widehat{x}(t) = \begin{cases} \phi(0) + \displaystyle\int_{t_0}^{t} f(s, x_s)\mathrm{d}s, & t \in [t_0, t_0 + \Delta], \\ \phi(t - t_0), & t \in [t_0 - r, t_0]. \end{cases}$$

By (3.4) we have $y \in BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ and it is not difficult to show that

$$\lim_{k\to\infty} \|T\widehat{x}^k - y\|_{BV} = \lim_{k\to\infty} \|T\widehat{x}^k - T\widehat{x}\|_{BV} = 0.$$

This immediately leads to the conclusion that every sequence in $T(Q)$ contains a subsequence which converges in $BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ and consequently, $T(Q)$ is sequentially compact.

All assumptions of the Schauder fixed-point theorem are satisfied we can conclude that there exists at least one $\widehat{x} \in Q$ such that $\widehat{x} = T\widehat{x}$, i.e.,

$$\widehat{x}(t) = T\widehat{x}(t) = \begin{cases} \phi(0) + \displaystyle\int_{t_0}^{t} f(s, x_s)\mathrm{d}s, & t \in [t_0, t_0 + \Delta], \\ \phi(t - t_0), & t \in [t_0 - r, t_0]. \end{cases}$$

Note that $x(t) = \widehat{x}(t)$ on $[t_0, t_0 + \Delta]$ and $\widehat{x}(t) = \phi(t - t_0)$ on $[t_0 - r, t_0]$. The RFDEs (1.1) with initial function ϕ therefore has a solution $\widehat{x} \in BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n) \subset G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$.

Now, we consider the case where t_0 is not a point of continuity of h. Define

$$\widetilde{h}(t) = \begin{cases} h(t), & t \le t_0, \\ h(t) - h(t_0+) + h(t_0) = h(t) - h(t_0+), & t_0 \le t \le t_0 + \sigma. \end{cases}$$

Then the function $\widetilde{h}$ is continuous at t_0, continuous from the left and nondecreasing. For $t \in [t_0, t_0 + \sigma]$, by (3.4), we have

$$\Big\| \int_{t_0}^{t} f(t, x_t)\mathrm{d}t \Big\| \le |\widetilde{h}(t) - \widetilde{h}(t_0)|.$$

As in the previous case, there is a $\Delta > 0$ such that if $t \in [t_0, t_0 + \Delta] \subset [t_0, t_0 + \sigma)$ and $x \in \mathbb{R}^n$ is such that $\|x(t) - \phi(0)\| = \|x(t) - x(t_0)\| < |\widetilde{h}(t) - \widetilde{h}(t_0)|$, then $(t, x_t) \in \Omega$.

Following the procedure of the previous case, it can be show that RFDEs (1.1) admits a unique solution $\widehat{x} \in BV^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n) \subset G^-([t_0 - r, t_0 + \Delta], \mathbb{R}^n)$ with initial function ϕ at t_0.

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滞后型泛函微分方程的有界变差解

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摘 要: 本文研究了一类滞后型泛函微分方程的有界变差解. 利用 Henstock-Kurzweil 积分与 Schauder 不动点定理, 在 Henstock-Kurzweil 积分下, 得到了这类滞后型泛函微分方程有界变差解的存在性定理, 推广了一些相关的结果.

关键词: Henstock-Kurzweil积分; 滞后型泛函微分方程; 有界变差解

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